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A LOOK AT PROXIMINAL AND CHEBYSHEV SETS IN BANACH SPACES

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The main aim of this survey is to present some classical as well as recent characterizations involving the notion of proximinal and Chebyshev sets in Banach spaces. In particular, we discuss the convexity of Chebyshev sets.

1. Introduction

One of the basic question in *Approximation Theory* concerns the existence of *best approximations*. Due to its applications, such as:

1. Solution to an over-determined system of equations.

2. Best least squares polynomial approximation to a function.

3. Some control problems,

the problem of best approximation has a long history and gives rise to a lot of notions and techniques useful in functional analysis. In fact: Since 1970, when [2] has gone to print, the theory of best approximation in Banach spaces has developed rapidly and the number of papers in this field is growing continuously.

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The structure of the paper is as follows. In section 2, after stating the main definitions, we gather some well-known facts which have been obtained by many authors concerning *proximinal* and *Chebyshev* sets, until now. In the last part of section 2, we provide some conditions under which a *metric projection* is continuous. In section 3, we discuss some results about the convexity of Chebyshev sets. All undefined terms and notation are standard and can be found, for example, in [1, 2, 6, 13, 23, 62].

2. Proximinal and Chebyshev sets; continuity of metric projections

Let K be a non-empty subset of a Banach space $(X, \|.\|)$ and let $x \in X$. The (possibly empty) set of *best approximations* to x from K is defined by:

$$P_K(x) = \{y \in K : \|x - y\| = d_K(x)\},$$

where $d_K(x) = inf\{\|x - y\| : y \in K\}$. The set K is called *proximinal* (resp. *Chebyshev*) if $P_K(x)$ contains at least (resp. exactly) one point for every $x \in X$. The term proximinal set was proposed by Killgrove and used first by Phelps [35]. Also, the concept of Chebyshev sets was introduced by Stechkin in honor of the founder of best approximation theory, Chebyshev (1821-1894). The mapping $P_K : X \to 2^K \equiv$ the set of all subsets of K, which associates with each $x \in X$, the set $P_K(x)$, is called the *metric projection* of X onto K. As addressed above, for any Chebyshev set K, the map P_K is single-valued. We have in the following list some examples and non-examples of proximinal and Chebyshev sets.

1. If $X = \mathbb{R}^n$ with its usual norm $\|(x_k)_{k=1}^n\|_2 = \left(\sum_{k=1}^n |x_k|^2\right)^{\frac{1}{2}}$, any closed convex set K in X is Chebyshev.

2. If $X = \mathbb{R}^2$ under the norm $\|(a,b)\|_\infty = max\{|a|, |b|\}$ and $K = \{(0,b) : b \in \mathbb{R}\}$, then K is not a Chebyshev set. In fact $P_K(x) = \{(0,b) : |b| \leq 1\}$, for $x = (1,0) \in \mathbb{R}^2$.

3. If $X = \mathbb{R}^2$ equipped with the l_1-norm $\|(a,b)\| = |a| + |b|$ and $K = \{(a,b) : b = \pm a\}$, then $P_K(x) = \{(a, |a|) : |a| \leq 1\}$, for $x = (0,1) \in \mathbb{R}^2$ and so K is not Chebyshev.

4. Let $X = \mathbb{R}^2$ equipped with the norm

$$\|(x,y)\| = |x - y| + (x^2 + y^2)^{\frac{1}{2}},$$

and $K = \{(x,0) : |x| \leq 1\}$. Then K is a Chebyshev set in X.

5. In the middle of 19$^{\text{th}}$ century, Chebyshev proved that in the space $C[0,1]$ the subspace of all polynomials of degree $\leq n$ and the subset R_{nm} of all rational functions $\frac{a_0+a_1x+...+a_nx^n}{b_0+b_1x+...+b_mx^m}$ with fixed $n,m \in \mathbb{N}$ are Chebyshev subsets.

