LE MATEMATICHE
Vol. LXIX (2014) – Fasc. I, pp. 57–69
doi: 10.4418/2014.69.1.4

EQUIVALENCE OF THE CONVERGENCES OF T-PICARD, T-MANN AND T-ISHIKAWA ITERATIONS FOR THE CLASS OF T-ZAMFIRESCU OPERATORS

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In this paper, we prove the equivalence between the convergences of T-Picard iteration, T-Mann iteration and T-Ishikawa iteration for the class of T-Zamfirescu operators in normed linear spaces. Our results extend and improve the results of Şoltuz [17] and Zhiqun [20].

1. Introduction and Preliminary Definitions

In 2009, Beiranvand et al. [1] introduced the concepts of T-Banach contraction and T-contractive mappings and then they extended Banach's contraction principle [2] and Edelstein's fixed point theorem [4]. T-Kannan contractive mappings were introduced by Moradi [6] which extended Kannan's fixed point theorem [5]. Followed by this, Morales and Rojas [7] introduced the notion of T-Chatterjea mapping and obtained sufficient conditions for the existence of a unique fixed point of these mappings in the framework of complete cone metric spaces. The same authors [9], then introduced the concept of T-Zamfirescu operators and obtained sufficient conditions for the existence of a unique fixed point of T-Zamfirescu operators in the setting of complete cone metric spaces. A new iteration scheme, namely T-Picard iteration was introduced by Morales and Rojas [8] in 2009 which is defined as follows:

Entrato in redazione: 18 gennaio 2013

AMS 2010 Subject Classification: 47H10.

Keywords: T-Picard iteration, T-Mann iteration, T-Ishikawa iteration, T-Zamfirescu operators.

Let E be a normed linear space. Let $T,S : E \to E$ be two mappings and let $p_0 \in E$. The sequence $\{Tp_n\}_{n=0}^{\infty} \in E$ defined by

$$Tp_{n+1} = TSp_n, \qquad n = 0,1,2,..., \tag{1}$$

is called the T-Picard iteration associated to S.
Here we note that when we take $T = I$, the identity map, (1) reduces to

$$p_{n+1} = Sp_n, \qquad n = 0,1,2,..., \tag{2}$$

which is Picard iteration.

Morales and Rojas [8] studied the existence of fixed points for T-Zamfirescu operators in complete metric spaces and proved the convegence of T-Picard iteration for the class of T-Zamfirescu operators. Inspired and motivated by the above said facts, the authors have introduced T-Mann iteration scheme and T-Ishikawa iteration scheme [10] and proved the convergence of these iteration procedures for the class of T-Zamfirescu operators in real Banach spaces. The new schemes are given as follows:
Let E be a normed linear space. Let $T,S : E \to E$ be two mappings and let $u_0, x_0 \in E$. The sequence $\{Tu_n\}_{n=0}^{\infty} \in E$ given by

$$Tu_{n+1} = (1-\alpha_n)Tu_n + \alpha_n TSu_n, \qquad n = 0,1,2,..., \tag{3}$$

where $\{\alpha_n\}_{n=0}^{\infty} \in (0,1)$ is called the T-Mann iteration associated to S.
When we substitute $T = I$, the identity map in (3), we get the definition of Mann iteration which is given by

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n Su_n, \qquad n = 0,1,2,..., \tag{4}$$

where $\{\alpha_n\}_{n=0}^{\infty} \in (0,1)$.

The T-Ishikawa iteration associated to S is defined by

$$Tx_{n+1} = (1-\alpha_n)Tx_n + \alpha_n STy_n \tag{5}$$

$$Ty_n = (1-\beta_n)Tx_n + \beta_n TSx_n, \qquad n = 0,1,2,...,$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \in (0,1)$.
If we take $T = I$, the identity map in (5), we get the definition of Ishikawa iteration which is given as

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Sy_n \tag{6}$$

$$y_n = (1-\beta_n)x_n + \beta_n Sx_n, \qquad n = 0,1,2,...,$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \in (0,1)$.

