LE MATEMATICHE
Vol. LXIX (2014) – Fasc. I, pp. 97–107
doi: 10.4418/2014.69.1.7

ON SOME INTERESTING PROPERTIES OF MULTIVALENT ANALYTIC FUNCTIONS INVOLVING A LIU-OWA OPERATOR

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In this paper, we introduce some new subclasses of multivalent analytic functions in the unit disc E, and investigate a number of inclusion relationships, radius problem, and some other interesting properties of p-valent functions which are defined here by means of a certain integral operator $Q^{\alpha}_{\beta,p}f(z)$.

1. Introduction

Let $\mathcal{A}(p)$denote the class of functions $f(z)$ normalized by

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p},\ (p \in \mathbb{N} = \{1,2,\ldots\}), \tag{1}$$

which are analytic and p-valent in the open unit disc $E = \{z : |z| < 1\}$.

Let $P_k(\rho)$ be the class of functions $p(z)$ analytic in E with $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\Re p(z) - \rho}{1-\rho} \right| d\theta \leq k\pi,\ z = re^{i\theta}, \tag{2}$$

Entrato in redazione: 9 febbraio 2012

AMS 2010 Subject Classification: 30C45, 30C50.
Keywords: Multivalent functions, Analytic functions, Liu and Owa operator, Functions with positive real part, Hadamard product.

where $k \geqslant 2$ and $0 \leq \rho < 1$. This class was introduced by Padmanabhan et al, see [6]. We note that $P_k(0) = P_k$ (see Pinchuk [7]), $P_2(\rho) = P(\rho)$, the class of analytic functions with positive real part greater than ρ and $P_2(0) = P$, the class of functions with positive real part. From (2) we can easily deduce that $p(z) \in P_k(\rho)$ if and only if there exists $p_1(z), p_2(z) \in P(\rho)$ such that for $z \in E$,

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z). \tag{3}$$

For functions $f_j(z) \in \mathcal{A}(p)$, given by

$$f_j(z) = z^p + \sum_{k=1}^{\infty} a_{k+p,j} z^{k+p} \quad (j = 1, 2), \tag{4}$$

we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 \star f_2)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p,1} a_{k+p,2} z^{k+p} = (f_2 \star f_1)(z) \ \ (z \in E). \tag{5}$$

Motivated by Jung et al. [2] Liu and Owa [3] considered the linear operator $Q^{\alpha}_{\beta,p} : \mathcal{A}(p) \longrightarrow \mathcal{A}(p)$ defined as follows:

$$Q^{\alpha}_{\beta,p} f(z) = \binom{p+\alpha+\beta-1}{p+\beta-1} \frac{\alpha}{z^{\beta}} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)dt$$

$$\text{for } \alpha > 0, \beta > -1, \tag{6}$$

and

$$Q^{0}_{\beta,p} f(z) = f(z) \text{ for } \alpha = 0,\ \beta > -1. \tag{7}$$

We note that if $f \in \mathcal{A}(p)$ then, from (6) and (7), it follows that

$$Q^{\alpha}_{\beta,p} f(z) = z^p + \frac{\Gamma(p+\alpha+\beta)}{\Gamma(p+\beta)} \sum_{k=p+1}^{\infty} \frac{\Gamma(k+\beta)}{\Gamma(k+\alpha+\beta)} a^k z^k,$$

whenever $\alpha \geq 0$ and $\beta > -1$. Using the above relation, it is easy to verify that

$$z(Q^{\alpha}_{\beta,p} f(z))' = (p+\alpha+\beta-1) Q^{\alpha-1}_{\beta,P} f(z) - (\alpha+\beta-1) Q^{\alpha}_{\beta,P} f(z). \tag{8}$$

For the interested readers we refer to the work done by the authors $[1,3]$.

Using the operator $Q^{\alpha}_{\beta,p}$, we now define a subclass of $\mathcal{A}(p)$ as follows ([9–11]):

Definition 1.1. Let $\alpha \geq 0$, $\beta > -1$, $\mu > 0$, $\lambda \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, $p \in \mathbb{N}$, we say that a function $f(z) \in \mathcal{A}(p)$ is in the class $\mathcal{T}^{\alpha}_{\beta,p,k}(\lambda,\mu,\rho)$ if it satisfies:

$$\left\{(1-\lambda)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}g(z)}\right)^{\mu}+\lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha-1}_{\beta,p}g(z)}\right)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}g(z)}\right)^{\mu-1}\right\} \in P_k(\rho), z \in E, \tag{9}$$

where $k \geq 2$, $0 \leq \rho < 1$ and $g \in \mathcal{A}(p)$ satisfies the condition

$$\left(\frac{Q^{\alpha-1}_{\beta,p}\ g(z)}{Q^{\alpha}_{\beta,p}\ g(z)}\right) \in P(\eta), z \in E, \text{ with } 0 \leq \eta < 1. \tag{10}$$

In the present paper, we investigate a number of inclusion relationships, radius problem, and some other interesting properties of p-valent functions which are defined here by means of a certain integral operator $Q^{\alpha}_{\beta,p}f(z)$.