6. The reader can find some examples of Chebyshev sets in matrix spaces in [58].

7. It is known that if Y is either a reflexive or a separable proximinal subspace of a Banach space X then $L^p(I,Y)$ is proximinal in $L^p(I,X)$ for $0 \leq p < \infty$, where I is the unit interval with the Lebesgue measure. However there is an example of a proximinal subspace Y in X for which $L^p(I,Y)$ is not proximinal in $L^p(I,X)$. Recently, Khalil in [60] presented a class of proximinal subspaces Y in a Banach space X (which includes reflexive subspaces properly) such that $L^p(I,Y)$ is proximinal in $L^p(I,X)$, for all $0 \leq p < \infty$.

It is not hard to show that proximinal sets (and so, Chebyshev sets) are closed. For the converse, every non-empty closed set in a Minkowski space (a finite-dimensional Banach space) is proximinal [2]. Also, each non-empty compact subset of a Banach space is proximinal [2]. The following list of examples is intended to be a representative sampling of some of the more useful known proximinal sets.

1. Any reflexive subspace, e.g., a finite-dimensional subspace [3].

2. Any weakly closed subset of a reflexive Banach space [2].

3. Any closed convex subset of a reflexive Banach space [3, 31]. Note that a closed convex set is not proximinal, in general. Let $X = l^1$. For any $n \in \mathbb{N}$, let $e_n \in X$ be such that its nth entry is $\frac{n+1}{n}$ and all other entries are 0. Let $K = \overline{\text{co}}\{e_1, e_2, \cdots\}$. Then K is a closed convex subset of X and is not proximinal.

4. Any weak* closed subset of a dual Banach space [4].

5. Any linear subspace G of a Banach space $(X, \|.\|)$ such that $\{g \in G : \|g\| \leq 1\}$ is sequentially compact for the weak topology $\sigma(X, X^*)$ [2, page 708].

It is also a well-known classical result that: A Banach space X is reflexive if and only if every closed convex non-empty subset of X is proximinal if and only if for every non-empty closed convex subset K of X, $P_K(x) \neq \phi$ for at least one $x \notin K$ [61, page 188].
Concerning Chebyshevity, we have the following theorem [55, page 33].

Theorem 2.1. *Any finite-dimensional subspace Y in a strictly convex Banach space X is Chebyshev.*

Proof. At first Y is proximinal in X. Now, suppose $y_1, y_2 \in P_Y(x)$ for some $x \in X$. Hence, $\|y_1 - x\| = \|y_2 - x\| = d_Y(x)$ and so,

$$\|\frac{1}{2}(y_1 + y_2) - x\| \leq \frac{1}{2}\|y_1 - x\| + \frac{1}{2}\|y_2 - x\| = d_Y(x).$$

Since Y is a linear subspace, $\frac{1}{2}(y_1 + y_2) \in Y$; $\|\frac{1}{2}(y_1 + y_2) - x\| \geq d_Y(x)$. Now if $d_Y(x) = 0$, it is clear that $y_1 = x = y_2$. If $d_Y(x) \neq 0$, then the vectors $\frac{y_1 - x}{d_Y(x)}$, $\frac{y_2 - x}{d_Y(x)}$, and the midpoint are all of norm 1, and so by the strict convexity, $y_1 = y_2$. □

Here we state some facts under which closed sets are Chebyshev:
In 1961, Efimov and Stechkin, proved that any weak* closed convex set in a locally uniformly convex dual Banach space is Chebyshev [5]. Also, it was shown by Day [6] that every closed convex set in a reflexive strictly convex Banach space (e.g., a uniformly convex Banach space) is Chebyshev. For example, any closed subspace of a Hilbert space is Chebyshev. In fact, Day improved a result obtained by Efimov and Stechkin which state: Any closed convex set in a reflexive locally uniformly convex Banach space is Chebyshev [5].
Next we provide a condition under which a proximinal set is Chebyshev.
Let K be a non-empty set in a Banach space $(X, \|.\|)$ and $\widehat{K} = \{x \in X : \|x\| = d_K(x)\}$. Some authors call the set $\widehat{K}$ the metric complement of K. For example, let $X = \mathbb{R}^2$ equipped with the l_1-norm and $K = \{(x,y) : y = x\}$. Then

$$\widehat{K} = \{(x,y) : (x \geq 0 \text{ and } y \leq 0), \text{ or } (x \leq 0 \text{ and } y \geq 0)\}.$$

