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ON THE TWO-DIMENSIONAL SAIGO-MAEDA FRACTIONAL CALCULUS ASSOCIATED WITH TWO-DIMENSIONAL ALEPH TRANSFORM

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This paper deals with the study of two-dimensional Saigo-Maeda operators of Weyl type associated with Aleph function defined in this paper. Two theorems on these defined operators are established. Some interesting results associated with the H-functions and generalized Mittag-Leffler functions are deduced from the derived results. One dimensional analog of the derived results is also obtained.

1. Introduction and Preliminaries

The Aleph-function is defined in terms of the Mellin-Barnes type integral in the following manner [16, 17]:

$$\aleph[z] = \aleph^{m,n}_{p_i,q_i,\tau_i,r}\left[z\left|\begin{matrix}(a_j,A_j)_{1,n},\ldots,[\tau_j(a_j,A_j)]_{n+1,p_i}\\(b_j,B_j)_{1,m},\ldots,[\tau_j(b_j,B_j)]_{m+1,q_i}\end{matrix}\right.\right] := \frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(s)z^{-s}\,\mathrm{d}s, \quad (1)$$

where $z \neq 0,\ i = \sqrt{-1}$ and

$$\Omega^{m,n}_{p_i,q_i,\tau_i;r}(s) = \frac{\{\prod_{j=1}^{m}\Gamma(b_j+B_js)\}\{\prod_{j=1}^{n}\Gamma(1-a_j-A_js)\}}{\sum_{i=1}^{r}\tau_i\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-B_{ji}s)\}\{\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)\}}. \quad (2)$$

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An account of the convergence conditions for the defining integral can be found in the paper by Saxena and Pogány [16] (also see [19]).
The object of this paper is to derive certain properties of two-dimensional Saigo-Maeda operators of Weyl type. The results obtained are of general nature and includes as special cases, the results given earlier by Arora et al. [1], Saxena et al. [18, 21], Nishimoto and Saxena [6], Raina and Kiryakova [9] and Saigo et al. [12].

Remark 1.1. The fractional integration of the Aleph function is obtained by Saxena and Pogány [17], Ram and Kumar [8].

2. Generalized Fractional Integrals

We present below the definitions of the following generalized fractional integration operators of arbitrary order involving Appell function F_3 as a kernel,introduced by Saigo and Maeda [[11], p. 393, Eqn. (4.12)].

Let $\gamma > 0$ and $\alpha, \alpha^{'}, \beta, \beta^{'}, \gamma \in C$, then following Saigo and Maeda [11], we define the Saigo-Maeda operators $I_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\gamma}$ and $I_{x,\infty}^{\alpha,\alpha^{'},\beta,\beta^{'},\gamma}$ in the following manner:

$$\left(I_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha^{'}} F_3\left(\alpha, \alpha^{'}, \beta, \beta^{'}; \gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)\mathrm{d}t, \quad (3)$$

where $\Re(\gamma) > 0$ and

$$\left(I_{x,\infty}^{\alpha,\alpha^{'},\beta,\beta^{'},\gamma} f\right)(x) = \frac{x^{-\alpha^{'}}}{\Gamma(\gamma)} \int_x^\infty (t-x)^{\gamma-1} t^{-\alpha} F_3\left(\alpha, \alpha^{'}, \beta, \beta^{'}; \gamma; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)\mathrm{d}t, \quad (4)$$

where $\Re(\gamma) > 0$.
Here the function $F_3(\alpha, \alpha^{'}, \beta, \beta^{'}; \gamma; z; \xi)$ is the familiar Appell hypergeometric function of two variables defined by

$$F_3\left(\alpha, \alpha^{'}, \beta, \beta^{'}; \gamma; z, \xi\right) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m (\alpha^{'})_n (\beta)_m (\beta^{'})_n}{(\gamma)_{m+n}} \frac{z^m}{m!} \frac{\xi^n}{n!} \quad (|z| < 1,\ |\xi| < 1). \quad (5)$$

These operators reduce to the Saigo fractional integral operators [10] due to the following relations:

$$I_{0,x}^{\alpha,0,\beta,\beta',\gamma} f(x) = I_{0,x}^{\gamma,\alpha-\gamma,-\beta} f(x) \quad (\gamma \in C), \tag{6}$$

and

$$I_{x,\infty}^{\alpha,0,\beta,\beta',\gamma} f(x) = I_{x,\infty}^{\gamma,\alpha-\gamma,-\beta} f(x) \quad (\gamma \in C). \tag{7}$$

Lemma 2.1 ([11] p. 394, eqns. (4.18) and (4.19)). *Let* $\alpha, \alpha', \beta, \beta', \gamma \in C$, *then there holds the following power function formulae:*

1. *If* $\Re(\gamma) > 0$, $\Re(\rho) > \max\left[0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')\right]$, *then*

$$I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1} = x^{\rho-\alpha-\alpha'+\gamma-1} \times \frac{\Gamma(\rho)\,\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\,\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\gamma-\alpha-\alpha')\,\Gamma(\rho+\gamma-\alpha'-\beta)\,\Gamma(\rho+\beta')}, \tag{8}$$

2. *If* $\Re(\gamma) > 0$, $\Re(\rho) < 1 + \min\left[\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)\right]$, *then*

$$I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1} = x^{\rho-\alpha-\alpha'+\gamma-1} \times \frac{\Gamma(1-\beta-\rho)\,\Gamma(1+\alpha+\alpha'-\gamma-\rho)\,\Gamma(1+\alpha+\beta'-\gamma-\rho)}{\Gamma(1+\alpha+\alpha'+\beta'-\gamma-\rho)\,\Gamma(1+\alpha-\beta-\rho)\,\Gamma(1-\rho)}. \tag{9}$$

Remark 2.2. A detailed account of fractional claculus operators can be found in the monograph by Samko et al. [13] and in a survey paper by Srivastava and Saxena [23] and Haubold-Mathai-Saxena [2].