2. Preliminaries

In this section we recall some known results.

Lemma 2.1 ([4]). *Let $u = u_1 + iu_2$, $v = v_1 + iv_2$ and $\Psi(u,v)$ be a complex valued function satisfying the conditions:*
(i) $\Psi(u,v)$ is continuous in a domain $D \subset \mathbb{C}^2$,
(ii) $(1,0) \in D$ and $\Re\Psi(1,0) > 0$,
(iii) $\Re\Psi(iu_2,v_1) \leq 0$, whenever $(iu_2,v_1) \in D$ and $v_1 \leq -\frac{1}{2}\left(1+u_2^2\right)$.

If $h(z) = 1 + c_1 z + \cdots$ is a function analytic in E such that $(h(z), zh'(z)) \in D$ and $\Re\Psi(h(z), zh'(z)) > 0$ for $z \in E$, then $\Re h(z) > 0$ in E.

Lemma 2.2 ([8]). *If $p(z)$ is analytic in E with $p(0) = 1$ and if λ_1 is a complex number satisfying $\Re(\lambda_1) \geq 0$ $(\lambda_1 \neq 0)$, then*

$$\Re\left\{p(z) + \lambda_1 z p'(z)\right\} > \sigma \qquad (0 \leq \sigma < 1).$$

Implies

$$\Re p(z) > \sigma + (1-\sigma)(2\gamma_1 - 1),$$

where γ_1 is given by

$$\gamma_1 = \gamma_1(\Re\lambda_1) = \int_0^1 (i + t^{\Re\lambda_1})dt,$$

which is an increasing function of $\Re\lambda_1$ and $\frac{1}{2} \leq \gamma_1 < 1$. The estimate is sharp in the sense that the bound cannot be improved.

Lemma 2.3 ([1]). *If $q(z)$ be analytic in E with $q(0) = 1$ and $\Re q(z) > 0$, $z \in E$. Then, for $|z| = r$, $z \in E$,*

(i) $\frac{1-r}{1+r} \leq \Re q(z) \leq |q(z)| \leq \frac{1+r}{1-r}$,

(ii) $|q'(z)| \leq \frac{2\Re q(z)}{1-r^2}$.

3. Main Results

Theorem 3.1. *Let $f \in \mathcal{T}^{\alpha}_{\beta,p,k}(\lambda,\mu,\rho)$ and $\Re\lambda > 0$. Then $\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}\,g(z)}\right)^{\mu} \in P_k(\gamma)$, where*

$$\gamma = \frac{2\mu(p+\alpha+\beta-1)\rho+\lambda\delta}{2\mu(p+\alpha+\beta-1)+\lambda\delta}, \tag{11}$$

and $g \in \mathcal{A}(p)$ satisfies the condition (10) and

$$\delta = \frac{\Re h_0(z)}{|h_0(z)|^2}, \quad h_0(z) = \left(\frac{Q^{\alpha-1}_{\beta,p}\ g(z)}{Q^{\alpha-1}_{\beta,p}\ g(z)}\right).$$

Proof. Set

$$\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}\,g(z)}\right)^{\mu} = (1-\gamma)h(z)+\gamma, \tag{12}$$

$h(0) = 1$, and $h(z)$ is analytic in E and we can write

$$h(z) = (\frac{k}{4}+\frac{1}{2})h_1(z) - (\frac{k}{4}-\frac{1}{2})h_2(z). \tag{13}$$

