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HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES WHEN THE POWER OF THE ABSOLUTE VALUE OF THE FIRST DERIVATIVE OF THE INTEGRAND IS PREINVEX

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In the paper, the authors establish some new Hermite-Hadamard type integral inequalities when the power of the absolute value of the first derivative of the integrand is preinvex.

1. Introduction

We first recite some definitions of various convex functions.

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty,\infty) \to \mathbb{R}$ is said to be convex if

$$f(\lambda x+(1-\lambda)y) \le \lambda f(x)+(1-\lambda)f(y) \tag{1}$$

holds for all $x,y \in I$ and $\lambda \in [0,1]$.

Definition 1.2 ([1, 5, 21])**.** A set $S \subseteq \mathbb{R}^n$ is said to be invex with respect to the map $\eta : S\times S \to \mathbb{R}^n$ if for every $x,y \in S$ and $t \in [0,1]$ we have

$$y+t\eta(x,y) \in S. \tag{2}$$

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Definition 1.3. Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. For every $x, y \in S$ the η-path P_{xv} joining the points x and $v = x + \eta(y, x)$ is defined by

$$P_{xv} = \{z \mid z = x + t\eta(y,x), t \in [0,1]\}. \tag{3}$$

Definition 1.4 ([5])**.** Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. A function $f : S \to \mathbb{R}$ is said to be preinvex with respect to η if for every $x, y \in S$ and $t \in [0,1]$ we have

$$f(y + t\eta(x,y)) \le tf(x) + (1-t)f(y). \tag{4}$$

We now formulate some inequalities of Hermite-Hadamard type for the above mentioned convex functions.

Theorem 1.5 ([8, Theorem 2.2])**.** *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping and $a, b \in I^\circ$ with $a < b$. If $|f'(x)|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{8}\big(|f'(a)| + |f'(b)|\big). \tag{5}$$

Theorem 1.6 ([11, Theorems 2.3 and 2.4])**.** *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $|f'(x)|^p$ is s-convex on $[a,b]$ for some $s \in (0,1]$ and $p > 1$, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p}\big(|f'(a)| + |f'(b)|\big) \tag{6}$$

and

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\left(\frac{4}{p+1}\right)^{1/p}\Big\{\big[|f'(a)|^{p/(p-1)}\\ + 3|f'(b)|^{p/(p-1)}\big]^{1-1/p} + \big[3|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}\big]^{1-1/p}\Big\}.\end{aligned} \tag{7}$$

Theorem 1.7 ([5, Theorem 2.1])**.** *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta : A \times A \to \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a differentiable function. If $|f'|$ is preinvex on A, then for every $a, b \in A$ with $\theta(a,b) \ne 0$ we have*

$$\begin{aligned}\left|\frac{f(b) + f(b+\theta(a,b))}{2} - \frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}x\right|\\ \le \frac{|\theta(a,b)|}{8}\big[|f'(a)| + |f'(b)|\big].\end{aligned} \tag{8}$$

For more information on Hermite-Hadamard type inequalities for various convex functions, please refer to recently published articles [2–4, 6, 7, 9, 10, 12–20, 22, 23] and closely related references therein.

In this article, we will establish some new Hermite-Hadamard type integral inequalities when the power of the absolute value of the first derivative of the integrand is preinvex.

2. A Lemma

In order to establish new integral inequalities of Hermite-Hadamard type, we need the following integral identity.

Lemma 2.1. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta : A \times A \to \mathbb{R}$, $a, b \in A$ with $\theta(a,b) \neq 0$, and let $f : A \to \mathbb{R}$ be a differentiable function. If the first derivative f' is integrable on the θ-path P_{bc}, then*

$$\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}x - f\left(\frac{2b+\theta(a,b)}{2}\right) = \theta(a,b)\left[\int_{1/2}^{1}(1-t)f'(b+t\theta(a,b))\,\mathrm{d}t - \int_0^{1/2} tf'(b+t\theta(a,b))\,\mathrm{d}t\right]. \quad (9)$$

