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GROUPS WITH FINITELY MANY NORMALIZERS OF NON-SUBNORMAL SUBGROUPS

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It is proved that a group G has finitely many normalizers of non-subnormal subgroups if and only if each subgroup of G either is subnormal or has finitely many conjugates; groups with this latter property have been completely described in [8]. Moreover, groups with finitely many normalizers of infinite non-subnormal subgroups are described.

1. Introduction.

A subgroup X of a group G is called *almost normal* if it has finitely many conjugates in G, or equivalently if its normalizer $N_G(X)$ has finite index in G. In a famous paper of 1955, B. H. Neumann [13] proved that all subgroups of a group G are almost normal if and only if the centre $Z(G)$ has finite index, and the same conclusion holds if the restriction is imposed only to abelian subgroups (see [6]). Thus central-by-finite groups are precisely those groups in which all the normalizers of (abelian) subgroups have finite index, and this result suggests that the behaviour of normalizers has a strong influence on the structure of the group. In fact, Y. D. Polovickiĭ [14] has shown that if a group G has finitely many

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normalizers of abelian subgroups, then the factor group $G/Z(G)$ is finite. Recently, groups have been considered with finitely many normalizers of subgroups with a given property (see [4], [5]).

It is well known that a finite group is nilpotent if and only if all its subgroups are subnormal, but there exist infinite metabelian primary groups with trivial centre in which every subgroup is subnormal (see for instance [9]). The structure of groups in which all subgroups are subnormal has been investigated by W. Möhres; among many other deep results, Möhres proved that such groups are soluble (see [11]). Some other relevant results on groups in which every subgroup is subnormal have more recently been obtained by C. Casolo ([1], [2], [3]) and H. Smith [16].

In the last few years there has also been considerable activity in the study of groups with few non-subnormal subgroups (see for instance [7], [10], [17]). In this article we will describe groups with finitely many normalizers of non-subnormal subgroups; it will be proved that such groups coincide with those in which every subgroup either is subnormal or has finitely many conjugates, and groups with this latter property have been completely characterized in [8]. Moreover, we will prove that if a group G has finitely many normalizers of infinite non-subnormal subgroups, then each infinite subgroup of G is either subnormal or almost normal, provided that G is locally finite or non-periodic. Our results actually holds for groups in which all but finitely many normalizers of (infinite) non-subnormal subgroups have finite index.

Most of our notation is standard and can for instance be found in [15].

2. Normalizers of non-subnormal subgroups.

The following result plays a central role in the study of groups with finitely many normalizers of subgroups with a given property. It was proved by B. H. Neumann [12] in the more general case of groups covered by finitely many cosets.

Lemma 2.1. *Let the group $G = X_1 \cup \ldots \cup X_t$ be the union of finitely many subgroups $X_1, \ldots, X_t$. Then any X_i of infinite index can be omitted from this decomposition; in particular, at least one of the subgroups $X_1, \ldots, X_t$ has finite index in G.*

Our first lemma shows that every group in which all but finitely many normalizers of (infinite) non-subnormal subgroups have finite index contains a subgroup of finite index whose (infinite) non-subnormal subgroups are almost normal.

Lemma 2.2. *Let G be a group in which all but finitely many normalizers of (infinite) non-subnormal subgroups have finite index. Then G contains a characteristic subgroup M of finite index such that each (infinite) subgroup of M is either subnormal or almost normal in G.*

Proof. If X is any (infinite) subgroup of G which neither is subnormal nor almost normal, the normalizer $N_G(X)$ has obviously finitely many images under automorphisms of G; in particular, the subgroup $N_G(X)$ has finitely many conjugates in G and so the index $|G : N_G(N_G(X))|$ is finite. It follows that also the characteristic subgroup

$$M(X) = \bigcap_{\alpha \in \, Aut \, G} N_G(N_G(X))^{\alpha}$$

has finite index in G. Let $\mathcal{H}$ be the set of all (infinite) non-subnormal subgroups of G whose normalizer has infinite index. If X and Y are elements of $\mathcal{H}$ such that $N_G(X) = N_G(Y)$, then $M(X) = M(Y)$ and hence also

$$M = \bigcap_{X \in \mathcal{H}} M(X)$$

is a characteristic subgroup of finite index of G. Let X be any (infinite) subgroup of M such that the index $|G : N_G(X)|$ is infinite; then

$$M \leq M(X) \leq N_G(N_G(X)),$$

so that $N_M(X) = N_G(X) \cap M$ is a normal subgroup of M and in particular X is subnormal in G. □

