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ON A DIFFERENTIAL SUBORDINATION AND SUPERORDINATION OF NEW CLASS OF MEROMORPHIC FUNCTIONS

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In this paper, we investigate differential subordination and superordination properties of a new class of meromorphic analytic functions in the punctured unit disc. We derive some sandwich theorems.

1. Introduction

Let $\mathcal{H}(E)$ denote the class of analytic functions in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$, let $\mathcal{H}[a,1] = \{f \in \mathcal{H}(E) : f(z) = a + a_1 z + a_2 z^2 + \ldots, z \in E\}$.

Also, let $\sum$ denote the class of functions of the form:

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the punctured unit disk

$$E^* = \{z \ : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = E \backslash \{0\}.$$

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If $f, g \in \mathcal{H}(E)$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w in E with $w(0) = 1$ and $|w(z)| < 1$ $(z \in E)$ such that $f(z) = g(w(z))$.

For a complex parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ $(\beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-,\ \mathbb{Z}_0^- = \{0, -1, -2, \ldots\};\ j = 1, \ldots, s)$, we now define the generalized hypergeometric function, see [25,33] as follows:

$$ {}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_q)_n}{(\beta_1)_n \ldots (\beta_s)_n \ n!} z^n, \tag{2} $$

$$ (q \leq s+1; s \in \mathbb{N} \cup \{0\}; \mathbb{N} = \{1, 2, \ldots\}; z \in E), $$

where $(v)_k$ is the Pochhammer symbol (or shifted factorial) defined (in terms of the Gamma function) by

$$ (v)_n = \frac{\Gamma(v+n)}{\Gamma(v)} = \begin{cases} 1 & \text{if } n = 0 \text{ and } v \in \mathbb{C}\backslash\{0\}; \\ v(v+1)\ldots(v+n-1) & \text{if } n \in \mathbb{N} \text{ and } v \in \mathbb{C}. \end{cases} $$

Corresponding to a function

$$ \mathcal{F}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = z^{-1}\, {}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z). \tag{3} $$

Liu and Srivastava [16] consider a linear operator
$H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) : \sum \longrightarrow \sum$ defined by the following Hadamard product (or convolution):

$$ H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) f(z) = \mathcal{F}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) * f(z). \tag{4} $$

We note that the linear operator $H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s)$ was motivated essentially by Dziok and Srivastava [9]. Some interesting developments with the generalized hypergeometric function were considered recently by Dziok and Srivastava [10,11] and Liu and Srivastava [14,15]. Corresponding to the function $\mathcal{F}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ defined by (3), we introduce a function

$$ \mathcal{F}_\lambda(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)\, z \in E^*, $$

given by

$$ \mathcal{F}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) * \mathcal{F}_\lambda(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \frac{1}{z(1-z)^\lambda} \quad (\lambda > 0). \tag{5} $$

Analogous to $H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s)$ defined by (4), we now define the linear operator $H_\lambda(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s)$ on $\sum$ as follows:

$$ H_\lambda(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) f(z) = \mathcal{F}_\lambda(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) * f(z) \tag{6} $$

$$ (\alpha_i, \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-; \quad i = 1, \ldots, q;\ j = 1, \ldots, s; \quad \lambda > 0; z \in E^*\ f \in \textstyle\sum). $$

For convenience, we write

$$H_{\lambda,q,s}(\alpha_1) = H_\lambda(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s).$$

It is easily verified from the definition (5) and (6) that

$$z(H_{\lambda,q,s}(\alpha_1+1)f(z))' = \alpha_1 H_{\lambda,q,s}(\alpha_1)f(z) - (\alpha_1+1)H_{\lambda,q,s}(\alpha_1+1)f(z), \quad (7)$$

and

$$z(H_{\lambda,q,s}(\alpha_1)f(z))' = \lambda H_{\lambda+1,q,s}(\alpha_1)f(z) - (\lambda+1)H_{\lambda,q,s}(\alpha_1)f(z). \quad (8)$$

We note that the operator $H_{\lambda,q,s}(\alpha_1)$ is closely related to the Choi-Saigo-Srivastava operator [8] for analytic functions, which includes the integral operator studied by Liu [12] and Noor et al. [19,22].