The following conjecture has been given by Rhoades and Şoltuz [11]: "Whenever T is a function for which Mann iteration converges, so does the Ishikawa iteration". They further remarked that, given the large variety of functions and spaces, such a global statement is, of course, not provable. In a series of papers, like [12], [13], [14], [15] and [16] the same authors have given a positive answer to this conjecture, showing the equivalence between Mann and Ishikawa iterations for strongly pseudocontractive maps, uniformly pseudocontractive maps and asymptotically nonexpansive maps in normed linear spaces. In 2005, Şoltuz [17] studied the equivalence of the convergences of Picard, Mann and Ishikawa iterations when applied to Zamfirescu operators and proved the following results:

Theorem 1.1 ([17], Theorem 1)**.** *Let X be a normed space, D a nonempty, closed, convex subset of X and $T : D \to D$ be a Zamfirescu operator. Suppose that x^* is a fixed point of T. If $u_0 = x_0 \in D$, let $\{u_n\}_{n=0}^{\infty}$ be defined by (4) for $u_0 \in D$, and let $\{x_n\}_{n=0}^{\infty}$ be defined by (6) for $x_0 \in D$ with $\{\alpha_n\}_{n=0}^{\infty} \in (0,1)$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the following are equivalent:*
(i) the Mann iteration (4) converges to x^* ;
(ii) the Ishikawa iteration (6) converges to x^*.

Theorem 1.2 ([17], Theorem 2)**.** *Let X be a normed space, D a nonempty, closed, convex subset of X and $T : D \to D$ be a Zamfirescu operator. Suppose that x^* is a fixed point of T. If $u_0 = p_0 \in D$, let $\{p_n\}_{n=0}^{\infty}$ be defined by (2) for $p_0 \in D$, and let $\{u_n\}_{n=0}^{\infty}$ be defined by (4) for $u_0 \in D$ with $\{\alpha_n\}_{n=0}^{\infty} \in (0,1)$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the following are equivalent:*
(i) if the Mann iteration (4) converges to x^* and $\lim_{n\to\infty} \frac{\|u_{n+1} - u_n\|}{\alpha_n} = 0$, then the Picard iteration (2) converges to x^*;
(ii) the Picard iteration (2) converges to x^* and $\lim_{n\to\infty} \frac{\|p_{n+1} - p_n\|}{\alpha_n} = 0$, then the Mann iteration (4) converges to x^*.

In 2007, Zhiqun [20] studied the equivalence between the convergences of Picard iteration and Mann iteration for Zamfirescu operators in normed linear spaces and improved the result of Şoltuz [17, Theorem 2] in the following sense:
(i) Both hypotheses $\lim_{n\to\infty} \frac{\|u_{n+1} - u_n\|}{\alpha_n} = 0$ and $\lim_{n\to\infty} \frac{\|p_{n+1} - p_n\|}{\alpha_n} = 0$ have been removed.
(ii) The assumption that $u_0 = p_0$ was found to be superfluous.
More precisely, he proved the following results:

Theorem 1.3 ([20] Theor. 2.1)**.** *Let E be a normed linear space, D a nonempty, closed, convex subset of E, and $T : D \to D$ a Zamfirescu operator. Suppose that T has a fixed point in D. Let $\{p_n\}_{n=0}^{\infty}$ be defined by (2) for $p_0 \in D$ and let $\{u_n\}_{n=0}^{\infty}$ be defined by (4) for $u_0 \in D$ with $\{\alpha_n\}_{n=0}^{\infty} \in [0,1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the following are equivalent:*
(i) the Picard iteration (2) converges to the fixed point of T;
(ii) the Mann iteration (4) converges to the fixed point of T.

Theorem 1.4 ([20], Theor. 2.3)**.** *Let E be a normed linear space, D a nonempty, closed, convex subset of E, and $T : D \to D$ a Zamfirescu operator. Suppose that T has a fixed point in D. Let $\{p_n\}_{n=0}^{\infty}$ be defined by (2) for $p_0 \in D$ and let $\{x_n\}_{n=0}^{\infty}$ be defined by (6) for $x_0 \in D$ with $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \in [0,1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the following are equivalent:*
(i) the Picard iteration (2) converges to the fixed point of T;
(ii) the Ishikawa iteration (6) converges to the fixed point of T.

Here we recall the definitions of the following classes of generalized T-contraction type mappings as given by Morales and Rojas [8]:

Definition 1.5. Let (M,d) be a metric space and $T,S : M \to M$ be two functions. A mapping S is said to be *T-Banach contraction* (TB contraction) if there exists $a \in [0,1)$ such that

$$d(TSx, TSy) \leq ad(Tx, Ty), \qquad \text{for all } x,y \in M.$$

When we substitute $T = I$, the identity map, in the above definition we obtain the definition of *Banach's contraction* [2].