Of course, arbitrary groups with finitely many normalizers of infinite subgroups need not be soluble-by-finite, as for instance any Tarski group (i.e. any infinite simple group whose proper non-trivial subgroups have prime order) has this property. However, the following holds.

Corollary 2.3. *Let G be a group in which all but finitely many normalizers of infinite non-subnormal subgroups have finite index. If G is either locally finite or non-periodic, then G is soluble-by-finite.*

Proof. By Lemma 2.2, G contains a subgroup M of finite index in which every infinite subgroup is either subnormal or almost normal. Then M is soluble-by-finite (see [8], Theorem 3.6 and Theorem 3.15), and so G is likewise soluble-by-finite. □

The assumption that the group G is either locally finite or non-periodic can be dropped out in the case of groups with few normalizers of non-subnormal subgroups.

Corollary 2.4. *Let G be a group in which all but finitely many normalizers of non-subnormal subgroups have finite index. Then G is soluble-by-finite.*

Proof. By Lemma 2.2, G contains a subgroup M of finite index in which every subgroup is either subnormal or almost normal. Then either M is an FC-group or all its subgroups are subnormal (see [8], Proposition 2.2). In the first situation M is soluble-by-finite (see [8], Lemma 2.4) while in the second case M is even soluble by Möhres' theorem. Therefore the group G is soluble-by-finite. □

Recall that the *FC-centre* of a group G is the subgroup consisting of all elements of G having only finitely many conjugates, and a group G is called an *FC-group* if G coincides with its FC-centre. Thus a group G has the property FC if and only if the centralizer $C_G(x)$ has finite index in G for each element x. Recall also that the *Baer radical* of a group G is the subgroup generated by all abelian subnormal subgroups of G, and G is called a *Baer group* if it coincides with its Baer radical. In particular, every Baer group is locally nilpotent and all its cyclic subgroups are subnormal.

Lemma 2.5. *Let G be a Baer group in which all but finitely many normalizers of infinite non-subnormal subgroups have finite index. Then all subgroups of G are subnormal.*

Proof. Assume for a contradiction that G contains a non-subnormal subgroup X. Then X is infinite and the normalizer $N_G(X)$ has finitely many conjugates in G; it follows that $N_G(N_G(X))$ has finite index in G, so that it is subnormal and hence X itself is subnormal in G. This contradiction shows that all subgroups of G are subnormal. □

Lemma 2.6. *Let G be an FC-group in which all but finitely many normalizers of infinite non-subnormal subgroups have finite index. Then every subgroup of G is either subnormal or almost normal.*

Proof. By Lemma 2.5, it can be assumed that the Baer radical B of G is a proper subgroup. Let x be any element of $G \setminus B$ and consider the centralizer $C = C_G(x)$. Clearly, every subgroup of C containing x is not subnormal in G, so that in $C/\langle x\rangle$ all but finitely many normalizers of subgroups have finite index and it follows that $A/\langle x\rangle = Z(C/\langle x\rangle)$ has finite index in $C/\langle x\rangle$ (see [5], Theorem A). Since G is an FC-group, the index $|G : A|$ is finite and so also the core $N_x = A_G$ has finite index in G. Thus the group G/N_x' is abelian-by-finite, so that it is central-by-finite and hence G'/N_x' is finite by Schur's theorem; moreover, $N_x' \leq A' \leq \langle x\rangle$ is cyclic and so even finite, since G' is periodic. Therefore G' is finite and in particular B is nilpotent. If G is not periodic, it is generated by its elements of infinite order, so that x can be chosen to have infinite order; in this case we have $N_x' = \{1\}$, so that $G/Z(G)$ is finite. Suppose now that G is periodic, and let X be any non-subnormal subgroup of G. Then X is not contained in B, and we may consider an element x of $X \setminus B$; since $N_x' \leq \langle x\rangle \leq X$ and G/N_x' is central-by-finite, it follows that X is almost normal in G. The lemma is proved. □

We can now prove the main theorem of this section.

