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THERMO-ELASTIC PLANE DEFORMATIONS IN DOUBLY-CONNECTED DOMAINS WITH TEMPERATURE AND PRESSURE WHICH DEPEND ON THE THERMAL CONDUCTIVITY

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We propose a new weak formulation for the plane problem of thermoelastic theory in multiply-connected domains. This permits to avoid the difficulties connected with the Cesaro-Volterra boundary conditions in the related elliptic boundary-value problem. In the second part we consider a nonlinear version of the problem assuming that the thermal conductivity depends not only on the temperature but also on the pressure. Recent studies reveal that this situation can occur in practice. A theorem of existence and uniqueness is proved for this problem.

1. Introduction

The thermal conductivity in elastic bodies is usually taken as dependent on the temperature but not on the pressure [1]. This, however, is not always the case. The accurate measurements made by Sundqvist and Bäckstöm [13] and by Gerlich [5] show a quite significant dependence of the thermal conductivity from the pressure in aluminum and in insulators. In this paper we reconsider the problem of plane doubly-connected thermo-elasticity taking into consideration this

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effect. Whereas the classical boundary value problem of plane thermo-elasticity is relatively simple since it is uncoupled [1], in the present case we have to deal with a nonlinear coupled system of partial differential equations. We consider a long cylinder of cross section Ω with a cylindrical hole of cross section Ω_1, $\Omega_1 \subset \Omega$ and define $\Omega^* = \Omega \setminus \Omega_1$.
We denote with Γ_1 the boundary of Ω and with Γ_2 the boundary of Ω_1, $\Gamma_1 \cap \Gamma_2 = \emptyset$. We assume the plane deformation theory [6], [1]. This permits the introduction of the Airy's stress function $\varphi(x,y)$ which gives the non-vanishing components of the stress tensor in the form

$$\sigma_{xx} = \frac{\partial^2 \varphi}{\partial y^2}, \ \tau_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y}, \ \sigma_{yy} = \frac{\partial^2 \varphi}{\partial x^2}. \tag{1}$$

If u is the temperature we have

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) - \alpha E u \tag{2}$$

and by (1)

$$\sigma_{zz} = \nu \Delta \varphi - \alpha E u$$

where α is the coefficient of linear thermal expansion, E the modulus of Young and ν the Poisson's ratio. For the mean pressure $p = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$ we have, in view of (1) and (2),

$$p = \frac{1}{3}(1+\nu)\Delta\varphi - \frac{1}{3}\alpha E u.$$

We assume that the inner and the outer surfaces are free from loads and kept at a constant temperature $u = 0$. Moreover, an internal heat source of density $f(x,y)$ acts in the body. For shorthand we put $k = \frac{1-\nu}{\alpha E}$, k is a positive constant in view of the thermodynamical restrictions on ν. For the determination of $\varphi(x,y)$ and $u(x,y)$ under steady condition of operation, we have the following boundary problem

$$\Delta^2 \varphi = -k \Delta u \ \text{ in } \Omega^* \tag{3}$$

$$\nabla \cdot (\kappa \nabla u) = f \ \text{ in } \Omega^* \tag{4}$$

$$\varphi = 0, \ \frac{\partial \varphi}{\partial n} = 0 \ \text{ on } \Gamma_1 \tag{5}$$

$$u = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \tag{6}$$

$$\int_{\Gamma_2} \frac{\partial \Delta\varphi}{\partial n} ds = -k \int_{\Gamma_2} \frac{\partial u}{\partial n} ds \tag{7}$$

$$\int_{\Gamma_2} (y\frac{\partial \Delta\varphi}{\partial n} - x\frac{\partial \Delta\varphi}{\partial s}) ds = -k \int_{\Gamma_2} (y\frac{\partial u}{\partial n} - x\frac{\partial u}{\partial s}) ds \tag{8}$$

$$\int_{\Gamma_2} (y\frac{\partial \Delta\varphi}{\partial s} + x\frac{\partial \Delta\varphi}{\partial n}) ds = -k \int_{\Gamma_2} (y\frac{\partial u}{\partial s} + x\frac{\partial u}{\partial n}) ds. \tag{9}$$