Definition 1.6. Let (M,d) be a metric space and $T,S : M \to M$ be two functions. A mapping S is said to be *T-Kannan contraction* (TK contraction) if there exists $b \in [0, \frac{1}{2})$ such that

$$d(TSx, TSy) \leq b[d(Tx, TSx) + d(Ty, TSy)], \qquad \text{for all } x,y \in M.$$

In Definition 1.6, if we take $T = I$, the identity map, then we get the definition of *Kannan operator* [5].

Definition 1.7. Let (M,d) be a metric space and $T,S : M \to M$ be two functions. A mapping S is said to be *T-Chatterjea contraction* (TC contraction) if there exists $c \in [0, \frac{1}{2})$ such that

$$d(TSx, TSy) \leq c[d(Tx, TSy) + d(Ty, TSx)] \qquad \text{for all } x,y \in M.$$

If we substitute $T = I$, the identity map, in Definition 1.7 we obtain the definition of Chatterjea operator [3].

Definition 1.8. Let (M,d) be a metric space and $T,S : M \to M$ be two functions. A mapping S is said to be T-*Zamfirescu operator* (TZ operator) if there are real numbers $0 \le a < 1, 0 \le b < \frac{1}{2}, 0 \le c < \frac{1}{2}$ such that for all $x,y \in M$ at least one of the conditions is true:
(TZ_1) : $d(TSx,TSy) \le ad(Tx,Ty)$,
(TZ_2) : $d(TSx,TSy) \le b[d(Tx,TSx) + d(Ty,TSy)]$,
(TZ_3) : $d(TSx,TSy) \le c[d(Tx,TSy) + d(Ty,TSx)]$.

When we take $T = I$, the identity map, in the above definition we obtain the definition of Zamfirescu operator [19].

In this paper, we prove that convergences of T-Picard iteration, T-Mann iteration and T-Ishikawa iteration are equivalent for the class of TZ-operators in normed linear spaces. Our results extend the results of Şoltuz [17] and Zhiqun [20].

In the sequel, we need the following lemmas:

Lemma 1.9 ([8]). *Let (M,d) be a complete metric space and $T,S : M \to M$ be two functions. If S is a TZ-operator, then there is $0 \le \delta < 1$ such that*

$$d(TSx,TSy) \le \delta d(Tx,Ty) + 2\delta d(Tx,TSx), \qquad \text{for all } x,y \in M.$$

Lemma 1.10 ([2]). *Let $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers and $0 \le q < 1$, so that $a_{n+1} \le qa_n + b_n, \qquad$ for all $n \ge 0$.*
(i) *If* $\lim\limits_{n\to\infty} b_n = 0$, *then* $\lim\limits_{n\to\infty} a_n = 0$,
(ii) *If* $\sum\limits_{n=0}^{\infty} b_n < \infty$, *then* $\sum\limits_{n=0}^{\infty} a_n < \infty$.

Lemma 1.11 ([20]). *Let $\{a_n\}$ and $\{\sigma_n\}$ be nonnegative real sequences satisfying the following inequality:*

$$a_{n+1} \le (1-\lambda_n)a_n + \sigma_n,$$

where $\lambda_n \in (0,1)$ for all $n \ge n_0$, $\sum\limits_{n=1}^{\infty} \lambda_n = \infty$ and $\sigma_n = O(\lambda_n)$. Then $\lim\limits_{n\to\infty} a_n = 0$.

2. Main Results

Theorem 2.1. *Let E be a normed linear space, K a nonempty, closed, convex subset of E and $T,S : K \to K$ be two mappings such that S is a TZ-operator.*

Suppose that S has a fixed point x^* in K. Let the T-Picard iteration be defined by (1) for $p_0 \in K$ and let the T- Mann iteration be defined by (3) for $u_0 \in K \in$ with $\{\alpha_n\} \in (0,1)$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the following are equivalent:
(i) The T-Picard iteration converges to Tx^*;
(ii) The T-Mann iteration converges to Tx^*.