Differentiating (12) with respect to z and using the identity (8), we have

$$\left\{(1-\lambda)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}\,g(z)}\right)^{\mu} + \lambda\left(\frac{Q^{\alpha-1}_{\beta,p}\ f(z)}{Q^{\alpha-1}_{\beta,p}\ g(z)}\right)\left(\frac{Q^{\alpha}_{\beta,p}\ f(z)}{Q^{\alpha}_{\beta,p}\,g(z)}\right)^{\mu-1}\right\}$$
$$= (\frac{k}{4}+\frac{1}{2})\left\{(1-\gamma)h_1(z)+\gamma-\rho+\frac{\lambda(1-\gamma)zh_1'(z)}{\mu(p+\alpha+\beta-1)h_0(z)}\right\}$$
$$-(\frac{k}{4}-\frac{1}{2})\left\{(1-\gamma)h_2(z)+\gamma-\rho+\frac{\lambda(1-\gamma)zh_2'(z)}{\mu(p+\alpha+\beta-1)h_0(z)}\right\}.$$

Now, we form the functional $\Psi(u,v)$ by choosing $u = h_i(z) = u_1+iu_2$ and $v = zh_i'(z) = v_1+iv_2$. Thus

$$\Psi(u,v) = \left\{(1-\gamma)u+\gamma-\rho+\frac{\lambda(1-\gamma)v}{\mu(p+\alpha+\beta-1)h_0(z)}\right\}.$$

The first two conditions of Lemma 2.1 are clearly satisfied. We verify the condition (iii) as follows:

$$\begin{aligned}\Psi(iu_2,v_1) &= \gamma-\rho+\frac{\lambda(1-\gamma)v_1\Re h_0(z)}{\mu(p+\alpha+\beta-1)\,|h_0(z)|^2}\\ &= \gamma-\rho+\frac{\lambda(1-\gamma)v_1\delta}{\mu(p+\alpha+\beta-1)}, \quad \text{where } \delta=\frac{\Re h_0(z)}{|h_0(z)|^2}.\end{aligned}$$

Now, for $v_1 \leq -\frac{1}{2}(1+u_2^2)$, we have

$$\begin{aligned}\Re\Psi(iu_2,v_1) &\leq \gamma-\rho-\frac{1}{2}\frac{\lambda(1-\gamma)(1+u_2^2)\delta}{\mu(p+\alpha+\beta-1)}\\ &= \frac{2\mu(p+\alpha+\beta-1)(\gamma-\rho)-\lambda\delta(1-\gamma)-\lambda\delta(1-\gamma)u_2^2}{2\mu(p+\alpha+\beta-1)}\\ &= \frac{A+Bu_2^2}{2C}, \qquad C>0,\\ A &= 2\mu(p+\alpha+\beta-1)(\gamma-\rho)-\lambda\delta(1-\gamma),\\ B &= -\lambda\delta(1-\gamma)\leq 0.\end{aligned}$$

Now, $\Re\Psi(iu_2,v_1)\leq 0$ if $A\leq 0$ and this gives us γ as defined by (11). We now applying Lemma 2.1 to conclude that $h_i \in P$ for $z\in E$ and thus $h\in P_k$ which gives us the required result. □

We note that $\gamma=\rho$ when $\eta=0$.

Theorem 3.2. *For $\lambda\geq 1$, let $f\in \mathcal{T}^{\alpha}_{\beta,p,k}(\lambda,\mu,\rho)$. Then*

$$\left(\frac{Q^{\alpha-1}_{\beta,p}\,f(z)}{Q^{\alpha-1}_{\beta,p}\,g(z)}\right)\in P_k(\rho), \ \text{for } z\in E.$$

Proof. We can write, for $\lambda\geq 1$,

$$\begin{aligned}\lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha-1}_{\beta,p}g(z)}\right) = \left\{(1-\lambda)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}g(z)}\right)+\lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha-1}_{\beta,p}g(z)}\right)\right\}&\\ +(\lambda-1)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}g(z)}\right).&\end{aligned}$$

This implies that

$$\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha-1}_{\beta,p}g(z)}\right)=\frac{1}{\lambda}\left\{(1-\lambda)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}g(z)}\right)+\lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha-1}_{\beta,p}g(z)}\right)\right\}$$
$$+(1-\frac{1}{\lambda})\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}g(z)}\right)=\frac{1}{\lambda}H_1(z)+(1-\frac{1}{\lambda})H_2(z).$$

Since $H_1(z)$, $H_2(z)\in P_k(\rho)$, by Theorem 3.1, Definition 1.1 and since $P_k(\rho)$ is a convex set (see [5]), we obtain the required result. □

Theorem 3.3. *Let $\lambda\in\mathbb{C}\backslash\{0\}$ with $\Re\lambda>0$. If $f\in\mathcal{A}(p)$ satisfies the following condition:*

$$\left\{(1-\lambda)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu}+\lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{z^p}\right)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu-1}\right\}\in P_k(\rho),$$

for $\mu>0$ ($z\in E$), then

$$\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu}\in P_k(\sigma),$$

where

$$\sigma=\rho+(1-\rho)(2\sigma_1-1)\text{ with }\sigma_1=\int_0^1(1+t^{\Re\frac{\lambda}{\mu(p+\alpha+\beta-1)}})dt.$$

The value of σ is the best possible and cannot be improved.

