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MAPPING PROPERTIES OF A GENERAL INTEGRAL OPERATOR DEFINED BY THE HADAMARD PRODUCT

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In the present paper, we introduce a general integral operator defined by Hadamard product and study mapping properties on some subclasses of analytic univalent functions. Relevant connections of the results presented here with various known results are briefly indicated.

1. Introduction

Let A denote the class of the functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$, and satisfy the normalisation condition $f(0) = f'(0) - 1 = 0$.

Further, we denote by S the subclass of A consisting of functions of the form (1) which are also univalent in U.

For $\beta > 1$ and $z \in U$, let

$$M(\beta) = \left\{ f \in A : \mathrm{Re}\frac{zf'(z)}{f(z)} < \beta \right\}$$

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and

$$N(\beta)=\left\{f\in A:\operatorname{Re}\{1+\frac{zf''(z)}{f'(z)}\}<\beta\right\}.$$

These classes were extensively studied by Uralegaddi et al. in [16], (see also Owa and Srivastava [9], Porwal and Dixit [14]).

Very recently Dixit and Chandra [4] generalizes these classes by introducing a new subclass $S_k^n(\beta)$ of analytic functions in the unit disk satisfying the condition

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\}<\beta,\quad n\in N_0,\ z\in U,1<\beta\leq 4/3, \tag{2}$$

where D^n stands for the Salagean-derivative introduced by Salagean in [15] and $f(z)$ of the form

$$f(z)=z+\sum_{j=k+1}^{\infty}a_j z^j. \tag{3}$$

It can be easily seen that $S_1^0(\beta)=M(\beta),\quad S_1^1(\beta)=N(\beta)$.

Further, we denote by $S_1^n(\beta)\equiv S^n(\beta)$.

Definition 1.1 (Hadamard product or convolutions)**.** Given two functions f and g in the class A, where f is given by (1) and g is given by

$$g(z)=z+\sum_{n=2}^{\infty}b_n z^n,$$

the Hadamard product (or convolution) $(f*g)$ is defined by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_n b_n z^n,\quad (z\in U). \tag{4}$$

Now, we denote the class $S(n,g,\beta)$ for which $f*g\in S^n(\beta)$.

Breaz [2] studied the mapping properties of the two integral operators on the classes $M(\beta)$ and $N(\beta)$. Recently, these results were generalized by Porwal [12].

In the present paper, we generalized and unified these results by introducing an interesting integral operator as follows

$$F_{m,n,\alpha}(z)=$$
$$\int_0^z\left(\frac{D^m(f_1*g_1)(t)}{t}\right)^{\alpha_1}\left(\frac{D^m(f_2*g_2)(t)}{t}\right)^{\alpha_2}\cdots\left(\frac{D^m(f_n*g_n)(t)}{t}\right)^{\alpha_n}dt \tag{5}$$

where $f_i(z)\in A$, $\alpha_i>0$, $\forall i\in\{1,2,\ldots n\}$ and $m\in N_0$.

Remark 1.2. For $g_i(z) = z + \sum_{n=2}^{\infty} z^n, f_i(z) \in A$, $\alpha_i > 0, \forall i \in \{1, 2, \ldots n\}$ and $m \in N_0$ the above operator becomes

$$F_{m,n,\alpha}(z) = \int_0^z \left(\frac{D^m f_1(t)}{t}\right)^{\alpha_1} \left(\frac{D^m f_2(t)}{t}\right)^{\alpha_2} \ldots \left(\frac{D^m f_n(t)}{t}\right)^{\alpha_n} dt \qquad (6)$$

studied by Porwal [12].

Remark 1.3. For $m = 0, n = 1, \alpha_1 = 1, \alpha_2 = \ldots = \alpha_n = 0$, $g_i(z) = z + \sum_{n=2}^{\infty} z^n$ and $f(z) \in A$, we obtain Alexander integral operator introduced in 1915 in [1]

$$I(z) = \int_0^z \frac{f(t)}{t} dt, \quad z \in U.$$

Remark 1.4. For $m = 0, n = 1, \alpha_1 = \alpha, \alpha_2 = \ldots = \alpha_n = 0$, $g_i(z) = z + \sum_{n=2}^{\infty} z^n$ and $f(z) \in A$, we obtain the integral operator

$$I_\alpha(z) = \int_0^z \left[\frac{f(t)}{t}\right]^\alpha dt, \quad z \in U,$$

studied in [8], (see also ([5], [7], [13]).

Remark 1.5. For $m = 1, n = 1, \alpha_1 = 1, \alpha_2 = \ldots = \alpha_n = 0$, $g_i(z) = z + \sum_{n=2}^{\infty} z^n$ and $f(z) \in A$, we obtain the integral operator

$$I(z) = \int_0^z f'(t) dt$$

studied by various authors in ([6], [11]).

Remark 1.6. For $m = 1, n = 1, \alpha_1 = \alpha, \alpha_2 = \ldots = \alpha_n = 0$, $g_i(z) = z + \sum_{n=2}^{\infty} z^n$ and $f(z) \in A$, we obtain the integral operator

$$I_\alpha(z) = \int_0^z \left[f'(t)\right]^\alpha dt, \quad z \in U$$

studied in [10].

Remark 1.7. For $m = 0, \alpha_i > 0, i \in \{1, 2, \ldots, n\}$, and $g_i(z) = z + \sum_{n=2}^{\infty} z^n$ we obtain the integral operator

$$I_n(z) = \int_0^z \left[\frac{f_1(t)}{t}\right]^{\alpha_1} \ldots \left[\frac{f_n(t)}{t}\right]^{\alpha_n} dt$$

studied in ([2], [3]).