Proof. Since S is a TZ-operator, by Lemma 1.9 there is $0 \leq \delta < 1$ such that

$$\|TSx - TSy\| \leq \delta \|Tx - Ty\| + 2\delta \|Tx - TSx\|, \qquad \text{for all } x, y \in K. \tag{7}$$

First, we shall prove that $(ii) \Rightarrow (i)$. Assume that

$$\|Tu_n - Tx^*\| \to 0 \text{ as } n \to \infty.$$

Now,

$$\begin{aligned} \|Tu_{n+1} - Tp_{n+1}\| &= \|(1-\alpha_n)Tu_n + \alpha_n TSu_n - TSp_n\| \\ &\leq (1-\alpha_n)\|Tu_n - TSp_n\| + \alpha_n \|TSu_n - TSp_n\| \\ &\leq (1-\alpha_n)\|Tu_n - TSu_n\| + \|TSu_n - TSp_n\|. \end{aligned} \tag{8}$$

Taking $x = u_n$ and $y = p_n$ in (7) we get,

$$\|TSu_n - TSp_n\| \leq \delta \|Tu_n - Tp_n\| + 2\delta \|Tu_n - TSu_n\|.$$

Using the above inequality, (8) becomes

$$\begin{aligned} &\|Tu_{n+1} - Tp_{n+1}\| \\ &\leq \delta \|Tu_n - Tp_n\| + (1-\alpha_n+2\delta)\|Tu_n - TSu_n\| \\ &\leq \delta \|Tu_n - Tp_n\| + (1-\alpha_n+2\delta)(\|Tu_n - Tx^*\| + \|TSu_n - TSx^*\|) \\ &\leq \delta \|Tu_n - Tp_n\| + (1-\alpha_n+2\delta)(1+\delta)\|Tu_n - Tx^*\|. \end{aligned}$$

Denote

$$\begin{aligned} a_n &= \|Tu_n - Tp_n\|, \\ q &= \delta \in [0,1), \\ b_n &= (1-\alpha_n+2\delta)(1+\delta)\|Tu_n - Tx^*\|. \end{aligned}$$

Now, by applying Lemma 1.10 we get that

$$\|Tu_n - Tp_n\| \to 0 \text{ as } n \to \infty.$$

Thus,

$$\|Tp_n - Tx^*\| \le \|Tu_n - Tp_n\| + \|Tu_n - Tx^*\| \to 0 \text{ as } n \to \infty.$$

To prove that $(i) \Rightarrow (ii)$, assume that

$$\|Tp_n - Tx^*\| \to 0 \text{ as } n \to \infty.$$

Consider

$$\begin{aligned}
&\|Tu_{n+1} - Tp_{n+1}\| \\
&\le (1-\alpha_n)\|Tu_n - TSp_n\| + \alpha_n \|TSu_n - TSp_n\| \\
&\le (1-\alpha_n)\|Tu_n - Tp_n\| + (1-\alpha_n)\|Tp_n - TSp_n\| + \alpha_n \|TSu_n - TSp_n\| \\
&\le (1-\alpha_n)\|Tu_n - Tp_n\| + (1-\alpha_n)(\|Tp_n - Tx^*\| + \|TSp_n - Tx^*\|) \\
&\qquad\qquad\qquad\qquad + \alpha_n \|TSu_n - TSp_n\|. \qquad (9)
\end{aligned}$$

By taking $x = p_n$ and $y = u_n$ in (7) we get

$$\|TSp_n - TSu_n\| \le \delta \|Tp_n - Tu_n\| + 2\delta \|Tp_n - TSp_n\|,$$

which gives

$$\begin{aligned}
\alpha_n \|TSu_n - TSp_n\| &\le \alpha_n\delta \|Tu_n - Tp_n\| + 2\alpha_n\delta \|Tp_n - TSp_n\| \\
&\le \alpha_n\delta \|Tu_n - Tp_n\| \\
&\quad + 2\alpha_n\delta(\|Tp_n - Tx^*\| + \|TSp_n - TSx^*\|). \qquad (10)
\end{aligned}$$