Suppose that h and k are two analytic functions in E, let

$$\varphi(r,s,t;z) : \mathbb{C}^3 \times E \longrightarrow \mathbb{C}.$$

If h and $\varphi(h(z),zh'(z),z^2h''(z);z)$ are univalent functions in E and if h satisfies the second order superordination

$$k(z) \prec \varphi(h(z),zh'(z),z^2h''(z);z), \quad (9)$$

then k is said to be a solution of the differential superordination (9). An analytic function $q \in \mathcal{H}(E)$ is called a subordinant to (9), if $q(z) \prec h(z)$ for all the functions h satisfying (9).

A univalent subordinant $\widetilde{q}$ that satisfies $q(z) \prec \widetilde{q}(z)$ for all of the subordinants q of (9), is said to be the best subordinant.

Miller and Mocanu [18] obtained sufficient conditions on the functions k,q and φ for which the following implications hold:

$$k(z) \prec \varphi(h(z),zh'(z),z^2h''(z);z) \Longrightarrow q(z) \prec h(z).$$

Using these results, the authors in [3] considered certain classes of first-order differential superordinations, see also [5], as well as superordination-preserving integral operators [4]. Aouf et al. [2,3], obtained sufficient conditions for certain normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent normalized functions in E. Very recently, Shanmugam et al. [28,29] obtained the such called sandwich results for certain classes of analytic functions. For interested readers we refer to the work done by the authors [1,2,7,17,21,23,24,27,30,31,32].

2. Preliminary Results

Definition 2.1 ([18]). Let $\mathcal{Q}$ be the set of all functions f that are analytic and injective on $\overline{E}\backslash U(f)$, where

$$U(f) = \left\{ \zeta \in \partial E : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial E \backslash U(f)$.

To establish our main results we need the following Lemmas.

Lemma 2.2 (Miller and Mocanu [17]). *Let q be univalent in the unit disc E, and let θ and φ be analytic in a domain D containing $q(E)$, with $\varphi(w) \neq 0$ when $w \in q(E)$. Set $Q(z) = zq'(z)\varphi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*
(i) Q is a starlike function in E,
(ii) $\Re \frac{zh'(z)}{Q(z)} > 0$, $z \in E$.
If p is analytic in E with $p(0) = q(0)$, $p(E) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{10}$$

then $p(z) \prec q(z)$, and q is the best dominant of (10).

Lemma 2.3 (Shanmugam et al.[29]). *Let ν, $\gamma \in \mathbb{C}$ with $\gamma \neq 0$, and let q be a convex function in E with*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -\Re\frac{\nu}{\gamma}\right\}, \ z \in E.$$

If p is analytic in E and

$$\nu p(z) + \gamma z p'(z) \prec \nu q(z) + \gamma z q'(z), \tag{11}$$

then $p(z) \prec q(z)$, and q is the best dominant of (11).

Lemma 2.4 (Bulboacă [6]). *Let q be a univalent function in the unit disc E, and let θ and φ be analytic in a domain D containing $q(E)$. Suppose that*
(i) $\Re \frac{\theta'(q(z))}{\varphi(q(z))} > 0$ for $z \in E$,
(ii) $h(z) = zq'(z)\varphi(q(z))$ is starlike in E.
If $p \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$ with $p(E) \subseteq D$, $\theta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in E, and

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z)), \tag{12}$$

then $q(z) \prec p(z)$, and q is the best subordinant of (12).

Note that this result generalize a similar one obtained in [4].

Lemma 2.5 (Miller and Mocanu [18]). *Let q be convex in E and let $\gamma \in \mathbb{C}$, with $\Re\gamma > 0$. If $p \in \mathcal{H}[q(0),1] \cap \mathcal{Q}$ and $p(z)+\gamma zp'(z)$ is univalent in E, then*

$$q(z)+\gamma zq'(z) \prec p(z)+\gamma zp'(z), \tag{13}$$

implies $q(z) \prec p(z)$, and q is the best subordinant of (13).