3. The two-Dimensional Saigo-Maeda Operator of Weyl Type

Following Miller [[5],p. 82], we denote by u_1 the class of function $f(x)$ on R_+ which are infinitely differentiable with partial derivatives of any order behaving as $O\left(|x|^{-\xi}\right)$ when x tends to ∞ for all ξ. Similarly by u_2, we denote the class of functions $f(x,y)$ on $R_+ \times R_+$ which are infinitely differentiable with partial derivatives of any order behaving as $O\left(|x|^{-\xi_1}|y|^{-\xi_2}\right)$ when x and y both tends to ∞ for all $\xi_i (i = 1,2)$.
The two-dimensional Saigo-Maeda operator of Weyl type of orders $\Re(\gamma) >$

0, $\Re(\zeta) > 0$ is defined in the class u_2 of functions $f(x,y)$ by
[[21], p.815, Eqn.(2.19)]

$$I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma} I_{y,\infty}^{\eta,\eta',\delta,\delta',\zeta}[f(x,y)] = x^{\alpha+\alpha'-\gamma} y^{\eta+\eta'-\zeta} \frac{x^{-\alpha'} y^{-\eta'}}{\Gamma(\gamma)\Gamma(\zeta)}$$
$$\times \int_x^\infty \int_y^\infty (u-x)^{\gamma-1}(v-y)^{\zeta-1} u^{-\alpha} v^{-\eta}\, F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{x}{u},1-\frac{u}{x}\right)$$
$$\times F_3\left(\eta,\eta',\delta,\delta';\zeta;1-\frac{y}{v},1-\frac{v}{y}\right) f(u,v)\mathrm{d}u\,\mathrm{d}v. \quad (10)$$

In view of the relations (6) and (7), the above equation reduces to the two-dimensional Saigo operator of Weyl type studied by Saigo et al. [[12], p.64, Eqn. (2.11)].

4. Two-Dimensional Laplace and Aleph Transforms

The Laplace transform [1] $\hbar(p,q)$ of a function $f(x,y) \in u_2$ is defined as

$$\hbar(p,q) = L[f(x,y);p,q] = \int_0^\infty \int_0^\infty \mathrm{e}^{-px-qy} f(x,y)\mathrm{d}x\,\mathrm{d}y,\ (\Re(p) > 0,\ \Re(q) > 0). \quad (11)$$

Analogously, the Laplace transform of $f[a\sqrt{x^2-b^2}\overset{*}{H}(x-b), c\sqrt{y^2-d^2}\overset{*}{H}(y-d)]$ is defined by the Laplace transform of $F(x,y)$, as

$$F(x,y) = f\left[a\sqrt{x^2-b^2}\overset{*}{H}(x-b), c\sqrt{y^2-d^2}\overset{*}{H}(y-d)\right],\ x > b > 0;\ y > d > 0, \quad (12)$$

where $\overset{*}{H}(.)$ denotes Heaviside's unit step function.

Definition 4.1. By two-dimensional Aleph function transform $\hbar(p,q)$ of a function $F(x,y)$, we mean the following repeated integral involving two different Aleph functions.

$$\hbar(p,q) = \aleph_{P_i,Q_i,\tau_i;P_i',Q_i',\tau_i';r}^{M_1,N_1;M_2,N_2}[F(x,y);\rho,\sigma;p,q]$$
$$= \int_b^\infty \int_d^\infty (px)^{\rho-1}(qy)^{\sigma-1} \aleph_{P_i,Q_i,\tau_i;r}^{M_1,N_1}\left[(px)^u \middle| \begin{matrix} (a_j,A_j)_{1,N_1},\cdots,[\tau_j(a_j,A_j)]_{N_1+1,P_i} \\ (b_j,B_j)_{1,M_1},\cdots,[\tau_j(b_j,B_j)]_{M_1+1,Q_i} \end{matrix}\right]$$
$$\times \aleph_{P_i',Q_i',\tau_i';r}^{M_2,N_2}\left[(qy)^v \middle| \begin{matrix} (c_j,C_j)_{1,N_2},\cdots,[\tau_j'(c_j,C_j)]_{N_2+1,P_i'} \\ (d_j,D_j)_{1,M_2},\cdots,[\tau_j'(d_j,D_j)]_{M_2+1,Q_i'} \end{matrix}\right] F(x,y)\mathrm{d}x\,\mathrm{d}y. \quad (13)$$