Theorem 2.7. *For a group G the following statements are equivalent:*

(i) *G has finitely many normalizers of non-subnormal subgroups.*

(ii) *All but finitely many normalizers of non-subnormal subgroups of G have finite index.*

(iii) *Every subgroup of G is either subnormal or almost normal.*

Proof. Suppose that all but finitely many normalizers of non-subnormal subgroups of G have finite index, and let B and F be the Baer radical and the FC-centre of G, respectively. If x is any element of the set $G\setminus(B\cup F)$, the subgroup $\langle x\rangle$ is not subnormal and the index $|G : N_G(\langle x\rangle)|$ is infinite. Therefore

$$G = B \cup F \cup N_G(X_1) \cup \ldots \cup N_G(X_k),$$

where $N_G(X_1), \dots, N_G(X_k)$ are all normalizers of infinite index of non-subnormal subgroups of G. It follows from Lemma 2.1 that $G = B \cup F$, so that either $G = B$ is a Baer group or $G = F$ is an FC-group and hence each subgroup of G is either subnormal or almost normal by Lemma 2.5 and Lemma 2.6.

Assume now that every subgroup of G is either subnormal or almost normal, and assume for a contradiction that (i) is false, so that in particular G contains some non-subnormal subgroup and $G/Z(G)$ is infinite. Then G is a periodic FC-group and contains a nilpotent normal subgroup N of finite index such that $N' \leq \langle x \rangle$ for each cyclic non-subnormal subgroup $\langle x \rangle$ of G (see [8], Proposition 2.2 and Theorem 2.8). In particular, $Z/N' = Z(G/N')$ has finite index in G/N', so that G' is finite and the Baer radical of G is nilpotent. Let X be any non-subnormal subgroup of G. Then X contains a cyclic non-subnormal subgroup $\langle x \rangle$ and hence $N' \leq \langle x \rangle \leq X$; it follows that Z is contained in $N_G(X)$, so that G has finitely many normalizers of non-subnormal subgroups. □

The above theorem allows us to give a complete description of groups with finitely many normalizers of non-subnormal subgroups. In fact, it follows from Proposition 2.2 and Theorem 2.8 of [8] that a group G has this property if and only if satisfies one of the following conditions:

(a) all subgroups of G are subnormal;

(b) the factor group $G/Z(G)$ is finite;

(c) G is periodic and contains a nilpotent normal subgroup N of finite index and class 2 whose commutator subgroup is cyclic with prime-power order p^k; moreover, the Fitting subgroup F of G has index a power of p and $N' \leq \langle x \rangle$ for each element x of $G \setminus F$.

Since a torsion-free group in which all subgroups are subnormal is nilpotent (see [1] or [16]), we have the following consequence.

Corollary 2.8. *Let G be a torsion-free group in which all but finitely many normalizers of non-subnormal subgroups have finite index. Then G is nilpotent.*

The *Wielandt subgroup* $\omega(G)$ of a group G is defined as the intersection of all normalizers of subnormal subgroups of G; here we shall denote by $\omega^*(G)$ the intersection of all the normalizers of non-subnormal subgroups of G (with the stipulation that $\omega^*(G) = G$ if all subgroups of G are subnormal). Clearly, $\omega(G) \cap \omega^*(G)$ is the well-known *norm* of G, i.e. the set of all elements inducing by conjugation power automorphisms on G.