Now using (7) with $x = x^*$ and $y = p_n$, we obtain

$$\|TSx^* - TSp_n\| \le \delta \|Tx^* - Tp_n\|. \qquad (11)$$

Using (10) and (11), (9) becomes

$$\begin{aligned}
&\|Tu_{n+1} - Tp_{n+1}\| \\
&\le [1-(1-\delta)\alpha_n]\|Tu_n - Tp_n\| + (1-\alpha_n + 2\alpha_n\delta)(1+\delta)\|Tp_n - Tx^*\| \\
&\le [1-(1-\delta)\alpha_n]\|Tu_n - Tx^*\| \\
&+ [(1-\alpha_n + 2\alpha_n\delta)(1+\delta) + (1-\alpha_n(1-\delta))]\|Tp_n - Tx^*\| \\
&\le [1-(1-\delta)\alpha_n]\|Tu_n - Tx^*\| \\
&+ (1-\alpha_n + 2\alpha_n\delta)(2+\delta)\|Tp_n - Tx^*\| \le
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\lambda\alpha_n)\left[(1-\alpha_{n-1})\|Tu_{n-1}-Tx^*\|+\alpha_{n-1}\|TSu_{n-1}-TSx^*\|\right]\\
&+(1-\alpha_n+2\alpha_n\delta)(2+\delta)\|Tp_n-Tx^*\|\\
&\leq (1-\lambda\alpha_n)\left[1-(1-\delta)\alpha_{n-1}\right]\|Tu_{n-1}-Tx^*\|\\
&+(1-\alpha_n+2\alpha_n\delta)(2+\delta)\|Tp_n-Tx^*\|\\
&\leq (1-\lambda\alpha_n)(1-\lambda\alpha_{n-1})...(1-\lambda\alpha_0)\|Tu_0-Tx^*\|\\
&+(1-\alpha_n+2\alpha_n\delta)(2+\delta)\|Tp_n-Tx^*\|\\
&\leq \exp\left(-\lambda\sum_{i=0}^{n}\alpha_i\right)\|Tu_0-Tx^*\|+(1-\alpha_n+2\alpha_n\delta)(2+\delta)\|Tp_n-Tx^*\|,
\end{aligned}
$$

where $1-\delta=\lambda$. Since $\sum_{n=0}^{\infty}\alpha_n=\infty$ and $\|Tp_n-Tx^*\|\to 0$ as $n\to\infty$,
we get $\|Tu_n-Tp_n\|\to 0$ as $n\to\infty$.
Hence,

$$\|Tu_n-Tx^*\|\leq\|Tu_n-Tp_n\|+\|Tp_n-Tx^*\|\to 0 \text{ as } n\to\infty.$$

□

Corollary 2.2 ([20], Theor. 2.1)**.** *Let E be a normed linear space, K a nonempty closed, convex subset of E and $T : K \to K$ be a Z-operator. Suppose that T has a fixed point x^* in K. Let the Picard iteration be defined by* (2) *for $p_0 \in K$ and let the Mann iteration be defined by* (4) *for $u_0 \in K$ with $\{\alpha_n\} \in (0,1)$ satisfying $\sum_{n=0}^{\infty}\alpha_n=\infty$. Then the following are equivalent:*
(i) *The Picard iteration converges to x^*;*
(ii) *The Mann iteration converges to x^*.*

Theorem 2.3. *Let E be a normed linear space, K a nonempty, closed, convex subset of E and $T,S : K \to K$ be two commuting mappings such that S is a TZ-operator. Suppose that S has a fixed point x^* in K. Let the T-Mann iteration for $u_0 \in K$ be defined by (3) and the T-Ishikawa iteration for $x_0 \in K$ be defined by (5) with $\{\alpha_n\},\{\beta_n\} \in (0,1)$ satisfying $\sum_{n=0}^{\infty}\alpha_n=\infty$. Then the following are equivalent:*
(i) *The T-Mann iteration converges to Tx^*;*
(ii) *The T-Ishikawa iteration converges to Tx^*.*

Proof. As in the proof of Theorem 2.1, since S is a TZ-operator, by applying Lemma 1.9 there is $0\leq\delta<1$ such that (7) holds for all $x,y\in K$.
First we will prove $(i)\Rightarrow(ii)$. Now, suppose that

$$\|Tu_n-Tx^*\|\to 0 \text{ as } n\to\infty.$$

Consider

$$\begin{aligned} \|Tu_{n+1} - Tx_{n+1}\| &= \|(1-\alpha_n)(Tu_n - Tx_n) + \alpha_n(TSu_n - TSy_n)\| \\ &\le (1-\alpha_n)\|Tu_n - Tx_n\| + \alpha_n \|TSu_n - TSy_n\|. \end{aligned} \tag{12}$$