Proof. We set

$$\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu}=h(z)=(\frac{k}{4}+\frac{1}{2})h_1(z)-(\frac{k}{4}-\frac{1}{2})h_2(z),$$

where $h(0)=1$ and h is analytic in E. Then by simple computations together with (8), we have

$$\left\{(1-\lambda)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu}+\lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{z^p}\right)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu-1}\right\}$$
$$=\left\{h(z)+\frac{\lambda zh'(z)}{\mu(p+\alpha+\beta-1)}\right\}\in P_k(\rho),\ z\in E.$$

Using Lemma 2.2, we note that $h_i(z) \in P(\sigma)$, where

$$\sigma = \rho + (1-\rho)(2\sigma_1 - 1), \quad \sigma_1 = \int_0^1 (1 + t^{\Re \frac{\lambda}{\mu(p+\alpha+\beta-1)}})dt, \tag{14}$$

and consequently $h(z) \in P_k(\sigma)$ and this gives the required result. □

We note that σ_1 given by (14) can be expressed in terms of hypergeometric function as

$$\begin{aligned} \sigma_1 &= \int_0^1 (1 + t^{\Re \frac{\lambda}{\mu(p+\alpha+\beta-1)}})dt \\ &= \frac{\mu(p+\alpha+\beta-1)}{\lambda_1} \int_0^1 u^{\frac{\mu(p+\alpha+\beta-1)}{\lambda_1}-1}(1+u)^{-1}du, \quad (\lambda_1 = \Re\lambda > 0) \\ &= {}_2F_1(1, \frac{\mu(p+\alpha+\beta-1)}{\lambda_1}; 1 + \frac{\mu(p+\alpha+\beta-1)}{\lambda_1}; -1) \\ &= {}_2F_1(1,1;1+\frac{\mu(p+\alpha+\beta-1)}{\lambda_1}; \frac{1}{2}). \end{aligned}$$

Consider the operator defined by

$$F_c = \left(\frac{p\mu + c}{z^c} \int_0^z t^{c-1} \left(f(t) \right)^{\mu} dt \right)^{\frac{1}{\mu}} \quad z \in E. \tag{15}$$

It is clear that the function $F_c \in \mathcal{A}(p)$ and

$$z^c (Q^{\alpha}_{\beta,p} f(z))^{\mu} = (p\mu + c) \int_0^z t^{c-1} \left(Q^{\alpha}_{\beta,P} f(t) \right)^{\mu} dt, \ \ z \in E. \tag{16}$$

Theorem 3.4. *Let $\lambda > 0$, $\mu > 0$ and $c > -p\mu$. If $f \in \mathcal{A}(p)$ satisfies the following condition:*

$$\left\{ (1-\lambda) \left(\frac{Q^{\alpha}_{\beta,p} f(z)}{z^p} \right)^{\mu} + \lambda \frac{(Q^{\alpha}_{\beta,p} f(z))'}{pz^{p-1}} \left(\frac{Q^{\alpha}_{\beta,p} f(z)}{z^p} \right)^{\mu-1} \right\} \in P_k(\rho),$$

for $\mu > 0$ $(z \in E)$, (17)

then the function defined by

$$\left\{(1-\lambda)\left(\frac{Q^{\alpha}_{\beta,p}\,F_c(z)}{z^p}\right)^{\mu}+\lambda\frac{(Q^{\alpha}_{\beta,p}\,F_c(z))'}{pz^{p-1}}\left(\frac{Q^{\alpha}_{\beta,p}\,F_c(z)}{z^p}\right)^{\mu-1}\right\}\in P_k(\alpha_1), \tag{18}$$

where

$$\alpha_1=\rho+(1-\rho)(2\sigma_2-1) \text{ with } \sigma_2=\int_0^1(1+t^{\Re\frac{1}{p\mu+c}})dt.$$

The value of α_1 is best possible and cannot be improved.