Proof. Since $a, b \in A$ and A is an invex set with respect to θ, for every $t \in [0,1]$, we have $b + t\theta(a,b) \in A$. Integrating by part implies that

$$\begin{aligned}
&\theta(a,b)\left[\int_0^{1/2}(-t)f'(b+t\theta(a,b))\,\mathrm{d}t + \int_{1/2}^{1}(1-t)f'(b+t\theta(a,b))\,\mathrm{d}t\right] \\
&= -f(b+t\theta(a,b))t\Big|_0^{1/2} + \int_0^{1/2} f(b+t\theta(a,b))\,\mathrm{d}t \\
&\quad + f(b+t\theta(a,b))(1-t)\Big|_{1/2}^{1} + \int_{1/2}^{1} f(b+t\theta(a,b))\,\mathrm{d}t \\
&= -\frac{1}{2}f\left(b+\frac{1}{2}\theta(a,b)\right) + \int_0^{1/2} f(b+t\theta(a,b))\,\mathrm{d}t \\
&\quad -\frac{1}{2}f\left(b+\frac{1}{2}\theta(a,b)\right) + \int_{1/2}^{1} f(b+t\theta(a,b))\,\mathrm{d}t \\
&= -f\left(b+\frac{1}{2}\theta(a,b)\right) + \int_0^{1} f(b+t\theta(a,b))\,\mathrm{d}t \\
&= \frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}x - f\left(\frac{2b+\theta(a,b)}{2}\right).
\end{aligned}$$

Lemma 2.1 is proved. □

3. Some new integral inequalities of Hermite-Hadamard type

We are now in a position to establish some new integral inequalities of Hermite-Hadamard type.

Theorem 3.1. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta : A \times A \to \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a differentiable function. If $|f'|^q$ is preinvex on A for $q \geq 1$, then for every $a, b \in A$ with $\theta(a,b) \neq 0$ we have*

$$\left|\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}x - f\left(\frac{2b+\theta(a,b)}{2}\right)\right| \leq \frac{|\theta(a,b)|}{8}\left[\left(\frac{|f'(a)|^q+2|f'(b)|^q}{3}\right)^{1/q} + \left(\frac{2|f'(a)|^q+|f'(b)|^q}{3}\right)^{1/q}\right]. \tag{10}$$

Proof. Since A is an invex set with respect to θ, for every $t \in [0,1]$, we have $b + t\theta(a,b) \in A$. By Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}x - f\left(\frac{2b+\theta(a,b)}{2}\right)\right| \\
&\leq |\theta(a,b)|\left[\int_0^{1/2} t\big|f'(b+t\theta(a,b))\big|\,\mathrm{d}t + \int_{1/2}^1 (1-t)\big|f'(b+t\theta(a,b))\big|\,\mathrm{d}t\right] \\
&\leq |\theta(a,b)|\left\{\left(\int_0^{1/2} t\,\mathrm{d}t\right)^{1-1/q}\left[\int_0^{1/2} t\big|f'(b+t\theta(a,b))\big|^q\,\mathrm{d}t\right]^{1/q}\right. \\
&\quad \left. + \left(\int_{1/2}^1 (1-t)\,\mathrm{d}t\right)^{1-1/q}\left[\int_{1/2}^1 (1-t)\big|f'(b+t\theta(a,b))\big|^q\,\mathrm{d}t\right]^{1/q}\right\} \\
&\leq |\theta(a,b)|\left\{\left(\int_0^{1/2} t\,\mathrm{d}t\right)^{1-1/q}\left[\int_0^{1/2} t\left(t|f'(a)|^q + (1-t)|f'(b)|^q\right)\mathrm{d}t\right]^{1/q}\right. \\
&\quad \left. + \left(\int_{1/2}^1 (1-t)\,\mathrm{d}t\right)^{1-1/q}\left[\int_{1/2}^1 (1-t)\left(t|f'(a)|^q + (1-t)|f'(b)|^q\right)\mathrm{d}t\right]^{1/q}\right\} \\
&= \frac{|\theta(a,b)|}{8}\left[\left(\frac{|f'(a)|^q+2|f'(b)|^q}{3}\right)^{1/q} + \left(\frac{2|f'(a)|^q+|f'(b)|^q}{3}\right)^{1/q}\right].
\end{aligned}$$

The proof of Theorem 3.1 is completed. □

Corollary 3.2. *Under the conditions of Theorem 3.1, if $q = 1$, we have*

$$\left|\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}x - f\left(\frac{2b+\theta(a,b)}{2}\right)\right| \leq \frac{|\theta(a,b)|}{8}[|f'(a)| + |f'(b)|].$$

Theorem 3.3. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta : A \times A \to \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a differentiable function. If $q > 1$, $q \geq r, s \geq 0$ and $|f'|$ is* *preinvex on A, then for every $a,b \in A$ with $\theta(a,b) \neq 0$ we have*

$$\left|\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}x - f\left(\frac{2b+\theta(a,b)}{2}\right)\right| \le \frac{|\theta(a,b)|}{4}$$
$$\times\left\{\left(\frac{1}{r+1}\right)^{1/q}\left(\frac{q-1}{2q-r-1}\right)^{1-1/q}\left[\frac{(r+1)|f'(a)|^q+(r+3)|f'(b)|^q}{2(r+2)}\right]^{1/q}\right.$$
$$\left.+\left(\frac{1}{s+1}\right)^{1/q}\left(\frac{q-1}{2q-s-1}\right)^{1-1/q}\left[\frac{(s+3)|f'(a)|^q+(s+1)|f'(b)|^q}{2(s+2)}\right]^{1/q}\right\}.$$