Here, we assume that $b>0,\ d>0,\ u>0,\ v>0$; $\hbar(p,q)$ exists and belongs to u_2. Further let

$$|\arg(p^u)|<\frac{\pi}{2}\phi_i, |\arg(q^v)|<\frac{\pi}{2}\psi_i, (\phi_i\geq 0,\ \psi\geq 0, i=\overline{1,r}); \tag{14}$$

where

$$\phi_i=\sum_{j=1}^{N_1}A_j+\sum_{j=1}^{M_1}B_j-\tau_i\left(\sum_{j=N_1+1}^{P_i}A_{ji}+\sum_{j=M_1+1}^{Q_i}B_{ji}\right), \tag{15}$$

$$\psi_i=\sum_{j=1}^{N_2}C_j+\sum_{j=1}^{M_2}D_j-\tau_i^{'}\left(\sum_{j=N_2+1}^{P_i^{'}}C_{ji}+\sum_{j=M_2+1}^{Q_i^{'}}D_{ji}\right) \tag{16}$$

and

$$\Re(\xi_i)+1<0, \Re(\zeta_i)+1<0\ \ (i=\overline{1,r}), \tag{17}$$

with

$$\xi_i=\sum_{j=1}^{M_1}b_j-\sum_{j=1}^{N_1}a_j+\tau_i\left(\sum_{j=M_1+1}^{Q_i}b_{ji}-\sum_{j=N_1+1}^{P_i}a_{ji}\right)+\frac{1}{2}(P_i-Q_i), \tag{18}$$

$$\zeta_i=\sum_{j=1}^{M_2}d_j-\sum_{j=1}^{N_2}c_j+\tau_i^{'}\left(\sum_{j=M_2+1}^{Q_i^{'}}d_{ji}-\sum_{j=N_2+1}^{P_i^{'}}c_{ji}\right)+\frac{1}{2}(P_i^{'}-Q_i^{'}). \tag{19}$$

Due to the generality of the $\aleph$-function, the integral transform (13) provides a generalization of a number of integral transforms such as, the two-dimensional Laplace transform, Stieltjes transform, Hankel transform, Whittaker transform, H-transform and I-transform etc.

5. Relationship Between Two-Dimensional Aleph Function Transforms in Terms of Two-Dimensional Saigo-Maeda Operator of Weyl Type

For proving the main results, we define the two-dimensional Aleph function transform $\hbar(p,q)$ of $F(x,y)$ as

$$\hbar(p,q) = \aleph^{M_1+3,N_1;M_2+3,N_2}_{P_i+3,Q_i+3,\tau_i;P_i'+3,Q_i'+3,\tau_i';r}[F(x,y);\rho,\sigma;p,q]$$
$$= \int_b^\infty \int_d^\infty (px)^{\rho-1}(qy)^{\sigma-1}\,\aleph^{M_1+3,N_1}_{P_i+3,Q_i+3,\tau_i;r}$$
$$\times \left[(px)^u \left| \begin{matrix} (a_j,A_j)_{1,N_1},(1-\rho,u),(1+\alpha-\beta-\rho,u),(1+\alpha+\alpha'+\beta'-\gamma-\rho,u),\cdots,[\tau_j(a_j,A_j)]_{N_1+1,P_i} \\ (1-\rho-\beta,u),(1+\alpha+\beta'-\gamma-\rho,u),(1+\alpha+\alpha'-\gamma-\rho,u),(b_j,B_j)_{1,M_1},\cdots,[\tau_j(b_j,B_j)]_{M_1+1,Q_i} \end{matrix}\right.\right]$$
$$\times \aleph^{M_2+3,N_2}_{P_i'+3,Q_i'+3,\tau_i';r}$$
$$\times \left[(qy)^v \left| \begin{matrix} (c_j,C_j)_{1,N_2},(1-\sigma,v),(1+\eta-\delta-\sigma,v),(1+\eta+\eta'+\delta'-\zeta-\sigma,v),\cdots,[\tau_j'(c_j,C_j)]_{M_2+1,Q_i'} \\ (1-\sigma-\delta,v),(1+\eta+\delta'-\zeta-\sigma,v),(1+\eta+\eta'-\zeta-\sigma,v)(d_j,D_j)_{1,M_2},\cdots,[\tau_j'(d_j,D_j)]_{M_2+1,Q_i'} \end{matrix}\right.\right]$$
$$\times F(x,y)\mathrm{d}x\,\mathrm{d}y, \quad (20)$$

where it is assumed that $\hbar_1(p,q)$ exists and belongs to u_2; $u>0,\ v>0$ and other conditions on the parameters in which additional parameters $\alpha,\alpha',\beta,\beta',\gamma,\eta,\eta',\delta,\delta',\zeta$ included correspond to those in (10).

Theorem 5.1. *Let $\hbar(p,q)$ is given by (13), then for $\Re(\gamma)>0,\ \Re(\zeta)>0,\ b>0,\ d>0,\ k_1>0,\ k_2>0$, there holds the formula*

$$I^{\alpha,\alpha',\beta,\beta',\gamma}_{p,\infty}\, I^{\eta,\eta',\delta,\delta',\zeta}_{q,\infty}[\hbar(p,q)] = \hbar_1(p,q), \quad (21)$$

provided that $\hbar_1(p,q)$ exists and belongs to u_2.