Proof. It is clear that $F_c\in\mathcal{A}(p)$ and differentiating both sides of (16), we obtain

$$(p\mu+c)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu}=c\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu}+\mu\frac{(Q^{\alpha}_{\beta,p}F_c(z))'}{pz^{p-1}}\left(\frac{Q^{\alpha}_{\beta,p}F_c(z)}{z^p}\right)^{\mu-1}. \tag{19}$$

Letting

$$G(z)=\left\{(1-\lambda)\left(\frac{Q^{\alpha}_{\beta,p}F_c(z)}{z^p}\right)^{\mu}+\lambda\frac{(Q^{\alpha}_{\beta,p}F_c(z))'}{pz^{p-1}}\left(\frac{Q^{\alpha}_{\beta,p}F_c(z)}{z^p}\right)^{\mu-1}\right\}, z\in E, \tag{20}$$

where

$$G(z)=(\frac{k}{4}+\frac{1}{2})g_1(z)-(\frac{k}{4}-\frac{1}{2})g_2(z).$$

Then $G(z)$ is analytic in E with $G(0)=1$. Again differentiating (20) and using (19) in the resulting equation, we have

$$\left\{(1-\lambda)\left(\frac{Q^{\alpha}_{\beta,p}\,f(z)}{z^p}\right)^{\mu}+\lambda\frac{(Q^{\alpha}_{\beta,p}\,f(z))'}{pz^{p-1}}\left(\frac{Q^{\alpha}_{\beta,p}\,f(z)}{z^p}\right)^{\mu-1}\right\}$$
$$=\left\{G(z)+\frac{zG'(z)}{(p\mu+c)}\right\}\in P_k(\rho)\quad z\in E.$$

Using Lemma 2.2, we note that $g_i(z)\in P(\alpha_1)$, where

$$\alpha_1=\rho+(1-\rho)(2\sigma_2-1),\quad \sigma_2=\int_0^1(1+t^{\Re\frac{1}{p\mu+c}})dt, \tag{21}$$

and consequently $G(z)\in P_k(\alpha_1)$ and this gives the required result. □

In term of hypergeometric function σ_2 can be written as

$$\sigma_2 = {}_2F_1(1,1;p\mu+c+1;\frac{1}{2}).$$

Theorem 3.5. *For* $0 \leq \lambda_2 < \lambda_1$,

$$\mathcal{T}^{\alpha}_{\beta,p,k}(\lambda_1,\mu,\rho) \subset \mathcal{T}^{\alpha}_{\beta,p,k}(\lambda_2,\mu,\rho).$$

Proof. If $\lambda_2 = 0$, then the proof is immediate from Theorem 3.1. Let $\lambda_2 > 0$ and $f \in \mathcal{T}^{\alpha}_{\beta,p,k}(\lambda_1,\mu,\rho)$. Then there exist two functions H_1, $H_2 \in P_k(\rho)$ such that

$$(1-\lambda)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}g(z)}\right)^{\mu} + \lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha-1}_{\beta,p}g(z)}\right)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}g(z)}\right)^{\mu-1} = H_1(z),$$

and

$$\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}g(z)}\right)^{\mu} = H_2(z).$$

Then

$$(1-\lambda_2)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}g(z)}\right)^{\mu} + \lambda_2\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha-1}_{\beta,p}g(z)}\right)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}g(z)}\right)^{\mu-1}$$
$$= \frac{\lambda_2}{\lambda_1}H_1(z) + (1-\frac{\lambda_2}{\lambda_1})H_2(z), \quad (22)$$

and since $P_k(\rho)$ is a convex set, see [5] it follows that the right hand side of (22) belongs to $P_k(\rho)$ and this completes the proof. □

We next take the converse case of Theorem 3.1 as follows:

Theorem 3.6. *Let* $\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}g(z)}\right)^{\mu} \in P_k(\rho)$ *with* $\left(\frac{Q^{\alpha-1}_{\beta,p}g(z)}{Q^{\alpha}_{\beta,p}g(z)}\right) \in P(\eta)$, *for* $z \in E$. *Then* $f \in \mathcal{T}^{\alpha}_{\beta,p,k}(\lambda,\mu,\rho)$ *for* $|z| < r$, *where* r *is given by*

$$r = \mu(p+\alpha+\beta-1)\Big/\Big(\{(1-\eta)\mu(p+\alpha+\beta-1)+|\lambda|\}$$
$$+\sqrt{\eta\mu(p+\alpha+\beta-1)^2+|\lambda|^2+2|\lambda|(1-\eta)\mu(p+\alpha+\beta-1).}\Big) \quad (23)$$

Proof. Let

$$\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}g(z)}\right)^{\mu} = H, \quad \left(\frac{Q^{\alpha-1}_{\beta,p}g(z)}{Q^{\alpha}_{\beta,p}g(z)}\right) = H_0,$$

then $H \in P_k(\rho)$, $H_0 \in P(\eta)$.