Let X be a Banach space, and K is a subspace of X. We have the following facts about $\widehat{K}$ [36].
(i) $K \bigcap \widehat{K} = \{0\}$.
(ii) The set $\widehat{K}$ is closed.
(iii) K is Chebyshev if and only if $X = K \oplus \widehat{K}$ (where $\oplus$ means that the sum decomposition of each element $x \in X$ is unique).
We have the following characterization of Chebyshev sets in terms of metric complement.

Theorem 2.2. ([37]) *Let X be a Banach space, and K a proximinal subspace of X for which $\widehat{K}$ is convex. Then K is Chebyshev.*

Proof. Suppose $g_1, g_2 \in P_K(x)$ for some $x \in X$. Then $\widehat{g_1} = x - g_1 \in \widehat{K}$ and $\widehat{g_2} = x - g_2 \in \widehat{K}$. Since $\frac{1}{2}(\widehat{g_1} - \widehat{g_2}) \in \widehat{K}$, we have $(g_1 - g_2) \in K \bigcap \widehat{K} = \{0\}$; it follows that $g_1 = g_2$. □

Let K be a non-empty subset of a Banach space $(X, \|.\|)$ and let $x \in X \backslash K$. The sequence $(y_n)_{n=1}^{\infty} \subseteq K$ is called *minimizing* for x if $lim_{n\to\infty}\|x-y_n\| = d_K(x)$. A non-empty subset K of a Banach space X is said to be (weakly) *approximatively compact* if for any $x \in X\backslash K$, all minimizing sequences for x have a (weakly convergent subsequence) Cauchy subsequence. A Banach space X is called (weakly) *approximatively compact* if any non-empty closed and convex subset of X is (weakly) approximatively compact. A Banach space X is weakly approximatively compact if and only if it is reflexive [39]. Also, a Banach space X is approximatively compact if and only if X is reflexive and X has the Kadec-Klee property [38].
In 1970, Singer [2] proved that every approximatively compact set in a Banach space is proximinal. For the converse: Let $X = l_\infty$, $Y = c_0$. Then Y is proximinal in X. For $x_0 = (1, 1, 1, \cdots) \in X$, the sequence $y_n = (1, 1, \cdots, 1, 0, 0, \cdots)$ is minimizing. But $(y_n)_{n=1}^{\infty}$ has no convergent subsequence. Thus proximinality does not imply approximative compactness [32].

Theorem 2.3. ([50]) *Let K be a closed convex set of a nearly strongly convex Banach space X. The following are equivalent:*
(i) K is proximinal.
(ii) K is weakly approximatively compact.
(iii) K is approximatively compact.

In 1998, it was proved that in approximatively compact Banach spaces, any closed convex set is Chebyshev [7]. A non-empty subset K of a Banach space X is *boundedly compact* provided that $K \cap B_X[0;r]$ is compact in X for every $r \geq 0$ (in this definition $B_X[0;r] \equiv \{y \in X : \|y\| \leq r\}$). Every boundedly compact set is approximatively compact although the converse is not true, in general. Thus, every boundedly compact set is proximinal.
An essential notion among Approximation Theory is the continuity of metric projection (the reader is referred to [57] and [59] for two brief surveys). The continuity property of the metric projections is a natural object of study in understanding the nature of some problems in Approximation Theory. If K is a non-empty subset of a Banach space X and $x \in X\backslash K$, then P_K is said to be continuous at x if $lim_{n\to\infty}y_n = y \in P_K(x)$ whenever $y_n \in P_K(x_n)$ and $lim_{n\to\infty}x_n = x$ [26]. It is clear that P_K is continuous at x if every minimizing sequence for x converges; the converse is not valid in general. In the following we list some sets, which all of them have the continuous metric projection.