(3) is the only non-vanishing part of the Lamé compatibility equations, (4) is the energy equation, (5) reflects the fact that Γ_1 is free of loads. Finally (7), (8) and (9) are the Cesaro-Volterra equations [14], [2] expressing the fact that the local rotation and the components of the displacement are single-valued (we exclude the presence of Volterra dislocations). If we take into account (6) the conditions (8) and (9) can be restated in the following way

$$\int_{\Gamma_2} (y\frac{\partial \Delta\varphi}{\partial n} - \Delta\varphi\frac{\partial y}{\partial n}) ds = -k \int_{\Gamma_2} y\frac{\partial u}{\partial n} ds \tag{10}$$

$$\int_{\Gamma_2} (x\frac{\partial \Delta\varphi}{\partial n} - \Delta\varphi\frac{\partial x}{\partial n}) ds = -k \int_{\Gamma_2} x\frac{\partial u}{\partial n} ds. \tag{11}$$

For the boundary value problem (3), (4), (5), (6), (10) and (11) we consider four cases: (i) $\kappa = \kappa_0$ is a positive constant, (ii) the thermal conductivity is a given function of the temperature, $\kappa = \kappa(u)$, (iii) the thermal conductivity is a given function of the temperature and of the space variable (x,y), (iv) the thermal conductivity is a given function of the space variable (x,y), of the temperature and of the pressure: $\kappa = \kappa(x,y,u,p)$. This is the situation quoted in the introduction. Boley and Weiner [1] prove a theorem of uniqueness for the case (iii) when Ω and Ω_1 are concentric circle. The case without thermal effects is treated in [12], [10] and [11]. The study of case (iv) seems to be new. In Section 2 the difficulty represented by the integral boundary conditions (7), (10) and (11) is dealt with a weak formulation valid for all the four cases and in which the integral conditions disappear. When the thermal effects are neglected a similar weak formulation is used in [4]. For regular solutions we prove in Section 2 the equivalence of the classical and weak formulation. In Section 3 we prove the existence and uniqueness of solutions for the first three cases and a theorem of existence for the fourth case.

2. The weak formulation and the equivalence with the classical formulation

Let us define the space $E(\Omega,\Omega_1)=E$

$$E=\{\varphi\in H^2(\Omega),\ \varphi=0 \text{ on } \Gamma_1,\ \frac{\partial\varphi}{\partial n}=0 \text{ on } \Gamma_1,$$
$$\varphi=ax+by+c \text{ in } \Omega_1,\ a,b,c\in\mathbf{R}^1\}.$$

Let $\mathcal{Q}:H_0^1(\Omega^*)\to H_0^1(\Omega)$, $\mathcal{U}=\mathcal{Q}u$ be the linear and bounded operator which extends the functions of $H_0^1(\Omega^*)$ to functions of $H_0^1(\Omega)$ setting them equal to zero in Ω_1. Our weak formulation of problem (3), (4), (5), (6), (7), (10) and (11) is: to find $\varphi(x,y)$ and $u(x,y)$ such that

$$\varphi\in E,\ \int_\Omega \Delta\varphi\Delta w\, dxdy=k\int_\Omega \nabla\mathcal{U}\cdot\nabla w\, dxdy, \text{ for all } w\in E \tag{12}$$

$$u\in H_0^1(\Omega^*),\ \int_{\Omega^*}\kappa\nabla u\cdot\nabla v\, dxdy=<f,v>, \text{ for all } v\in H_0^1(\Omega^*), \tag{13}$$

where $f\in H^{-1}(\Omega)$, $\mathcal{U}=\mathcal{Q}u$ and the thermal conductivity κ corresponds to any of the cases (i), (ii), (iii) or (iv). The next Lemma legitimizes this weak formulation.

Lemma 2.1. *Let (φ,u), $\varphi\in C^3(\bar{\Omega}^*)\cap C^4(\Omega^*)$, $u\in C^0(\bar{\Omega}^*)\cap C^2(\Omega^*)$ be a regular solution of problem (12), (13). Then (φ,u) is a solution of problem (3), (4), (5), (6), (7), (10) and (11).*

Proof. Let $\eta\in C_0^\infty(\Omega^*)$ and extend η with zero in Ω_1. Choosing $w=\eta\in E$ in (12) we find

$$\int_{\Omega^*}\eta\Delta^2\varphi\, dxdy=k\int_{\Omega^*}\eta\Delta u\, dxdy.$$