Proof. Since A is an invex set with respect to θ, for every $t \in [0,1]$, then $b+t\theta(a,b) \in A$. By Lemma 2.1 and Hölder's inequality, we have

$$\left|\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}x - f\left(\frac{2b+\theta(a,b)}{2}\right)\right|$$
$$\le |\theta(a,b)|\left[\int_0^{1/2} t|f'(b+t\theta(a,b))|\,\mathrm{d}t + \int_{1/2}^1 (1-t)|f'(b+t\theta(a,b))|\,\mathrm{d}t\right]$$
$$\le |\theta(a,b)|\left\{\left[\int_0^{1/2} t^{(q-r)/(q-1)}\,\mathrm{d}t\right]^{1-1/q}\left[\int_0^{1/2} t^r|f'(b+t\theta(a,b))|^q\,\mathrm{d}t\right]^{1/q}\right.$$
$$\left.+\left[\int_{1/2}^1 (1-t)^{(q-s)/(q-1)}\,\mathrm{d}t\right]^{1-1/q}\left[\int_{1/2}^1 (1-t)^s|f'(b+t\theta(a,b))|^q\,\mathrm{d}t\right]^{1/q}\right\}$$
$$\le |\theta(a,b)|\left\{\left[\int_0^{1/2} t^{(q-r)/(q-1)}\,\mathrm{d}t\right]^{1-1/q}\left[\int_0^{1/2} t^r(t|f'(a)|^q\right.\right.$$
$$\left.+(1-t)|f'(b)|^q)\,\mathrm{d}t\right]^{1/q} + \left[\int_{1/2}^1 (1-t)^{(q-s)/(q-1)}\,\mathrm{d}t\right]^{1-1/q}$$
$$\left.\times\left[\int_{1/2}^1 (1-t)^s(t|f'(a)|^q+(1-t)|f'(b)|^q)\,\mathrm{d}t\right]^{1/q}\right\}$$
$$= \frac{|\theta(a,b)|}{4}\left\{\left(\frac{1}{r+1}\right)^{1/q}\left(\frac{q-1}{2q-r-1}\right)^{1-1/q}\right.$$
$$\times\left[\frac{(r+1)|f'(a)|^q+(r+3)|f'(b)|^q}{2(r+2)}\right]^{1/q}$$
$$\left.+\left(\frac{1}{s+1}\right)^{1/q}\left(\frac{q-1}{2q-s-1}\right)^{1-1/q}\left[\frac{(s+3)|f'(a)|^q+(s+1)|f'(b)|^q}{2(s+2)}\right]^{1/q}\right\}.$$

The proof of Theorem 3.3 is complete. □

Corollary 3.4. *Under the conditions of Theorem 3.3, when $r = s$, we have*

$$\left|\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}x - f\left(\frac{2b+\theta(a,b)}{2}\right)\right| \le \frac{|\theta(a,b)|}{4}$$
$$\times\left(\frac{q-1}{2q-r-1}\right)^{1-1/q}\left(\frac{1}{r+1}\right)^{1/q}\left\{\left[\frac{(r+1)|f'(a)|^q+(r+3)|f'(b)|^q}{2(r+2)}\right]^{1/q}\right.$$
$$\left.+\left[\frac{(r+3)|f'(a)|^q+(r+1)|f'(b)|^q}{2(r+2)}\right]^{1/q}\right\}.$$

Corollary 3.5. *Under the conditions of Theorem 3.3,*

1. *when $r = s = 0$, we have*

$$\left|\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}x - f\left(\frac{2b+\theta(a,b)}{2}\right)\right| \le \left(\frac{q-1}{2q-1}\right)^{1-1/q}$$
$$\times\frac{|\theta(a,b)|}{4}\left\{\left[\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right]^{1/q}+\left[\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right]^{1/q}\right\}.$$

2. *when $r = s = q$, we have*

$$\left|\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)\,\mathrm{d}x - f\left(\frac{2b+\theta(a,b)}{2}\right)\right|$$
$$\le\frac{|\theta(a,b)|}{4}\left(\frac{1}{q+1}\right)^{1/q}\left\{\left[\frac{(q+1)|f'(a)|^q+(q+3)|f'(b)|^q}{2(q+2)}\right]^{1/q}\right.$$
$$\left.+\left[\frac{(q+3)|f'(a)|^q+(q+1)|f'(b)|^q}{2(q+2)}\right]^{1/q}\right\}.$$

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