Using (7) with $x = u_n$ and $y = y_n$, we have

$$\|TSu_n - TSy_n\| \le \delta \|Tu_n - Ty_n\| + 2\delta \|Tu_n - TSu_n\|.$$

Using the above inequality in (12), we get

$$\begin{aligned} \|Tu_{n+1} - Tx_{n+1}\| \le (1-\alpha_n)&\|Tu_n - Tx_n\| \\ &+ \alpha_n\delta \|Tu_n - Ty_n\| + 2\alpha_n\delta \|Tu_n - TSu_n\|. \end{aligned} \tag{13}$$

Now applying (5) and (7), we have

$$\begin{aligned} \|Tu_n - Ty_n\| &= \|(1-\beta_n)(Tu_n - Tx_n) + \beta_n(Tu_n - TSx_n)\| \\ &\le (1-\beta_n)\|Tu_n - Tx_n\| + \beta_n\|Tu_n - TSu_n\| + \beta_n\|TSu_n - TSx_n\| \\ &\le (1-\beta_n)\|Tu_n - Tx_n\| + \beta_n\|Tu_n - TSu_n\| \\ &+ \beta_n\delta\|Tu_n - Tx_n\| + 2\beta_n\delta\|Tu_n - TSu_n\| \\ &= (1-\beta_n(1-\delta))\|Tu_n - Tx_n\| + \beta_n(1+2\delta)\|Tu_n - TSu_n\|. \end{aligned} \tag{14}$$

Combining (13) and (14), we obtain

$$\begin{aligned} \|Tu_{n+1} - Tx_{n+1}\| &\le (1-\alpha_n)\|Tu_n - Tx_n\| + \alpha_n\delta(1-\beta_n(1-\delta))\|Tu_n - Tx_n\| \\ &+ \alpha_n\beta_n\delta(1+2\delta)\|Tu_n - TSu_n\| + 2\alpha_n\delta\|Tu_n - TSu_n\| \\ &= (1-\alpha_n(1-\delta(1-\beta_n(1-\delta))))\|Tu_n - Tx_n\| \\ &+ \alpha_n\delta\,(\beta_n(1+2\delta)+2)\|Tu_n - TSu_n\|. \end{aligned}$$

Denote by

$$\begin{aligned} a_n &= \|Tu_n - Tx_n\|, \\ \lambda_n &= \alpha_n(1-\delta(1-\beta_n(1-\delta))) \subset (0,1), \\ \sigma_n &= \alpha_n\delta\,(\beta_n(1+2\delta)+2)\|Tu_n - TSu_n\|. \end{aligned}$$

Since $\lim_{n\to\infty}\|Tu_n - Tx^*\| = 0$ and $x^* \in F(S)$, from (7) we obtain

$$\begin{aligned} \|Tu_n - TSu_n\| &\le \|Tu_n - Tx^*\| + \|TSx^* - TSu_n\| \\ &\le \|Tu_n - Tx^*\| + \delta\|Tx^* - Tu_n\| + 2\delta\|Tx^* - TSx^*\| \\ &= (\delta+1)\|Tu_n - Tx^*\| \to 0 \text{ as } n \to \infty, \end{aligned}$$

which gives $\sigma_n = O(\lambda_n)$.
Hence, from Lemma 1.11 it follows that

$$\lim_{n\to\infty} \|Tu_n - Tx_n\| = 0.$$

Thus,

$$\|Tx^* - Tx_n\| \le \|Tu_n - Tx^*\| + \|Tu_n - Tx_n\| \to 0 \text{ as } n \to \infty.$$

Next, we will prove that $(ii) \Rightarrow (i)$. Suppose that

$$\|Tx_n - Tx^*\| \to 0 \text{ as } n \to \infty.$$

Consider

$$\begin{aligned} \|Tx_{n+1} - Tu_{n+1}\| &= \|(1-\alpha_n)(Tx_n - Tu_n) + \alpha_n(TSy_n - TSu_n)\| \\ &\le (1-\alpha_n)\|Tx_n - Tu_n\| + \alpha_n \|TSy_n - TSu_n\|. \end{aligned} \tag{15}$$