Proof. Let $\Re(\gamma)>0,\ \Re(\zeta)>0$, then by virtue of the results (10) and (13), it follows that

$$I^{\alpha,\alpha',\beta,\beta',\gamma}_{p,\infty}\, I^{\eta,\eta',\delta,\delta',\zeta}_{q,\infty}[\hbar(p,q)] = \frac{p^{\alpha-\gamma}q^{\eta-\zeta}}{\Gamma(\gamma)\Gamma(\zeta)}\int_p^\infty\int_q^\infty (u-p)^{\gamma-1}(v-q)^{\zeta-1}u^{-\alpha}v^{-\eta}$$
$$\times F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\tfrac{p}{u},1-\tfrac{u}{p}\right) F_3\left(\eta,\eta',\delta,\delta';\zeta;1-\tfrac{q}{v},1-\tfrac{v}{q}\right)\hbar(u,v)\mathrm{d}u\,\mathrm{d}v$$

$$= \frac{p^{\alpha-\gamma}q^{\eta-\zeta}}{\Gamma(\gamma)\Gamma(\zeta)}\int_p^\infty\int_q^\infty (u-p)^{\gamma-1}(v-q)^{\zeta-1}u^{-\alpha}v^{-\eta}$$
$$\times F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{p}{u},1-\frac{u}{p}\right) F_3\left(\eta,\eta',\delta,\delta';\zeta;1-\frac{q}{v},1-\frac{v}{q}\right)$$
$$\times \left\{\int_b^\infty\int_d^\infty (ux)^{\rho-1}(vy)^{\sigma-1}\,\aleph^{M_1,N_1}_{P_i,Q_i,\tau_i;r}\left[(ux)^{k_1}\left|\begin{matrix}(a_j,A_j)_{1,N_1},\cdots,[\tau_j(a_j,A_j)]_{N_1+1,P_i}\\(b_j,B_j)_{1,M_1},\cdots,[\tau_j(b_j,B_j)]_{M_1+1,Q_i}\end{matrix}\right.\right]\right.$$
$$\left.\times\ \aleph^{M_2,N_2}_{P_i',Q_i',\tau_i';r}\left[(vy)^{k_2}\left|\begin{matrix}(c_j,C_j)_{1,N_2},\cdots,[\tau_j'(c_j,C_j)]_{N_2+1,P_i'}\\(d_j,D_j)_{1,M_2},\cdots,[\tau_j'(d_j,D_j)]_{M_2+1,Q_i'}\end{matrix}\right.\right]F(x,y)\mathrm{d}x\,\mathrm{d}y\right\}\mathrm{d}u\,\mathrm{d}v. \quad (22)$$

On interchanging the order of integration, which is permissible under the given conditions, evaluating the $u-$ and $v-$ integrals, and applying Lemma 1, we obtain the L.H.S. of (22).

$$= \int_b^\infty \int_d^\infty (px)^{\rho-1}(qy)^{\sigma-1} \aleph^{M_1+3,N_1}_{P_i+3,Q_i+3,\tau_i;r} \times \left[(px)^{k_1} \middle| \begin{matrix} (a_j,A_j)_{1,N_1},(1-\rho,k_1),(1+\alpha-\beta-\rho,k_1),(1+\alpha+\alpha'+\beta'-\gamma-\rho,k_1),\cdots,[\tau_j(a_j,A_j)]_{N_1+1,P_i} \\ (1-\rho-\beta,k_1),(1+\alpha+\beta'-\gamma-\rho,k_1),(1+\alpha+\alpha'-\gamma\rho,k_1),(b_j,B_j)_{1,M_1},\cdots,[\tau_j(b_j,B_j)]_{M_1+1,Q_i} \end{matrix} \right] \times$$

$$\aleph^{M_2+3,N_2}_{P_i'+3,Q_i'+3,\tau_i';r}$$

$$\left[(qy)^{k_2} \middle| \begin{matrix} (c_j,C_j)_{1,N_2},(1-\sigma,k_2),(1+\eta-\delta-\sigma,k_2),(1+\eta+\eta'+\delta'-\zeta-\sigma,k_2),\cdots,[\tau_j'(c_j,C_j)]_{M_2+1,Q_i'} \\ (1-\sigma-\delta,k_2),(1+\eta+\delta'-\zeta-\sigma,k_2),(1+\eta+\eta'-\zeta-\sigma,k_2)(d_j,D_j)_{1,M_2},\cdots,[\tau_j'(d_j,D_j)]_{M_2+1,Q_i'} \end{matrix} \right]$$

$$\times F(x,y)\mathrm{d}x\,\mathrm{d}y$$

$$= \aleph^{M_1+3,N_1;M_2+3,N_2}_{P_i+3,Q_i+3,\tau_i;P_i'+3,Q_i'+3,\tau_i';r}[F(x,y);\rho,\sigma;p,q] = \hbar_1(p,q) = R.H.S.of(22).$$

As far as the two-dimensional Weyl type Saigo-Maeda operators $I^{\alpha,\alpha',\beta,\beta',\gamma}_{p,\infty} \times I^{\eta,\eta',\delta,\delta',\zeta}_{q,\infty}$ preserve the class u_2, it follows that $\hbar_1(p,q)$ also belongs to u_2. This completes the proof of Theorem 5.1. □

6. Special Cases of Theorem 5.1

If we put $\alpha' = \eta' = 0$ in Theorem 5.1 and use the relation

$$I^{\alpha+\beta,0,-\gamma,\beta',\alpha}_{p,\infty} f(x) = I^{\alpha,\beta,\gamma}_{p,\infty} f(x) \text{ (right-sided Saigo fractional integral operator)}, \tag{23}$$

then we arrive at the result concerning the two-dimensional Saigo fractional integral of Weyl type as given following:

Corollary 6.1. *Let $\hbar(p,q)$ be given by (13) then for $\Re(\alpha) > 0, \Re(\eta) > 0, b > 0, d > 0, u > 0, v > 0$, there hold the formula*

$$I^{\alpha,\beta,\gamma}_{p,\infty} I^{\eta,\delta,\zeta}_{q,\infty} [\hbar(p,q)] = \hbar_2(p,q), \tag{24}$$

provided that $\hbar_2(p,q)$ exists and belongs to u_2, where $\hbar_2$ is represented by the *repeated integral, given below:*

$$\hbar_2(p,q) = \int_b^\infty \int_d^\infty (px)^{\rho-1}(qy)^{\sigma-1} \aleph^{M_1+2,N_1}_{P_i+2,Q_i+2,\tau_i;r}$$

$$\times \left[(px)^u \left| \begin{matrix} (a_j,A_j)_{1,N_1},(1-\rho,u),(1+\alpha+\beta+\gamma-\rho,u),\cdots,[\tau_j(a_j,A_j)]_{N_1+1,P_i} \\ (1+\gamma-\rho,u),(1+\beta-\rho,u),(b_j,B_j)_{1,M_1},\cdots,[\tau_j(b_j,B_j)]_{M_1+1,Q_i} \end{matrix} \right. \right] \aleph^{M_2+2,N_2}_{P_i'+2,Q_i'+2,\tau_i';r}$$

$$\times \left[(qy)^v \left| \begin{matrix} (c_j,C_j)_{1,N_2},(1-\sigma,v),(1+\eta+\delta+\zeta-\sigma,v),\cdots,[\tau_j'(c_j,C_j)]_{M_2+1,Q_i'} \\ (1-\zeta-\sigma,v),(1+\delta-\sigma,v),(d_j,D_j)_{1,M_2},\cdots,[\tau_j'(d_j,D_j)]_{M_2+1,Q_i'} \end{matrix} \right. \right] F(x,y)\mathrm{d}x\,\mathrm{d}y. \quad (25)$$

Next, if we put $\tau_i = \tau_i' = 1, i = \overline{1,r}$ and set $r = 1$ in Theorem 5.1, then we see that the two-dimensional $\aleph$-transforms reduce to the corresponding two-dimensional H-transform $\widehat{H}(p,q)$, defined as [21].

$$\hat{H}(p,q) = H^{M_1,N_1;M_2,N_2}_{P_1,Q_1;P_2,Q_2}\left[F(x,y);\rho,\sigma;p,q\right]$$

$$\int_b^\infty \int_d^\infty (px)^{\rho-1}(qy)^{\sigma-1} H^{M_1,N_1}_{P_1,Q_1}\left[(px)^u \left| \begin{matrix} (a_j,A_j)_{1,P_1} \\ (b_j,B_j)_{1,Q_1} \end{matrix} \right. \right]$$

$$\times H^{M_2,N_2}_{P_2,Q_2}\left[(qy)^v \left| \begin{matrix} (c_j,C_j)_{1,P_2} \\ (d_j,D_j)_{1,Q_2} \end{matrix} \right. \right] F(x,y)\,dx\,dy, \quad (26)$$

where $u > 0, v > 0, b > 0, d > 0$; $\hat{H}(p,q)$ exists and belong to u_2,
The sufficient conditions for the absolute convergence of the equation (26) are given below:

$$|\arg(p^u)| < \frac{\pi}{2}\varphi,\ |\arg(q^v)| < \frac{\pi}{2}\psi,\ (\varphi > 0,\ \psi > 0), \quad (27)$$

where

$$\varphi = \sum_{j=1}^{N_1} A_j - \sum_{j=N_1+1}^{P_1} A_j + \sum_{j=1}^{M_1} B_j - \sum_{j=M_1+1}^{Q_1} B_j, \quad (28)$$

$$\psi = \sum_{j=1}^{N_2} C_j - \sum_{j=N_2+1}^{P_2} C_j + \sum_{j=1}^{M_2} D_j - \sum_{j=M_2+1}^{Q_2} D_j; \quad (29)$$

and

$$\sum_{j=1}^{Q_1} B_j - \sum_{j=1}^{P_1} A_j \geq 0,\ \sum_{j=1}^{Q_2} D_j - \sum_{j=1}^{P_2} C_j \geq 0. \quad (30)$$

Then we obtain the following result given by Saxena et al. [21]:

Corollary 6.2. *Let $\hat{H}(p,q)$ be given by (26), then for $Re(\alpha) > 0$, $Re(\eta) > 0$, $b > 0$, $d > 0$, $u > 0, v > 0$, there holds the formula which is obtained by Saxena et al. [21].*

$$I^{\alpha,\alpha',\beta,\beta',\gamma}_{p,\infty} I^{\eta,\eta',\delta,\delta',\zeta}_{q,\infty}\left[\hat{H}(p,q)\right] = \hat{H}_1(p,q), \quad (31)$$

provided that $\hat{H}_1(p,q)$ exists and belongs to u_2, where $\hat{H}_1$ is represented by the repeated integral:

$$\hat{H}_1(p,q) = \int_b^\infty \int_d^\infty (px)^{\rho-1}(qy)^{\sigma-1} H_{P_1+3,Q_1+3}^{M_1+3,N_1} \times \left[(px)^u \left| \begin{matrix} (a_j,A_j)_{1,P_1},(1-\rho,u),(1+\alpha-\beta-\rho,u),(1+\alpha+\alpha'+\beta'-\gamma-\rho,u) \\ (1-\beta-\rho,u),(1+\alpha+\beta'-\gamma-\rho,u),(1+\alpha+\alpha'-\gamma-\rho,u),(b_j,B_j)_{1,Q_1} \end{matrix} \right.\right] H_{P_2+3,Q_2+3}^{M_2+3,N_2} \times \left[(qy)^v \left| \begin{matrix} (c_j,C_j)_{1,P_2},(1-\sigma,v),(1+\eta-\delta-\sigma,v),(1+\eta+\eta'+\delta'-\zeta-\sigma,v) \\ (1-\delta-\sigma,v),(1+\eta+\delta'-\zeta-\sigma,v),(1+\eta+\eta'-\zeta-\sigma,v),(d_j,D_j)_{1,Q_2} \end{matrix} \right.\right] F(x,y)\,dx\,dy. \tag{32}$$

We now deduce the results for the two-dimensional Mittag-Leffler function transform from above Corollary 1.2.