Remark 1.8. For $m=1, \alpha_i>0,\ \forall i \in \{1,2,\ldots,n\}$ and $g_i(z)=z+\sum_{n=2}^{\infty} z^n$, we obtain the integral operator

$$I_{\alpha_1,\alpha_2,\ldots,\alpha_n}(z)=\int_0^z \left[f_1'(t)\right]^{\alpha_1}\ldots\left[f_n'(t)\right]^{\alpha_n} dt$$

studied in [2].

2. Main Results

We study the condition for the integral operator defined in (5) which map $S^m(\alpha_1) \times S^m(\alpha_2)\times\cdots\times S^m(\alpha_n)$ into $N(\mu)$.

Theorem 2.1. *Let $f_i \in S(m,\beta_i,g_i)$ where $g_i \in A$ for each $i=1,2,\ldots,n$ with $\beta_i>1, m\in N_0$. Then $F_{m,n,\alpha}(z)\in N(\mu)$ where*

$$\mu=1+\sum_{i=1}^{n}\alpha_i(\beta_i-1) \quad \textit{and} \quad \alpha_i>0.$$

Proof. Let

$$F_{m,n,\alpha}(z)=\int_0^z \left(\frac{D^m(f_1*g_1)(t)}{t}\right)^{\alpha_1}\left(\frac{D^m(f_2*g_2)(t)}{t}\right)^{\alpha_2}\ldots\left(\frac{D^m(f_n*g_n)(t)}{t}\right)^{\alpha_n} dt.$$

Differentiating it, we have

$$F'_{m,n,\alpha}(z)=\left(\frac{D^m(f_1*g_1)(t)}{t}\right)^{\alpha_1}\left(\frac{D^m(f_2*g_2)(t)}{t}\right)^{\alpha_2}\cdots\left(\frac{D^m(f_n*g_n)(t)}{t}\right)^{\alpha_n} \tag{7}$$

this equality implies that

$$F'_{m,n,\alpha}(z)=\alpha_1 ln\frac{D^m(f_1*g_1)(t)}{t}+\alpha_2 ln\frac{D^m(f_2*g_2)(t)}{t}+\ldots+\alpha_n ln\frac{D^m(f_n*g_n)(t)}{t}.$$

By differentiating the above equality, we get

$$\frac{zF''_{m,n,\alpha}(z)}{F'_{m,n,\alpha}(z)}=\alpha_1\left[\frac{(D^{m+1}(f_1*g_1)(z))}{D^m(f_1*g_1)(z)}-\frac{1}{z}\right]+\ldots+\alpha_n\left[\frac{(D^{m+1}(f_n*g_n)(z))}{D^m(f_n*g_n)(z)}-\frac{1}{z}\right]$$

We obtain from this equality that

$$\frac{zF''_{m,n,\alpha}(z)}{F'_{m,n,\alpha}(z)}=\sum_{i=1}^{n}\alpha_i\left[\frac{(zD^{m+1}(f_i*g_i)(z))}{D^m(f_i*g_i)(z)}-1\right]$$

or, equivalently

$$\text{Re}\left\{1+\frac{zF''_{m,n,\alpha}(z)}{F'_{m,n,\alpha}(z)}\right\}=\sum_{i=1}^{n}\alpha_i\text{Re}\left\{\frac{(zD^{m+1}(f_i*g_i)(z))}{D^m(f_i*g_i)(z)}\right\}-\sum_{i=1}^{n}\alpha_i+1$$

$$<\sum_{i=1}^{n}\alpha_i\beta_i-\sum_{i=1}^{n}\alpha_i+1=\sum_{i=1}^{n}\alpha_i(\beta_i-1)+1.$$

Now, since $\sum_{i=1}^{n} \alpha_i(\beta_i - 1) > 0$, we obtain $F_{m,n}(z) \in N(\mu)$, where $\mu = 1 + \sum_{i=1}^{n} \alpha_1(\beta_1 - 1)$. $\square$

If we put $m = 0$ and $g_i(z) = \frac{z}{1-z}$ in Theorem 2.1, we obtain the following result obtained by Porwal in [12].

Corollary 2.2. *Let $f_i \in S^m(\beta_i)$ for each $i = 1, 2, \ldots, n$ with $\beta_i > 1, m \in N_0$. Then $F_{m,n,\alpha}(z) \in N(\mu)$ where*

$$\mu = 1 + \sum_{i=1}^{n} \alpha_i(\beta_i - 1) \quad \textit{and} \quad \alpha_i > 0.$$

If we put $m = 0$ and $g_i(z) = \frac{z}{1-z}$ in Theorem 2.1, we obtain the following result obtained by Breaz in [2].

Corollary 2.3. *Let $f_i \in M(\beta_i)$ with $\beta_i > 1$, for each $i = 1, 2, \ldots, n$. Then $I_n(z) \in N(\mu)$, where $\mu = 1 + \sum_{i=1}^{n} \alpha_i(\beta_i - 1)$ and $\alpha_i > 0$, $(\forall i = 1, 2, \ldots, n)$.*

If we put $m = 1$ and $g_i(z) = \frac{z}{1-z}$ in Theorem 2.1 we obtain the following result obtained by Breaz in [2].

Corollary 2.4. *Let $f_i \in M(\beta_i)$,and $g_i(z) = \frac{z}{1-z}$ for each $i = 1, 2, \ldots, n$ with $\beta_i > 1$. Then $I_{\alpha_1,\ldots,\alpha_n}(z) \in N(\mu)$ with $\mu = 1 + \sum_{i=1}^{n} \alpha_i(\beta_i - 1)$ and $\alpha_i > 0$, $(\forall i = 1, 2, \ldots, n)$.*

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