1. Any finite-dimensional Chebyshev subspace of a Banach space [2, page 744].

2. Any weakly closed Chebyshev subset of a reflexive Banach space with the

Kadec-Klee property (e.g., a uniformly convex Banach space) [61, page 190].

3. Any closed convex set in a strictly convex reflexive space with the Kadec-Klee property [33].
   Brown [34] has shown that there exists a separable strictly convex and reflexive (real) Banach space X and a closed subspace Y such that P_Y is not continuous.

4. Any closed convex set in a reflexive locally uniformly convex Banach space (e.g., a uniformly convex Banach space) [5].

5. Any weak* closed convex set in a locally uniformly convex dual Banach space [5].

6. Any boundedly compact Chebyshev set [9].

In 1972, Oshman [40] discussed the relationship between approximative compactness of a non-empty subset K of a Banach space X and continuity of the P_K. In [42] the authors have provided a counterexample: There exists a midpoint locally uniformly convex Banach space X and a non-empty closed convex Chebyshev subset K of X such that P_K is continuous and K is not approximatively compact. In [39] it was proved that: Let X be a strongly convex Banach space and K be a non-empty closed and convex subset in X. Then K is a Chebyshev set and P_K is continuous if and only if K is approximatively compact in X. It is not known even in a Hilbert space whether for a Chebyshev set the associated metric projection is continuous or does there exist a Chebyshev set in a Hilbert space supporting a discontinuous metric projection? Brown [Abstract Approximation Theorem, Seminar in Analysis, 1969-70, Matscience, Madras (India)] gave an example supporting that even in a Hilbert space the metric projection on a Chebyshev set may fail to be weakly continuous.
In the end of this section, we introduce some sufficient conditions under which the metric projection P_K of a Banach space X onto a non-empty closed subset K of X, is continuous. We do it in terms of differentiability of distance function d_K on X which associates with each $x \in X$, the non-negative real number $d_K(x)$.

Lemma 2.4. *Suppose that K is a non-empty closed set in a Banach space $(X, \|.\|)$ and $x \in X \backslash K$. If d_K is Gateaux differentiable at x, then $d'_K(x)\left(\frac{x-y}{\|x-y\|}\right) = 1$, for all $y \in P_K(x)$.*

Proof. At first, from Gateaux differentiability of d_K, the limit

$$\liminf_{t \to 0^+} \frac{d_K(x+tz) - d_K(x)}{t},$$

exists for every $z \in X$. But for each $t > 0$

$$d_K(x+t(x-y)) - d_K(x) \leq t d_K(x).$$

Hence, in particular, for $z = x-y$

$$\liminf_{t\to 0^+} \frac{d_K(x+tz) - d_K(x)}{t} = d_K(x).$$

Now if $t' = \frac{t}{d_K(x)}$ (notice that $d_K(x) > 0$) then

$$\liminf_{t'\to 0^+} \frac{d_K(x+t'(x-y)) - d_K(x)}{t'} = d_K(x),$$

and consequently

$$\liminf_{t\to 0^+} \frac{d_K(x+t\frac{x-y}{\|x-y\|}) - d_K(x)}{t} = 1.$$

On the other hand, since distance functions are Lipschitz (with constant 1) we have

$$\limsup_{t\to 0^+} \frac{d_K(x+t\frac{x-y}{\|x-y\|}) - d_K(x)}{t} \leq 1,$$

as required. □

Let X be a Banach space. We say that a non-zero element $x^* \in X^*$ *strongly exposes* a subset C of X at a point x of C provided a sequence $(z_n)_{n=1}^\infty$ in C converges to x whenever $(x^*(z_n))_{n=1}^\infty$ converges to $x^*(x)$.
The next theorem is proved by Fitzpatrick in [8], but with some manipulation.