Since η is arbitrary we obtain (3). In a similar way we obtain (4). To prove (7) we take as test function in (12) $w\in E$ with $w=1$ in Ω_1. Thus $w=1$ and $\frac{\partial w}{\partial n}=0$ on Γ_2, and

$$\int_{\Omega^*}(\Delta\varphi\Delta w-w\Delta^2\varphi)dxdy=\int_{\Gamma_2}\frac{\partial\Delta\varphi}{\partial n}ds \tag{14}$$

by (3)

$$\int_{\Omega^*}(\Delta\varphi\Delta w+kw\Delta u)dxdy=-\int_{\Gamma_2}\frac{\partial\Delta\varphi}{\partial n}ds. \tag{15}$$

From (12), since $w = 1$ in Ω_1, we have

$$\int_{\Omega^*} \Delta\varphi\Delta w \, dxdy = k\int_{\Omega^*} \nabla u \cdot \nabla w \, dxdy. \tag{16}$$

Substituting (16) in (15) we have

$$k\int_{\Omega^*} (w\Delta u + \nabla u \cdot \nabla w) dxdy = -\int_{\Gamma_2} \frac{\partial \Delta\varphi}{\partial n} ds, \tag{17}$$

but

$$k\int_{\Omega^*} (w\Delta u + \nabla u \cdot \nabla w) dxdy = k\int_{\Gamma_2} \frac{\partial u}{\partial n} ds. \tag{18}$$

and (7) follows. To prove (11) we take in (12) $w = x$ in Ω_1, (16) still holds, but instead of (14) we now have

$$\int_{\Omega^*} (\Delta\varphi\Delta w - w\Delta^2\varphi) dxdy = \int_{\Gamma_2} \big(\Delta\varphi \frac{\partial x}{\partial n} - x\frac{\partial \Delta\varphi}{\partial n}\big) ds. \tag{19}$$

Plugging (16) and (18) in (19) we obtain

$$k\int_{\Omega^*} (w\Delta u + \nabla u \cdot \nabla w) dxdy = \int_{\Gamma_2} \big(\Delta\varphi \frac{\partial x}{\partial n} - x\frac{\partial \Delta\varphi}{\partial n}\big) ds. \tag{20}$$

On the other hand,

$$\int_{\Omega^*} \nabla u \cdot \nabla w \, dxdy = -\int_{\Omega^*} w\Delta u \, dxdy + \int_{\Gamma_2} x\frac{\partial u}{\partial n} ds. \tag{21}$$

Plugging (21) in (20) we obtain (11). With similar calculations, choosing as test function in (12) $w \in E$ such that $w = y$ in Ω_1, we can prove (10). □

3. Existence and uniqueness of weak and classical solutions

When the thermal conductivity is constant: $\kappa = \kappa_0 > 0$ the system (12), (13) is uncoupled and becomes

$$u \in H_0^1(\Omega^*), \ \int_{\Omega^*} \kappa_0 \nabla u \cdot \nabla v \, dxdy = < f, v > \ \text{for all } v \in H_0^1(\Omega^*) \tag{22}$$

$$\varphi \in E, \ \int_{\Omega} \Delta\varphi\Delta w \, dxdy = k\int_{\Omega} \nabla \mathcal{U} \cdot \nabla w \, dxdy, \ \text{for all } w \in E \tag{23}$$

(22) is immediately solvable using the Lax-Milgram lemma. On the other hand, the bilinear form

$$a(\varphi, w) = \int_\Omega \Delta\varphi \Delta w \, dxdy, \ \varphi \in E, \ w \in E$$

is coercive and continuous on $E \times E$ which is a linear subspace of $H_0^2(\Omega) \times H_0^2(\Omega)$. Thus, using again the Lax-Milgram lemma to solve (23), we obtain the unique solution (φ, u) of (22), (23). If f, Γ_1 and Γ_2 are regular (φ, u) is also regular and by Lemma 2.1 it is a classical solution of problem (3), (4), (5), (6), (10) and (11). When the thermal conductivity depends only on the temperature: $\kappa = \kappa(u)$ and

$$\kappa(u) \geq \bar{\kappa} > 0 \text{ for all } u \in \mathbf{R}^1 \tag{24}$$

we can reduce problem (12), (13) to the case $\kappa = \kappa_0$, a constant, with the aid of the Kirchhoff's transformation $\tau = \mathcal{F}(u)$

$$\mathcal{F}(u) = \int_0^u \kappa(t) dt \tag{25}$$

which is invertible in view of (24). If the body is not homogeneous the thermal conductivity depends on both the temperature and the space variable i.e. $\kappa = \kappa(x, y, u)$. In this case we may still prove existence and uniqueness for problem (12), (13) using the following theorem of M. Chipot [3] (page 99) and then proceeding as in the previous case.