Definition 6.3. The generalized Mittag-Leffler function introduced and studied by Prabhakar [7], is defined by

$$E_{\beta,\gamma}^{\delta}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_n z^n}{\Gamma(\beta n+\gamma)\, n!}, \quad (\beta,\gamma,\delta \in C, Re(\beta) > 0, Re(\gamma) > 0). \tag{33}$$

Its relation with the H-function is obtained by Saxena et al. [15] in the following form:

$$E_{\beta,\gamma}^{\delta}(z) = \frac{1}{\Gamma(\delta)} H_{1,2}^{1,1}\left[-z \left| \begin{matrix} (1-\delta,1) \\ (0,1),(1-\gamma,\beta) \end{matrix} \right.\right]. \tag{34}$$

Definition 6.4. By two-dimensional Mittag-Leffler function $\hat{E}(p,q)$ of a function $F(x,y)$, we mean the following repeated integral involving two different Mittag-Leffler functions.

$$\begin{aligned} \hat{E}(p,q) &= E_{\beta_1,\gamma_1;\beta_2,\gamma_2}^{\delta_1;\delta_2}[F(x,y);\rho,\sigma;p,q] \\ &= \int_b^\infty \int_d^\infty (px)^{\rho-1}(qy)^{\sigma-1} E_{\beta_1,\gamma_1}^{\delta_1}[(px)] E_{\beta_2,\gamma_2}^{\delta_2}[(qy)]\, F(x,y)\,dx\,dy, \end{aligned} \tag{35}$$

where $\beta_1,\beta_2,\gamma_1,\gamma_2,\delta_1,\delta_2 \in C$, $Re(\beta_1) > 0$, $Re(\beta_2) > 0$, $Re(\gamma_1) > 0$, $Re(\gamma_2) > 0$. Here, it is assumed that $b > 0$, $d > 0$; $E(p,q)$ exists and belongs to u_2.

If we use the identity (34) and make suitable changes in the parameters, then two-dimensional H-transform reduces to two-dimensional Mittag-Leffler function transform and we arrive at the following:

Corollary 6.5. *Let $\hat{E}(p,q)$ be given by (35), then for $Re(\alpha) > 0$, $Re(\eta) > 0$, $b > 0$, $d > 0$, then there holds the formula which is introduced as following:*

$$I_{p,\infty}^{\alpha,\alpha',\beta,\beta',\gamma} I_{q,\infty}^{\eta,\eta',\delta,\delta',\zeta}\left[\hat{E}(p,q)\right] = \hat{E}_1(p,q), \tag{36}$$

provided that $\hat{E}_1(p,q)$ exists and belongs to u_2, where $\hat{E}_1$ is represented by the repeated integral as following:

$$\hat{H}_1(p,q) = \frac{1}{\Gamma(\delta_1)\,\Gamma(\delta_2)} \int_b^\infty \int_d^\infty (px)^{\rho-1} (qy)^{\sigma-1}$$
$$\times\ H_{4,5}^{4,1}\left[-(px)^u \middle| \begin{matrix} (1-\delta_1,1),(1-\rho,u),(1+\alpha-\beta-\rho,u),(1+\alpha+\alpha'+\beta'-\gamma-\rho,u) \\ (0,1),(1-\beta-\rho,u),(1+\alpha+\beta'-\gamma-\rho,u),(1+\alpha+\alpha'-\gamma-\rho,u),(1-\gamma_1,\beta_1) \end{matrix}\right]$$
$$\times\ H_{4,5}^{4,1}\left[-(qy)^v \middle| \begin{matrix} (1-\delta_2,1),(1-\sigma,v),(1+\eta-\delta-\sigma,v),(1+\eta+\eta'+\delta'-\zeta-\sigma,v) \\ (0,1),(1-\delta-\sigma,v),(1+\eta+\delta'-\zeta-\sigma,v),(1+\eta+\eta'-\zeta-\sigma,v),(1-\gamma_2,\beta_2) \end{matrix}\right]$$
$$\times F(x,y)\,dx\,dy, \tag{37}$$

then from (35) - (37), we obtain

$$\hat{E}_1(p,q) = \int_b^\infty \int_d^\infty (px)^{\rho-1} (qy)^{\sigma-1} E_{\beta_1,\gamma_1}^{\delta_1}(px)^u E_{\beta_2,\gamma_2}^{\delta_2}(qy)^v F(x,y)\,dx\,dy. \tag{38}$$

7. One- Dimensional Analogue of Theorem 5.1

In this section we establish a theorem for the one-dimensional Aleph transform $\hbar(p)$ of $F(x)$ with similar proof as followed for Theorem 5.1.