Theorem 2.5. *Suppose that K is a non-empty closed set in a Banach space $(X, \|.\|)$ and d_K is Fréchet differentiable at $x \in X\setminus K$. Moreover $y \in P_K(x)$ and $d'_K(x)$ strongly exposes the closed unit ball $B(X)$ at $\|x-y\|^{-1}(x-y)$. Then every minimizing sequence $(y_n)_{n=1}^\infty$ in K for x converges to y.*

Proof. Choose a sequence $(a_n)_{n=1}^\infty$ of positive numbers such that $\lim_{n\to\infty} a_n = 0$ and

$$a_n^2 > \|x-y_n\| - d_K(x), \qquad n \in \mathbb{N}.$$

Hence, if $0 < t < 1$ then for each $n \in \mathbb{N}$

$$\begin{aligned} d_K(x+t(y_n-x)) &\leq \|x+t(y_n-x)-y_n\| \\ &= (1-t)\|x-y_n\| \\ &< (1-t)(a_n^2 + d_K(x)). \end{aligned}$$

Therefore

$$d_K(x) - d_K(x + t(y_n - x)) \geq t d_K(x) - 2a_n^2.$$

Fix $\varepsilon > 0$. By Fréchet differentiability of d_K, there is $\delta > 0$ such that if $\|y\| < \delta$ then

$$|d_K(x+y) - d_K(x) - d'_K(x)(y)| \leq \varepsilon \|y\| \tag{1}$$

Let $t_n = \frac{a_n}{\|x-y_n\|}$ and $a_n < \delta$ for large n. Replacing y by $t_n(y_n - x)$ in 1 we get

$$\begin{aligned}
\varepsilon t_n \|x - y_n\| - d'_K(x)\Big(t_n(y_n - x)\Big) &\geq d_K(x) - d_K(x + t_n(y_n - x)) \\
&\geq t_n d_K(x) - 2a_n^2,
\end{aligned}$$

whence

$$d'_K(x)\Big(t_n(x - y_n)\Big) \geq -\varepsilon a_n - 2a_n^2 + t_n d_K(x),$$

therefore

$$d'_K(x)\Big(\|x - y_n\|^{-1}(x - y_n)\Big) \geq -\varepsilon - 2a_n + \frac{d_K(x)}{\|x - y_n\|}.$$

Since $\varepsilon > 0$, $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} \|x - y_n\| = d_K(x)$, we have

$$1 \geq \liminf_{n\to\infty} d'_K(x)\Big(\|x - y_n\|^{-1}(x - y_n)\Big) \geq \liminf_{n\to\infty} \frac{d_K(x)}{\|x - y_n\|} = 1,$$

therefore by Lemma 2.4

$$\lim_{n\to\infty} d'_K(x)\Big(\|x - y_n\|^{-1}(x - y_n)\Big) = 1 = d'_K(x)\Big(\|x - y\|^{-1}(x - y)\Big).$$

Since $d'_K(x)$ strongly exposes $B(X)$ at $\|x - y\|^{-1}(x - y)$, we deduce that

$$\lim_{n\to\infty} \|x - y_n\|^{-1}(x - y_n) = \|x - y\|^{-1}(x - y),$$

which yields $\lim_{n\to\infty} y_n = y$. □

Theorem 2.6. ([8]) *Let X be a Banach space and $x^* \in X^*$. The dual norm of X^* is Fréchet differentiable at x^* if and only if x^* strongly exposes $B(X)$.*

Combining Theorems 2.5 and 2.6, we get the following corollaries by using the result that the dual norm of X^* is Fréchet differentiable if X is a uniformly convex Banach space.