Theorem 3.1. *Let* $A(\mathbf{x}, u) : \Omega \times \mathbf{R}^1 \to \mathbf{R}^1$, $\mathbf{x} \in \Omega$ *satisfy*

$$\textit{for all } u \in \mathbf{R}^1, \ \mathbf{x} \to A(\mathbf{x}, u) \textit{ is measurable} \tag{26}$$

$$M \geq A(\mathbf{x}, u) \geq m > 0, \ \textit{a.e. } \mathbf{x} \in \Omega, \textit{ for all } u \in \mathbf{R}^1 \tag{27}$$

$$u \to A(\mathbf{x}, u) \textit{ is Lipschitz continuous in } \mathbf{R}^1, \tag{28}$$

then the problem

$$\nabla \cdot (A(\mathbf{x}, u) \nabla u) = f \in H^{-1}(\Omega) \tag{29}$$

has a unique solution.

Hereafter we discuss the case $\kappa = \kappa(x, y, u, p)$ where $p = (1/3)(1 + \nu)\Delta\varphi - (1/3)\alpha E u$. As weak formulation of problem (12), (13) we have

$$\varphi \in E, \ \int_\Omega \Delta\varphi \Delta w \, dxdy = k \int_\Omega \nabla\mathcal{U} \cdot \nabla w \, dxdy, \text{ for all } w \in E \tag{30}$$

$$\mathcal{U} = \mathcal{Q}u$$

$$u \in H_0^1(\Omega^*),\ \int_{\Omega^*} \kappa(x,y,\frac{1}{3}((1+\nu)\Delta\varphi - \alpha E u)\nabla u \cdot \nabla v\ dxdy =< f, v > \quad (31)$$

for all $v \in H_0^1(\Omega^*)$. We assume

$$|\kappa(x,y,u,p) - \kappa(x,y,u',p')| \leq L_1(|u-u'| + |p-p'|). \quad (32)$$

Setting $\xi = \Delta\varphi$ and $a(x,y,u,\xi) = \kappa(x,y,\frac{1}{3}((1+\nu)\xi - \alpha E u))$ from (32) we have

$$|a(x,y,u,\xi) - a(x,y,u',\xi')| \leq L_2(|u-u'| + |\xi - \xi'|). \quad (33)$$

Therefore (30), (31) become

$$\varphi \in E,\ \int_\Omega \Delta\varphi \Delta w\ dxdy = k \int_\Omega \nabla\mathcal{U} \cdot \nabla w\ dxdy,\ \text{for all } w \in E \quad (34)$$

$$\mathcal{U} = \mathcal{Q}u$$

$$u \in H_0^1(\Omega^*),\ \int_{\Omega^*} a(x,y,\Delta\varphi)\nabla u \cdot \nabla v\ dxdy =< f, v >,\ \text{for all } v \in H_0^1(\Omega^*). \quad (35)$$

We use the following classical "a priori" estimate [7] to prove that the problem (34), (35) has a solution.

Theorem 3.2. *Let $\mathcal{O}$ be an open and bounded subset of $\mathbf{R}^2$ with a regular boundary $\partial\mathcal{O}$. Let $\zeta(x,y)$ be the weak solution of the problem*

$$\zeta \in H^2(\mathcal{O}),\ \Delta^2\zeta = f \text{ in } \mathcal{O},\ \zeta = g \text{ on } \partial\mathcal{O},\ \frac{\partial\zeta}{\partial n} = h \text{ on } \partial\mathcal{O}. \quad (36)$$

Assume

$$f \in H^{-1}(\mathcal{O}),\ g \in H^1(\mathcal{O}),\ h \in H^2(\mathcal{O}). \quad (37)$$

Then

$$\|\zeta\|_{H^3(\mathcal{O})} \leq C\big(\|f\|_{H^{-1}(\mathcal{O})} + \|g\|_{H^1(\mathcal{O})} + \|h\|_{H^2(\mathcal{O})}\big). \quad (38)$$

Theorem 3.3. *Let $a(x,y,u,\xi)$ satisfy*

$$|a(x,y,u,\xi) - a(x,y,u',\xi')| \leq L_2(|u-u'| + |\xi - \xi'|), \quad (39)$$

and

$$0 < m \leq a(x,y,u,\xi) \leq M \text{ a.e in } \Omega \text{ and for all } (u,\xi) \in \mathbf{R}^2. \quad (40)$$

Then there exists at least one solution to problem (34), (35).