Corollary 2.9. *Let G be a group with finitely many normalizers of non-subnormal subgroups. Then the factor group $G/\omega^*(G)$ is finite.*

Proof. Clearly, it can be assumed that $G/Z(G)$ is infinite and that G contains subgroups which are not subnormal. Then by the above description G contains a nilpotent normal subgroup N of finite index such that N' lies in every non-subnormal subgroup of G; as G/N' is central-by-finite, it follows that $G/\omega^*(G)$ is finite. □

3. Normalizers of infinite non-subnormal subgroups.

In this section we will describe the structure of groups in which all but finitely many normalizers of infinite non-subnormal subgroups have finite index. Suppose that G is a group with this property, and let $N_G(X_1), \dots, N_G(X_k)$ be the normalizers of infinite index of infinite non-subnormal subgroups of G. If G is either a Baer group or an FC-group, then by Lemma 2.5 and Lemma 2.6 each non-subnormal subgroup of G is almost normal. Thus it can be assumed that G neither is a Baer group nor an FC-group. If B and F are the Baer radical and the FC-centre of G, respectively, it follows from Lemma 2.1 that $B \cup F \cup N_G(X_1) \cup \dots \cup N_G(X_k)$ is a proper subset of G. Let x be an element of

$$G \setminus (B \cup F \cup N_G(X_1) \cup \dots \cup N_G(X_k)).$$

Clearly, x has finite order and every infinite subgroup of G containing x either is subnormal or almost normal. Most of the proofs given in the last section of [8], concerning groups in which every infinite non-subnormal subgroup is almost normal, actually work in our situation; in particular, we will refer here to Lemmas 3.3, 3.4, 3.8, 3.9, 3.10 and 3.12 of that paper, intending the corresponding slight generalizations of these results.

Lemma 3.1. *Let G be a non-periodic group in which all but finitely many* *normalizers of infinite non subnormal subgroups have finite index. If G neither is a Baer group nor an FC-group, then it is finitely generated and abelian-by-finite.*

Proof. Let $N_G(X_1), \dots, N_G(X_k)$ be the normalizers of infinite index of infinite non-subnormal subgroups of G. If B and F are the Baer radical and the FC-centre of G, respectively, we may consider an element x in the set

$$G \setminus (B \cup F \cup N_G(X_1) \cup \dots \cup N_G(X_k)).$$

Then the subgroup T consisting of all elements of finite order of B is finite and B/T is a finitely generated abelian group (see [8], Lemma 3.9 and Lemma 3.12). In particular $B/Z(B)$ is finite. By Lemma 2.2, G contains a characteristic subgroup M of finite index such that every infinite subgroup of M is either subnormal or almost normal, and of course $B \cap M$ is the Baer radical of M. Then either $M/B \cap M$ is finite or M is an FC-group (see [8], Lemma 3.13), and in the latter case all non-subnormal subgroups of M are almost normal; it follows that $M/B \cap M$ is finite (see [8], Theorem 2.8). Therefore G/B is finite and so G is a finitely generated abelian-by-finite group. □

The main result of this section describes groups with few normalizers of infinite non-subnormal subgroups.

Theorem 3.2. *Let G be a group in which all but finitely many normalizers of infinite non-subnormal subgroups have finite index. If G is either locally finite or non-periodic, then every infinite subgroup of G is either subnormal or almost normal.*

Proof. The group G is soluble-by-finite by Corollary 2.3. Moreover, by Lemma 2.5 and Lemma 2.6, it can be assumed that G neither is a Baer group nor an FC-group, and so there exists an element x of finite order such that every infinite subgroup of G containing x is either subnormal or almost normal. If G is locally finite, then each abelian $\langle x \rangle$-invariant subgroup of G satisfies the minimal condition (see [8], Lemma 3.4) and hence G itself is a Černikov group (see [18]). On the other hand, if G is non-periodic, then it is finitely generated and abelian-by-finite by Lemma 3.1. Therefore G contains an abelian normal subgroup A of finite index which is either periodic divisible or finitely generated and torsion-free. Let X be any infinite subgroup of G, so that $Y = A \cap X$ is also infinite and X/Y is finite. Moreover, $L = N_G(Y)$ has finite index in G and $X \leq Y$. Since in L/Y all but finitely many normalizers of non-subnormal subgroups have finite index, it follows from Theorem 2.7 that each subgroup of L/Y is either subnormal or almost normal. In particular, either X is subnormal in L or $N_L(X)$ has finite index in L and so also in G. Suppose that X is subnormal in L, so that