The Laplace transform $\hbar(p)$ of a function $f(x) \in u_1$ is defined by

$$\hbar(p) = L[f(x);p] = \int_0^\infty e^{-px} f(x)\ dx,\ \ (Re(p) > 0). \tag{39}$$

Analogously, the Laplace transform of $f\left[a\sqrt{x^2-b^2}\,\overset{*}{H}(x-b)\right]$ is defined by the Laplace transform of $F(x)$, where

$$F(x) = f\left[a\sqrt{x^2-b^2}\,\overset{*}{H}(x-b)\right],\ \ x > b > 0, \tag{40}$$

and $\overset{*}{H}(.)$ denotes Heaviside's unit step function.

Theorem 7.1. *Let $\hbar(p)$ be the one-dimensional Aleph-function transform of $F(x)$defined by*

$$\hbar(p) = \aleph_{P_i,Q_i,\tau_i;r}^{M,N}[F(x);\rho;p]$$
$$= \int_b^\infty (px)^{\rho-1} \aleph_{P_i,Q_i,\tau_i;r}^{M,N}\left[(px)^k \middle| \begin{matrix} (a_j,A_j)_{1,N},\ldots,[\tau_j(a_j,A_j)]_{N+1,P_i} \\ (b_j,B_j)_{1,M},\ldots,[\tau_j(b_j,B_j)]_{M+1,Q_i} \end{matrix}\right] F(x)\,dx, \tag{41}$$

where $\hbar(p)$ exists and belongs to u_1, where $k > 0$; together with the following conditions:

$$\left|\arg\left(p^k\right)\right| < \frac{\pi}{2}\phi_i, \quad \phi_i \geq 0\left(i = \overline{1,r}\right), \text{ where}$$

$$\phi_i = \sum_{j=1}^{N} A_j + \sum_{j=1}^{M} B_j - \tau_i\left(\sum_{j=N+1}^{P_i} A_{ji} + \sum_{j=M+1}^{Q_i} B_{ji}\right);$$

$$\text{and } \Re\{\xi_i\} + 1 < 0\left(i = \overline{1,r}\right), \tag{42}$$

$$\text{where } \xi_i = \sum_{j=1}^{M} b_j - \sum_{j=1}^{N} a_j + \tau_i\left(\sum_{j=M+1}^{Q_i} b_{ji} - \sum_{j=N+1}^{P_i} a_{ji}\right) + \frac{1}{2}(P_i - Q_i); \tag{43}$$

also $F(x) = f\left[a\sqrt{x^2 - b^2}\,\overset{*}{H}(x-b)\right]$, $x > b > 0$. *Then for* $Re(\alpha) > 0$, $b > 0$, $k > 0$, *there holds the formula*

$$\bar{I}_{p,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}[\hbar(p)] = \hbar_1(p) \tag{44}$$

provided that $\hbar_1(p)$ exists and belongs to u_1, where

$$\hbar_1(p) = \int_b^\infty (px)^{\rho-1}\,\aleph_{P_i+3,Q_i+3,\tau_i;r}^{M+3,N}$$
$$\times\left[(px)^k\left|\begin{matrix}\left(a_j,A_j\right)_{1,N},(1-\rho,k),(1+\alpha-\beta-\rho,k),(1+\alpha+\alpha'+\beta'-\gamma-\rho,k),\ldots,\left[\tau_j\left(a_j,A_j\right)\right]_{N+1,P_i}\\(1-\beta-\rho,k),(1+\alpha+\beta'-\gamma-\rho,k),(1+\alpha+\alpha'-\gamma-\rho,k),\left(b_j,B_j\right)_{1,M},\ldots,\left[\tau_j\left(b_j,B_j\right)\right]_{M+1,Q_i}\end{matrix}\right.\right]$$
$$\times F(x)\,dx, \tag{45}$$

and

$$\begin{aligned}&\bar{I}_{p,\infty}^{\alpha,\alpha',\beta,\beta',\gamma} f(p)\\&= \frac{p^{\alpha-\gamma}}{\Gamma(\gamma)}\int_p^\infty (u-p)^{\gamma-1} u^{-\alpha} F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{p}{u},1-\frac{u}{p}\right) f(u)\,du\\&= p^{\alpha+\alpha'-\gamma} I_{p,\infty}^{\alpha,\alpha',\beta,\beta',\gamma} f(p).\end{aligned} \tag{46}$$

8. Special Cases of Theorem 7.1

If we set $\tau_i = 1, i = \overline{1,r}$ and set $r = 1$ in Theorem 7.1, then we see that the one-dimensional $\aleph$-transform reduces to the corresponding one-dimensional H-transform $\hat{H}(p)$, defined by [21].

$$\begin{aligned}\hat{H}(p) &= H_{P,Q}^{M,N}[F(x);\rho;p]\\&= \int_b^\infty (px)^{\rho-1} H_{P,Q}^{M,N}\left[(px)^k\left|\begin{matrix}\left(a_j,A_j\right)_{1,P}\\\left(b_j,B_j\right)_{1,Q}\end{matrix}\right.\right] F(x)\,dx,\end{aligned} \tag{47}$$

where $k>0, b>0$; $\hat{H}(p)$ exists and belongs to u_1. Then under the conditions stated in Corollary 1.2, we obtain the following:

Corollary 8.1. *Let $\hat{H}(p)$ be given by (47), then for $Re(\alpha)>0$, $b>0$, k is being positive integer, there holds the formula [21].*

$$\bar{I}_{p,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}\left[\hat{H}(p)\right] = \hat{H}_1(p), \tag{48}$$

provided that $\hat{H}_1(p)$ exists and belongs to u_1, where $\hat{H}_1(p)$ is represented by

$$\hat{H}_1(p) = \int_b^\infty (px)^{\rho-1} H_{P+3,Q+3}^{M+3,N} \times \left[(px)^k \left| \begin{matrix} (a_j,A_j)_{1,P},(1-\rho,k),(1+\alpha-\beta-\rho,k),(1+\alpha+\alpha'+\beta'-\gamma-\rho,k) \\ (1-\beta-\rho,k),(1+\alpha+\beta'-\gamma-\rho,k),(1+\alpha+\alpha'-\gamma-\rho,k),(b_j,B_j)_{1,Q} \end{matrix} \right.\right] F(x)\,dx. \tag{49}$$

Definition 8.2. By one-dimensional Mittag-Leffler function transform $\hat{E}(p)$ of a function $F(x)$ we mean the following integral involving Mittag-Leffler function.

$$\hat{E}(p) = E_{\beta_1,\gamma_1}^{\delta_1}\left[F(x);\rho;p\right] = \int_b^\infty (px)^{\rho-1} E_{\beta_1,\gamma_1}^{\delta_1}\left[(px)\right] F(x)\,dx, \tag{50}$$

where $\beta_1,\gamma_1,\delta_1 \in C$, $Re(\beta_1)>0$, $Re(\gamma_1)>0$.

Here, we assume that $b>0$; $E(p)$ exists and belongs to u_1.

Corollary 8.3. *If we use the identity (34) and make suitable changes in the parameters, then one-dimensional H-transform reduces to Mittag-Leffler function transform and we obtain*

$$\hat{H}_1(p) = \frac{1}{\Gamma(\delta_1)} \int_b^\infty (px)^{\rho-1} H_{4,5}^{4,1} \times \left[-(px)^k \left| \begin{matrix} (1-\delta_1,1),(1-\rho,k),(1+\alpha-\beta-\rho,k),(1+\alpha+\alpha'+\beta'-\gamma-\rho,k) \\ (0,1),(1-\beta-\rho,k),(1+\alpha+\beta'-\gamma-\rho,k),(1+\alpha+\alpha'-\gamma-\rho,k),(1-\gamma_1,\beta_1) \end{matrix} \right.\right] F(x)\,dx. \tag{51}$$

Let $\hat{E}(p)$ be given by (50), then for $Re(\alpha)>0$, $b>0$, there holds the formula

$$\bar{I}_{p,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}\left[\hat{E}(p)\right] = \hat{E}_1(p), \tag{52}$$

provided that $\hat{E}_1(p)$ exists and belongs to u_1.

Thus by virtue of (34) the one-dimensional H-transform reduces to one-dimensional Mittag-Leffler function transform and it yields

$$\hat{E}_1(p) = \int_b^\infty (px)^{\rho-1} E_{\beta_1,\gamma_1}^{\delta_1}(px)^k F(x)\,dx. \tag{53}$$

Next, if we put $\alpha^{'} = 0$ in 7.1 and use the relation

$$\bar{I}_{p,\infty}^{\alpha+\beta,0,-\gamma,\beta',\alpha} f(x) = \bar{I}_{p,\infty}^{\alpha,\beta,\gamma} f(x) \text{ (right-sided Saigo fract. integral operator),} \tag{54}$$

then we obtain the following Corollary concerning one-dimensional Saigo fractional integral of Weyl type:

Corollary 8.4. *Let $\hbar(p)$ be the one-dimensional Aleph function transform of $F(x)$ as given in (41), $\hbar(p)$ exists and belongs to u_1. Then for $Re(\alpha) > 0$, $b > 0$, $k > 0$, there holds the formula*

$$\bar{I}_{p,\infty}^{\alpha,\beta,\gamma}[\hbar(p)] = \hbar_2(p), \tag{55}$$

provided that $\hbar_2(p)$ exists and belongs to u_1, where $\hbar_2$ is represented by

$$\hbar_2(p) = \int_b^\infty (px)^{\rho-1} \aleph_{P_i+2,Q_i+2,\tau_i;r}^{M+2,N} \times \left[(px)^k \left| \begin{matrix} (a_j,A_j)_{1,N},(1-\rho,k),(1+\alpha+\beta+\gamma-\rho,k),\ldots,[\tau_j(a_j,A_j)]_{N+1,P_i} \\ (1+\gamma-\rho,k),(1+\beta-\rho,k),(b_j,B_j)_{1,M},\ldots,[\tau_j(b_j,B_j)]_{M+1,Q_i} \end{matrix} \right. \right] F(x)\,dx, \tag{56}$$

and

$$\begin{aligned} &\bar{I}_{p,\infty}^{\alpha,\beta,\gamma} f(p) \\ &= \frac{p^\beta}{\Gamma(\alpha)} \int_p^\infty u^{-\alpha-\beta}(u-p)^{\alpha-1} \, {}_2F_1\left(\alpha+\beta,-\gamma;\alpha,1-\frac{p}{u}\right) f(u)\,du \\ &= p^\beta I_{p,\infty}^{\alpha,\beta,\gamma} f(p). \end{aligned} \tag{57}$$

9. Result and Discussions

In this paper we have obtained the two-dimensional $\aleph$-transforms involving Weyl type two-dimensional Saigo-Maeda operators. The two-dimensional H-transform and two-dimensional Mittag-Leffler function transform are special cases of our main findings.

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