Proof. For applying (38), in view of the considerations which can be seen in the Appendix , we define the space $F = \{\psi;\ \psi = \varphi_{|\Omega^*},\ \varphi \in E\}$ with norm

$$\|\psi\|_F = (\int_{\Omega^*} |\Delta\psi|^2\, dxdy)^{1/2}.$$

Let $\varphi \in F,\ f \in H^{-1}(\Omega^*)$ and let us consider the operator $\mathcal{G} : F \to H_0^1(\Omega^*)$, $u = \mathcal{G}(\varphi)$ defined via the problem

$$u \in H_0^1(\Omega^*),\ \nabla(a(x,y,u,\Delta\varphi)\nabla u) = f. \tag{41}$$

The problem (41) has a unique solution: simply define

$$A(x,y,u) = a(x,y,u,\Delta\varphi)$$

and use Theorem 3.1. Hence $\mathcal{G}$ is well-defined. Let $\mathcal{U} = \mathcal{Q}u$ and

$$\mathcal{H} : H_0^1(\Omega) \to E,\ \varphi = \mathcal{H}(\mathcal{U})$$

be the operator defined by the problem

$$\varphi \in E,\ \int_\Omega \Delta\varphi\Delta w\, dxdy = k\int_\Omega \nabla\mathcal{U}\cdot\nabla w\, dxdy, \text{ for all } w \in E.$$

By the Lax-Milgram lemma $\mathcal{H}$ is well-defined. Taking as test function in (35) $v = u$, using (40) and the Cauchy-Schwartz inequality, we obtain

$$\int_{\Omega^*} |\nabla u|^2\, dxdy \le C_1, \tag{42}$$

where C_1 does not depend on φ. Setting $w = \varphi$ in (34) we have

$$\int_\Omega |\Delta\varphi|^2\, dxdy = k\int_\Omega \nabla\mathcal{U}\cdot\nabla\varphi\, dxdy.$$

Using (42) and the inequality

$$\varphi \in E,\ \int_\Omega |\nabla\varphi|^2\, dxdy \le C_2\int_\Omega |\Delta\varphi|^2\, dxdy, \tag{43}$$

we infer

$$\int_\Omega |\Delta\varphi|^2\, dxdy \le C_3. \tag{44}$$

Thus, by the Sobolev's embedding theorem,

$$\sup_{\bar\Omega} |\varphi| \le C_4. \tag{45}$$

Since $\varphi \in E$ we have $\varphi(x,y) = ax+by+c$ in Ω_1 and by (45) the constants a,b,c depend only on the data. Let ψ be the weak solution of the problem

$$\Delta^2\psi = -k\Delta u \text{ in } \Omega^*, \ \psi = 0 \text{ on } \Gamma_1, \ \frac{\partial\psi}{\partial n} = 0 \text{ on } \Gamma_1 \tag{46}$$

$$\psi = ax+by+c \text{ on } \Gamma_2, \ \frac{\partial\psi}{\partial n} = \frac{\partial}{\partial n}(ax+by+c) \text{ on } \Gamma_2. \tag{47}$$

Since $\Delta u \in H^{-1}(\Omega^*)$, we can apply the a-priori estimate (38) to the problem (46), (47). Thus we have

$$\|\psi\|_{H^3(\Omega^*)} \leq C_5. \tag{48}$$

Let $\mathcal{P} : E \to F$, $\psi = \mathcal{P}\varphi$ be the isometry $\mathcal{P}\varphi = \varphi_{|\Omega^*}$ and consider the operator $\mathcal{S} : F \to F$, $\Phi = \mathcal{S}(\varphi)$ defined by

$$\Phi = \mathcal{S}(\varphi) = \mathcal{P}(\mathcal{H}(\mathcal{Q}(\mathcal{G})))(\varphi).$$