Corollary 2.7. *Let K be a non-empty closed set in a Banach space X and $x \in X \setminus K$. If d_K is Fréchet differentiable at x and the dual norm of X^* is Fréchet differentiable, then P_K is continuous at x.*

Corollary 2.8. *Suppose that K is a non-empty closed set in an uniformly convex Banach space X, $x \in X \setminus K$ and d_K is Fréchet differentiable at x. Then P_K is continuous at x.*

In [8] Fitzpatrick proved the following relations between properties of the metric projection and the distance function.

Theorem 2.9. *Suppose that K is a non-empty closed subset of a Banach space X such that the norm of X is both Fréchet differentiable and uniformly Gateaux differentiable and the dual norm of X^* is Fréchet differentiable. The following are equivalent for $x \in X \setminus K$:*
(i) d_K is Fréchet differentiable at x.
(ii) P_K is continuous at x.
(iii) Every minimizing sequence in K for x converges.

3. Convexity of Chebyshev sets

In finite-dimensional Hilbert spaces, we have the following theorem of Bunt [27].

Theorem 3.1. *Every Chebyshev subset of a finite-dimensional Hilbert space is convex.*

One of the most outstanding open problem of Approximation Theory is: Whether every Chebyshev set in an (infinite-dimensional) Hilbert space is convex? This problem was proposed by Klee in 1961 [9]. Klee conjectured that the answer is negative and proved that in every Hilbert space there exist non-convex closed semi-Chebyshev sets. In [16] Johnson gave an example: Let E denote the real inner product space that is the union of all finite-dimensional Euclidean spaces. There exists a bounded non-convex Chebyshev set S in E (Jiang completed the proof in 1993 [17]). Also, in [56], the question is raised: If H is the completion of E, then will the closure of E remain a Chebyshev set in H? A partial answer is given in the form of a sufficient condition for a point in H to have a unique best approximation in the closure of S. Recently, a conjecture aiming for the construction of a non-convex Chebyshev set in a Hilbert space was proposed in [43] by Faraci and Iannizzotto. On the other hand, much work has been done towards a positive answer. In 1951, Ficken showed that in a Hilbert space, every compact Chebyshev set is convex. Several partial answers to this problem are known in the literature [2, 9, 11–15, 18, 45, 52] but the problem is still unsolved. The characterization of those Banach spaces in which every Chebyshev set is convex is another open problem, and many sufficient conditions for a Chebyshev set to be convex have been obtained, until now. An old result of Motzkin [28] states that in a smooth and strictly convex Banach space X of finite-dimension, the class of closed convex sets coincides with the class of Chebyshev sets. Notice that, there are still some open problems in this subject, for instance: Is every Chebyshev set in a strictly convex reflexive Banach space convex [53]? All of the Chebyshev sets given in the following list are convex.

1. Any weakly closed Chebyshev set in a smooth and uniformly convex Banach space (e.g., a Hilbert space) [5, 30].

2. Every boundedly compact Chebyshev set in a smooth Banach space [18, 19].

3. Any approximatively compact Chebyshev set in a uniformly smooth Banach space [20].

4. Any Chebyshev set with continuous metric projection in a Banach space with strictly convex dual space (for example in a smooth reflexive Banach space) [21].
   For Hilbert spaces, Asplund [30] has shown that it is sufficient to assume that the metric projection is continuous from the norm topology to the weak topology. Also, Asplund [30] has proved: If K is a Chebyshev set in a Hilbert space such that every closed half-space intersects K in a proximinal set, then K is convex.
   Notice that, in (4) above, the assumption of continuity of the metric projection can be replaced by much weaker conditions. For example, Balaganskii proved that if K is a non-empty Chebyshev subset of a real Hilbert space and the set of discontinuities of P_K is countable, then K is convex [51]. Any non-convex Chebyshev set in a Hilbert space has a badly discontinuous metric projection [54].

5. Any Chebyshev set having continuous metric projection in a strongly smooth Banach space [21].
   Every reflexive and smooth Banach space with the Kadec-Klee property is strongly smooth. For example, each Hilbert space is strongly smooth. We noted in Section 2 that any weakly closed set in a reflexive Banach space with the Kadec-Klee property has continuous metric projection. Now, the following result is obtained from (5).