Proceeding as in Theorem 3.1, for $\alpha \geq 0$, $\beta > -1$, $\mu > 0$, $k \geq 2$, $\lambda \in \mathbb{C}\backslash\{0\}$, $0 \leq \rho$, $\eta < 1$, and

$$H = (1-\rho)h + \rho, \quad H_0 = (1-\eta)h_0 + \eta, \quad \text{with } h \in P_k,\ h_0 \in P,$$

we have

$$\begin{aligned}
&\frac{1}{1-\rho}\left\{(1-\lambda)\left(\frac{Q^{\alpha}_{\beta,p} f(z)}{Q^{\alpha}_{\beta,p}\, g(z)}\right)^{\mu} + \lambda\left(\frac{Q^{\alpha-1}_{\beta,p}\ f(z)}{Q^{\alpha-1}_{\beta,p}\ g(z)}\right)\left(\frac{Q^{\alpha}_{\beta,p} f(z)}{Q^{\alpha}_{\beta,p}\, g(z)}\right)^{\mu-1} - \rho\right\}\\
&= \left\{h(z) + \frac{\lambda}{\mu(p+\alpha+\beta-1)}\frac{zh'(z)}{(1-\eta)h_0(z)+\eta}\right\}\\
&= \left(\frac{k}{4}+\frac{1}{2}\right)\left[h_1(z) + \frac{\lambda}{\mu(p+\alpha+\beta-1)}\frac{zh_1'(z)}{\{(1-\eta)h_0(z)+\eta\}}\right]\\
&- \left(\frac{k}{4}-\frac{1}{2}\right)\left[h_2(z) + \frac{\lambda}{\mu(p+\alpha+\beta-1)}\frac{zh_2'(z)}{\{(1-\eta)h_0(z)+\eta\}}\right].
\end{aligned}$$

Using well known estimates, see [2], for $h_i \in P$,

$$\left|zh_i'(z)\right| \leq \frac{2r\Re h_i(z)}{1-r^2}, \qquad \frac{1-r}{1+r} \leq |h_i(z)| \leq \frac{1+r}{1-r},$$

we have

$$\begin{aligned}
&\Re\left[h_i(z) + \frac{\lambda}{\mu(p+\alpha+\beta-1)}\frac{zh_i'(z)}{\{(1-\eta)h_0(z)+\eta\}}\right]\\
&\geq \Re h_i(z)\left[1 - \frac{2|\lambda|r}{\mu(p+\alpha+\beta-1)}\frac{1}{1-r^2}\left(\frac{1+r}{(1-(1-2\eta)r)}\right)\right]\\
&\geq \Re h_i(z)\left[1 - \frac{2|\lambda|r}{\mu(p+\alpha+\beta-1)}\frac{1}{1-r}\left(\frac{1+r}{(1-(1-2\eta)r)}\right)\right]\\
&\geq \Re h_i(z)\left[\frac{\mu(p+\alpha+\beta-1)[(1-r-(1-2\eta)r+(1-2\eta)r^2]-2|\lambda|r}{\mu(p+\alpha+\beta-1)(1-r)\{1-(1-(1-2\eta)r\}}\right]\\
&\geq \Re h_i(z)\left[\frac{\mu(p+\alpha+\beta-1)(1-2\eta)r^2-2[(1-\eta)\mu(p+\alpha+\beta-1)}{\mu(p+\alpha+\beta-1)(1-r)\{1-(1-(1-2\eta)r\}}\right.\\
&\times\left.\frac{+|\lambda|]r+\mu(p+\alpha+\beta-1)}{\mu(p+\alpha+\beta-1)(1-r)\{1-(1-(1-2\eta)r\}}\right]. \qquad (24)
\end{aligned}$$

Right hand side of (24) is positive for $|z| < r$, where r is given by (23). □

We note that , for $p = 1 = \mu$, $\alpha = 0$, $\beta > -1$, $\eta = 0$ and $\lambda = 1$, $\left(\frac{f}{g}\right) \in P_k(\rho)$, for $z \in E$ implies $\left(\frac{f'}{g'}\right) \in P_k(\rho)$ for $|z| < R = \frac{1}{2+\sqrt{3}}$.

Acknowledgments

The author is grateful to the Editorial Board in typesetting this paper. The author would also like to express gratitude to Olusola Amodu, Kwara State Nigeria, for her support and help.

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