We apply the Schauder fixed point theorem to the operator $\mathcal{S}$. Let

$$B = \left\{\varphi \in F, \int_{\Omega^*} |\Delta\varphi|^2 \, dxdy \leq C_3\right\}.$$

By (44) we have $\mathcal{S}(B) \subseteq B$. Moreover, by (48) $\mathcal{S}$ is compact. It remains to prove that $\mathcal{S}$ is continuous. $\mathcal{Q}$ and $\mathcal{P}$ are clearly continuous. $\mathcal{H}$ is also continuous. For, let $\mathcal{U}_n \to \mathcal{U}$ in $H^1_0(\Omega)$ and $\varphi_n = \mathcal{H}(\mathcal{U}_n)$ i.e.

$$\varphi_n \in E, \int_\Omega \Delta\varphi_n \Delta w \, dxdy = k\int_\Omega \nabla\mathcal{U}_n \cdot \nabla w \, dxdy, \text{ for all } w \in E, \tag{49}$$

and $\varphi = \mathcal{H}(\mathcal{U})$ i.e.

$$\varphi \in E, \int_\Omega \Delta\varphi \Delta w \, dxdy = k\int_\Omega \nabla\mathcal{U} \cdot \nabla w \, dxdy, \text{ for all } w \in E. \tag{50}$$

By difference from (49) and (50), setting $w = \varphi_n - \varphi \in E$ in the resulting equation and recalling (43), we have

$$\int_\Omega |\Delta(\varphi_n - \varphi)|^2 \, dxdy \leq C_2 \int_\Omega |\nabla(\mathcal{U} - \mathcal{U})|^2 \, dxdy.$$

Thus $\mathcal{H}$ is continuous. $\mathcal{G} : F \to H^1_0(\Omega^*)$, $u = \mathcal{G}(\varphi)$ is also continuous. For, let $\varphi_n \to \varphi$ and $\xi_n = \Delta\varphi_n$ and $u_n = \mathcal{G}(\varphi_n)$ i.e.

$$u_n \in H^1_0(\Omega^*), \ \int_{\Omega^*} a(x,y,\xi_n)\nabla u_n \cdot \nabla v \, dxdy = < f, v >, \text{ for all } v \in H^1_0(\Omega^*). \tag{51}$$

Let $\xi = \Delta\varphi$ and $u = \mathcal{G}(\varphi)$ i.e.

$$u \in H_0^1(\Omega^*), \int_{\Omega^*} a(x,y,\xi)\nabla u \cdot \nabla v \, dxdy =< f,v >, \text{ for all } v \in H_0^1(\Omega^*). \quad (52)$$

We claim that $u_n \to u$ in $H_0^1(\Omega^*)$ strongly. Choose $v = u_n$ in (51). We have

$$\int_{\Omega^*} |\nabla u_n|^2 \, dxdy \leq C_6.$$

Hence for a sub-sequence u_{n_k} and for some $u_\infty \in H_0^1(\Omega^*)$ we have

$$u_{n_k} \to u_\infty \text{ weakly in } H_0^1(\Omega^*),\ u_{n_k} \to u_\infty \ \text{ in } L^2(\Omega^*),\ u_{n_k} \to u_\infty \text{ a.e in } \Omega^*.$$

Since $\xi_{n_k} \to \xi$ in $L^2(\Omega^*)$ we have, recalling (39) and (40)

$$a(x,y,u_{n_k},\xi_{n_k}) \to a(x,y,u_\infty,\xi) \text{ in } L^q(\Omega^*) \text{ for all } 1 < q < \infty. \quad (53)$$

Moreover, for all $v \in C_0^\infty(\Omega^*)$ we have

$$a(x,y,u_{n_k},\xi_{n_k})\nabla v \to a(x,y,u_\infty,\xi)\nabla v \text{ in } L^2(\Omega^*).$$