6. Any weakly closed Chebyshev set in a strongly smooth Banach space.

7. Any weakly closed Chebyshev set in a reflexive Banach space whose norm is Gateaux differentiable and has the Kadec-Klee property [61, page 193].

8. Every weakly compact Chebyshev set in a smooth Banach space [19].

In 2000, Kanellopoulos [29] extended (8): If X is an almost smooth Banach space then every weakly compact Chebyshev subset of X is convex. As an immediate consequence, Kanellopoulos obtained that: If X is a finite dimensional Banach space such that every exposed point of $B(X)$ is a smooth point then every bounded Chebyshev subset of X is convex.
The convexity of Chebyshev sets depends on the structural constraints imposed on the set. For example, Tsarkov [46, 47] showed that the class of finite-dimensional Banach spaces in which any Chebyshev set is convex differs from the class of spaces in which any bounded Chebyshev set is convex. The next theorem originates from results of Tsarkov [47], who constructed a norm on a Banach space X with $3 \leq \dim X < \infty$ such that any Chebyshev set in X which is bounded with respect to this norm is convex, whereas unbounded Chebyshev sets in X may be non-convex (for more results see [48, 49]).

Theorem 3.2. ([48]) *Let X be a finite-dimensional Banach space and $H \subset X$ is a hyperplane in X with $\dim H \geq 3$. Then there exists a norm on X such that any Chebyshev set $M \subset H$ in X which is bounded with respect to this norm is convex and there is an unbounded non-convex Chebyshev set $M_1 \subset H$ in X.*

Differentiability properties of the distance function have been of great interest in Approximation Theory as it relates to the famous problem of convexity of Chebyshev sets. It is not hard to show that for a non-empty closed subset K of a Banach space X, d_K is a convex function on X if and only if K is a convex set in X. The following theorem give us sufficient conditions for the convexity of d_K.

Theorem 3.3. ([22]) *In a Banach space $(X, \|.\|)$ where X^* has a strictly convex dual norm, given a non-empty closed set K in X, if $\limsup_{\|y\|\to 0} \frac{d_K(x+y)-d_K(x)}{\|y\|} = 1$ for all $x \in X \backslash K$, then d_K is a convex function on X.*

Let X be a Banach space, $x \in X$ and $x^* \in X^*$. Then x^* is called the *subdifferential* of a real-valued function f on X at x, if

$$x^*(y-x) \leq f(y) - f(x), \qquad y \in X.$$

The set of all subdifferentials of f at x is denoted by $\partial f(x)$.

Theorem 3.4. ([23]) *Let f be a real-valued convex function on X and continuous at $x \in X$ such that $\partial f(x)$ is a singleton. Then f is Gateaux differentiable at x.*

In the theorem above notice that the continuity of f at x is an essential condition. For example, if $f(x) = 1+\sin(\frac{1}{x})$ for all $x \neq 0$ and $f(0) = 0$, then f is not continuous at $x = 0$. Also, $\partial f(0) = \{0\}$, while f is not Gateaux differentiable at $x = 0$.
The next theorem give us a condition for the convexity of Chebyshev sets.

Theorem 3.5. *Let $(X, \|.\|)$ be a Banach space with a strictly convex dual norm, K is a non-empty Chebyshev set in X, $x \in X \backslash K$ and $\partial d_K(x)$ is a singleton. The following are equivalent:*
(i) K is convex.
(ii) d_K is Gateaux differentiable at x.
(iii) There is $z \in X$ such that $\|z\| = 1$ and $\lim_{t\to 0^+} \frac{d_K(x+tz)-d_K(x)}{t} = 1$.
(iv) $\limsup_{\|y\|\to 0} \frac{d_K(x+y)-d_K(x)}{\|y\|} = 1$.