Using (7) with $x = y_n$ and $y = u_n$, we obtain

$$\|TSy_n - TSu_n\| \le \delta \|Ty_n - Tu_n\| + 2\delta \|Ty_n - TSy_n\|.$$

Using the above inequality in (15), we get

$$\begin{aligned} &\|Tx_{n+1} - Tu_{n+1}\| \\ &\le (1-\alpha_n)\|Tx_n - Tu_n\| + \alpha_n\delta \|Ty_n - Tu_n\| + 2\alpha_n\delta \|Ty_n - TSy_n\|. \end{aligned} \tag{16}$$

Consider

$$\begin{aligned} &\|Ty_n - Tu_n\| \\ &= \|(1-\beta_n)(Tx_n - Tu_n) + \beta_n(TSx_n - Tu_n)\| \\ &\le (1-\beta_n)\|Tx_n - Tu_n\| + \beta_n \|TSx_n - Tx_n\| + \beta_n \|Tx_n - Tu_n\| \\ &= \|Tx_n - Tu_n\| + \beta_n \|TSx_n - Tx_n\|. \end{aligned} \tag{17}$$

Now, from (16) and (17) it results that

$$\begin{aligned} &\|Tx_{n+1} - Tu_{n+1}\| \\ &\le (1-\alpha_n)\|Tx_n - Tu_n\| + \alpha_n\delta(\|Tx_n - Tu_n\| + \beta_n \|TSx_n - Tx_n\|) \\ &+ 2\alpha_n\delta \|Ty_n - TSy_n\| \\ &= (1-\alpha_n(1-\delta))\|Tx_n - Tu_n\| + \alpha_n\beta_n\delta \|TSx_n - Tx_n\| \\ &+ 2\alpha_n\delta \|Ty_n - TSy_n\|. \end{aligned}$$

Denote by

$$\begin{aligned}
a_n &= \|Tx_n - Tu_n\|, \\
\lambda_n &= \alpha_n(1-\delta) \subset (0,1), \\
\sigma_n &= \alpha_n\beta_n\delta\,\|TSx_n - Tx_n\| + 2\alpha_n\delta\,\|Ty_n - TSy_n\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\|Tx_n - Tx^*\| = 0$ and $x^* \in F(S)$, from (7) we get

$$\begin{aligned}
\|Tx_n - TSx_n\| &\le \|Tx_n - Tx^*\| + \|TSx^* - TSx_n\| \\
&\le (\delta+1)\,\|Tx_n - Tx^*\| \to 0 \text{ as } n\to\infty. \qquad (18)
\end{aligned}$$

Again since $\lim_{n\to\infty}\|Tx_n - Tx^*\| = 0$ and $x^* \in F(S)$, from (5) and (7) we get

$$\begin{aligned}
\|Ty_n - TSy_n\| &\le \|Ty_n - Tx^*\| + \|Tx^* - TSy_n\| \\
&\le (\delta+1)\,\|Ty_n - Tx^*\| \\
&\le (\delta+1)[(1-\beta_n)\,\|Tx_n - Tx^*\| + \beta_n\,\|TSx_n - Tx^*\|] \\
&\le (\delta+1)[(1-\beta_n)\,\|Tx_n - Tx^*\| + \beta_n\delta\,\|Tx_n - Tx^*\|] \\
&= (\delta+1)(1-\beta_n(1-\delta))\,\|Tx_n - Tx^*\| \to 0 \text{ as } n\to\infty. \qquad (19)
\end{aligned}$$

From (18) and (19), it follows that $\sigma_n = O(\lambda_n)$. Now, by applying Lemma 1.11 we get

$$\lim_{n\to\infty}\|Tx_n - Tu_n\| = 0.$$

Hence

$$\|Tx^* - Tu_n\| \le \|Tx_n - Tu_n\| + \|Tx_n - Tx^*\| \to 0 \text{ as } n\to\infty.$$

□

Since the condition $u_0 = x_0$ is superfluous, taking $T = I$, the identity map in Theorem 2.3 we get the result proved by Şoltuz [[17], Theorem 1] as a corollary to our result.

Corollary 2.4. *Let E be a normed linear space, K a nonempty, closed, convex subset of E and $T : K \to K$ be a Z-operator. Suppose that T has a fixed point x^* in K. Let the Mann iteration for $u_0 \in K$ be defined by (4) and the Ishikawa iteration for $x_0 \in K$ be defined by (6) with $\{\alpha_n\}, \{\beta_n\} \in (0,1)$ satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then the following are equivalent:*
(i) The Mann iteration converges to x^*;
(ii) The Ishikawa iteration converges to x^*.

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