Therefore

$$\int_{\Omega^*} a(x,y,u_\infty,\xi)\nabla u_\infty \cdot \nabla v \, dxdy =< f,v >$$

for all $v \in C_0^\infty(\Omega^*)$ and, by density, also for all $v \in H_0^1(\Omega^*)$. By Theorem 3.1 the solution of problem (52) is unique, thus the entire sequence u_n converges weakly to $u_\infty = u$. It remains to prove that $u_n \to u$ strongly in $H_0^1(\Omega^*)$. Recalling (40) and a result of N. Meyers [8] we have from (52)

$$\int_{\Omega^*} |\nabla u|^p \, dxdy \leq C_7,\ p > 2. \quad (54)$$

Taking as test function $v = u_n - u$ in (51) and in (52), by difference and using (54), we obtain

$$\begin{aligned} m\int_{\Omega^*} |\nabla(u_n-u)|^2 \, dxdy &\leq \int_{\Omega^*} a(x,y,u_n,\xi_n)|\nabla(u_n-u)|^2 \, dxdy \\ &= -\int_{\Omega^*} [a(x,y,u_n,\xi_n) - a(x,y,u_n,\xi)]\nabla u \cdot \nabla(u_n-n) \, dxdy \\ &\quad - \int_{\Omega^*} [a(x,y,u_n,\xi) - a(x,y,u,\xi)]\nabla u \cdot \nabla(u_n-n) \, dxdy \leq \end{aligned}$$

$$
\begin{aligned}
&\leq \Big(\int_{\Omega*} |\nabla(u_n - u)|^2 \,dxdy\Big)^{1/2} \times \int_{\Omega*} |\nabla u|^p \,dxdy\Big)^{1/p} \\
&\qquad\qquad \times \Big(\int_{\Omega^*} [a(x,y,u_n,\xi_n) - a(x,y,u_n,\xi)]^q \,dxdy\Big)^{1/q} \\
&\qquad\qquad + \Big(\int_{\Omega*} |\nabla(u_n - u)|^2 \,dxdy\Big)^{1/2} \times \int_{\Omega*} |\nabla u|^p \,dxdy\Big)^{1/p} \\
&\quad \times \Big(\int_{\Omega^*} [a(x,y,u_n,\xi) - a(x,y,u,\xi)]^q \,dxdy\Big)^{1/q}, \ (1/p) + (1/q) = 1. \quad (55)
\end{aligned}
$$

Recalling (53) and (54) we conclude that $u_n \to u$ in $H_0^1(\Omega^*)$ strongly. Therefore $\mathcal{G}$ is continuous and by the Schauder's theorem, there exists a fixed point of $\mathcal{S}$ which gives a solution to problem (34), (35). $\square$

4. Appendix

The "a-priori" estimate (38) does not hold in the framework of spaces like $E(\Omega, \Omega_1)$. As an example, let us consider the weak solution of the problem

$$\varphi \in E(B, B_1), \ \Delta^2 \varphi = 1, \tag{56}$$

where $B = \{(x,y); 0 \leq \sqrt{x^2+y^2} < 1\}$, $B_1 = \{(x,y); 0 \leq \sqrt{x^2+y^2} < 1/2\}$ and $E(B,B_1) = \{\varphi(x,y) \in H^2(B),\ \varphi = 0,\ \frac{\partial \varphi}{\partial n} = 0 \text{ on } \Gamma_1,\ \varphi = ax+by+c \text{ in } B_1\}$, $\Gamma_1 = \{(x,y), x^2+y^2 = 1\}$. In view of the uniqueness of the solution of problem (56) and of its rotational symmetry, we may consider the auxiliary one-dimensional problem for the function $\psi(\rho)$

$$\Delta^2 \psi = 1 \text{ in } B \setminus B_1,\ \psi(1) = 0,\ \psi'(1) = 0,\ \psi(1/2) = C,\ \psi'(1/2) = 0.$$

We have $\psi(\rho) = (1/64)\rho^4 + C_1\rho^2 + C_2 \log(\rho) + C_3 \rho^2 \log \rho + C_4$. The constants C_1, C_2, C_3 and C_4 are easily determined with the conditions $\psi(1/2) = C$, $\psi'(1/2) = 0$, $\psi(1) = 0$, $\psi'(1) = 0$. We obtain $\psi(\rho, C)$. The remaining constant C is found minimizing the quadratic function

$$f(C) = 2\pi \int_{1/2}^{1} \Big[(1/\rho)(\rho\psi'(\rho,C))'\Big]^2 \,d\rho + 4\pi \int_{1/2}^{1} \psi(\rho,C)\,d\rho.$$

If $f'(\bar{C}) = 0$, the solution of problem (56) is given by $\varphi(\rho) = \psi(\rho, \bar{C})$ in $B \setminus B_1$, $\varphi(\rho) = \bar{C}$ in B_1 and $\varphi \notin H^3(B)$.

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