Proof. By Theorems 3.3 and 3.4, it suffices to show $(ii \Rightarrow iii)$ and $(iii \Rightarrow iv)$.
$(ii \Rightarrow iii)$ At first, by the Chebishevity of K there exists a unique element $\bar{x} \in K$ such that $\|x - \bar{x}\| = d_K(x)$. It follows from Gateaux differentiability of d_K that $\liminf_{t\to 0^+} \frac{d_K(x+ty)-d_K(x)}{t}$ exists for every $y \in X$. For each $t > 0$ we have $d_K(x+t(x-\bar{x})) - d_K(x) \leq t d_K(x)$. Hence if $y = x - \bar{x}$ then

$$\liminf_{t\to 0^+} \frac{d_K(x+t(x-\bar{x}))-d_K(x)}{t} = d_K(x) \tag{2}$$

Since $x \in X \backslash K$, $d_K(x) > 0$. Let $t' = \dfrac{t}{d_K(x)}$. Then by 2 above

$$\liminf_{t'\to 0^+} \frac{d_K(x+t'(x-\bar{x}))-d_K(x)}{t'} = d_K(x).$$

If now $z = \dfrac{x-\bar{x}}{\|x-\bar{x}\|}$ then $\|z\| = 1$ and we have $\liminf_{t\to 0^+} \frac{d_K(x+tz)-d_K(x)}{t} = 1$. On the other hand, d_K is a Lipschitz function and so $\limsup_{t\to 0^+} \frac{d_K(x+tz)-d_K(x)}{t} \leq 1$.
$(iii \Rightarrow iv)$ Since d_K is a Lipschitz function, $\limsup_{\|y\|\to 0} \frac{d_K(x+y)-d_K(x)}{\|y\|} \leq 1$. On the other hand, for each $v \in X$ with $\|v\| = 1$

$$\lim_{t\to 0^+} \frac{d_K(x+tv)-d_K(x)}{t} \leq \limsup_{\|y\|\to 0} \frac{d_K(x+y)-d_K(x)}{\|y\|}.$$

In particular for $v = z$ in (iii), we have $1 \leq \limsup_{\|y\|\to 0} \frac{d_K(x+y)-d_K(x)}{\|y\|}$. □

We have the following theorems for the convexity of d_K (equivalently, for the convexity of K) if $\partial d_K(x)$ is not a singleton set:

Theorem 3.6. ([22]) *In a Banach space X with a strictly convex dual norm, if d_K, generated by a non-empty closed set K in X, is Gateaux differentiable on $X \backslash K$ and $\|d'_K(x)\| = 1$ (in the norm topology of X^*) for all $x \in X \backslash K$ then d_K is a convex function on X.*

Theorem 3.7. ([25]) *In a Banach space X with a strictly convex dual norm, if for a given non-empty closed set K in X, d_K is Fréchet differentiable at each $x \in X \backslash K$, then d_K is a convex function on X.*

Theorem 3.8. ([26]) *Let K be a non-empty closed subset of a Banach space $(X, \|.\|)$ such that for each $x \in X \backslash K$ there exists $z \in X$ such that $\|z\| = 1$ and $\lim_{t \to 0} \frac{d_K(x+tz) - d_K(x)}{t} = 1$. If the norms of X and X^* are Fréchet differentiable, then d_K is a convex function on X.*

Recently, the following nice characterizations have been obtained.

Theorem 3.9. ([24]) *Suppose the norms on a Banach space X and its dual X^* are locally uniformly convex. Then a non-empty Chebyshev set K is convex in X if and only if $\partial d_K(x)$ is a singleton for all $x \in X \backslash K$.*

Theorem 3.10. ([61, page 190]) *Let H be a Hilbert space and suppose K is a non-empty weakly closed subset of H. The following are equivalent:*
(i) K is convex.
(ii) K is a Chebyshev set.
(iii) d_K^2 is Fréchet differentiable.
(iv) d_K^2 is Gateaux differentiable.

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