$$[A, \underbrace{X, \ldots, X}_{n}] \leq A \cap X = Y$$

for some positive integer n. If A is divisible, then $[A, X] \leq Y$ (see [15] Part 1, Lemma 3.13); on the other hand, if A is finitely generated, there is a positive integer e such that $A^e Y/Y$ is torsion-free and so $[A^e, X] \leq Y$. Therefore, in both cases A contains a subgroup of finite index normalizing X, and hence X is almost normal in G. The theorem is proved. □

The consideration of the infinite dihedral group shows that a soluble non-periodic group in which all infinite subgroups are almost normal can have infinitely many normalizers of infinite non-subnormal subgroups. However, the situation is different in the case of locally finite groups.

Corollary 3.3. *Let G be a locally finite group. Then the following statements are equivalent:*

(i) *G has finitely many normalizers of infinite non-subnormal subgroups.*

(ii) *All but finitely many normalizers of infinite non-subnormal subgroups of G have finite index.*

(iii) *Every infinite subgroup of G is either subnormal or almost normal.*

Proof. By Theorem 3.2 it is enough to prove that if every infinite non-subnormal subgroup of G is almost normal, then G has finitely many normalizers of infinite non-subnormal subgroups. If G is either a Baer group or an FC-group, then all its non-subnormal subgroups are almost normal, and hence G has finitely many normalizers of non-subnormal subgroups by Theorem 2.7. Suppose that G neither is a Baer group nor an FC-group, so that it contains normal subgroups D, E such that E is finite, D/E is a divisible primary group of finite rank, G/D is cyclic and D/E does not contain infinite proper $\langle y \rangle$-invariant subgroups for each element y of $G \setminus D$ (see [8], Theorem 3.6). In particular, G is a Černikov group, and its divisible part J is contained in $Z(D)$. Let X be any infinite subgroup of G. If X is contained in D, then J lies in $N_G(X)$. Assume that X is not contained in D; as $X \cap D$ is an infinite X-invariant subgroup of D, it follows that $D = (X \cap D)E$ and hence $J \leq X \cap D \leq X$. Therefore J is contained in all the normalizers of infinite subgroups of G, and in particular G has finitely many normalizers of infinite non-subnormal subgroups. □

Our last result shows that there are only the extreme possibilities for non-periodic groups with few normalizers of infinite non-subnormal subgroups.

Corollary 3.4. *Let G be a non-periodic group. Then G has finitely many normalizers of infinite non-subnormal subgroups if and only if either $G/Z(G)$ is finite or all subgroups of G are subnormal.*

Proof. Suppose that G has finitely many normalizers of infinite non-subnormal subgroups and assume for a contradiction that $G/Z(G)$ is infinite and G has non-subnormal subgroups. Then G contains normal subgroups A, E such that E is finite, A/E is a finitely generated torsion-free abelian group and $G/E = \langle xE\rangle \ltimes A/E$, where xE has prime order p and $C_{G/E}(A/E) = A/E$ (see [8], Theorem 3.15). Clearly, there exist infinitely many prime numbers $q \notin \pi(E) \cup \{p\}$ such that $[A, x]$ is not contained in A^q. If q is any such prime, $\langle x, A^q\rangle$ is an infinite non-subnormal subgroup of G and the index $|G : N_G(\langle x, A^q\rangle)|$ is a non-trivial power of q, a contradiction, since G has finitely many normalizers of infinite non-subnormal subgroups. □

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