Lemma 2.6 (Royster [26]). *The function $q(z)=\frac{1}{(1-z)^{2ab}}$ is univalent in E if and only if $|2ab-1| \le 1$ or $|2ab+1| \le 1$.*

3. Main Results

Theorem 3.1. *Let p be univalent in E with $p(0)=1$, and suppose that*

$$\Re\left(1+\frac{zp''(z)}{p'(z)}\right) > \max\left\{0;\ -\mu\lambda\Re\frac{1}{\alpha}\right\},\ z \in E, \tag{14}$$

where $\alpha \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, $0<\mu<1$ and $zH_{\lambda,q,s}(\alpha_1)f(z) \neq 0$. If $f \in \sum$ satisfies the subordination

$$\begin{aligned}&(1+\alpha)(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu}-\alpha z(H_{\lambda+1,q,s}(\alpha_1)f(z))(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu-1}\\ &\prec p(z)+\frac{\alpha zp'(z)}{\mu\lambda},\end{aligned} \tag{15}$$

then

$$(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu} \prec p(z),$$

and p is the best dominant of Eq. (15).

Proof. We begin by setting

$$(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu}=h(z),$$

where $h(z)$ is analytic in E with $h(0)=1$.

A simple computation together with (8) shows that

$$\begin{aligned}&(1+\alpha)(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu}-\alpha z(H_{\lambda+1,q,s}(\alpha_1)f(z))(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu-1}\\ &=h(z)+\frac{\alpha}{\mu\lambda}zh'(z),\end{aligned}$$

hence the subordination (15) becomes

$$h(z)+\frac{\alpha}{\mu\lambda}zh'(z) \prec p(z)+\frac{\alpha}{\mu\lambda}zp'(z).$$

Combining this last relation together with Lemma 2.3 for special case $\gamma=\frac{\alpha}{\mu\lambda}$ and $\nu=1$, we obtain our result. $\square$

Taking $p(z) = \frac{1+Az}{1+Bz}$ in Theorem 3.1, where $-1 \leq B < A \leq 1$, the condition (14) reduces to

$$\Re \frac{1-Bz}{1+Bz} > \max\left\{0;\ -\mu\lambda\Re\frac{1}{\alpha}\right\},\ z \in E. \tag{16}$$

It is easy to verify that the function $\varphi(\zeta) = \frac{(1-\zeta)}{(1+\zeta)}$, $|\zeta| < |B|$, is convex in E, and since $\varphi(\overline{\zeta}) = \overline{\varphi(\zeta)}$ for all $|\zeta| < |B|$, it follows that $\varphi(E)$ is a convex domain symmetric with respect to the real axis, hence

$$\inf\left\{\Re\frac{1-Bz}{1+Bz} :\ z \in E\right\} = \frac{1-|B|}{1+|B|} > 0. \tag{17}$$

Then, the inequality (16) is equivalent to

$$\mu\lambda\Re\frac{1}{\alpha} \geq \frac{|B|-1}{|B|+1},$$

hence, we have the following result.

Corollary 3.2. *Let* $0 < \mu < 1$, $zH_{\lambda,q,s}(\alpha_1)f(z) \neq 0$, $-1 \leq B < A \leq 1$ *and* $\alpha \in \mathbb{C}^*$ *with*

$$\max\left\{0;\ -\mu\lambda\Re\frac{1}{\alpha}\right\} \leq \frac{1-|B|}{1+|B|}.$$

If $f \in \sum$, *and*

$$\begin{aligned}(1+\alpha)&(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu} - \alpha z(H_{\lambda+1,q,s}(\alpha_1)f(z))(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu-1}\\ &\prec \frac{1+Az}{1+Bz} + \frac{\alpha}{\mu\lambda}\frac{(A-B)z}{(1+Bz)^2},\end{aligned} \tag{18}$$

then

$$(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu} \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant of (18).*

For $A = 1$ and $B = -1$, the above corollary reduces to

Corollary 3.3. *Let* $0 < \mu < 1$, $zH_{\lambda,q,s}(\alpha_1)f(z) \neq 0$ *and* $\alpha \in \mathbb{C}^*$ *with* $\Re\frac{1}{\alpha} \geq 0$. *If* $f \in \sum$, *and*

$$\begin{aligned}(1+\alpha)&(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu} - \alpha z(H_{\lambda+1,q,s}(\alpha_1)f(z))(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu-1}\\ &\prec \frac{1+z}{1-z} + \frac{\alpha}{\mu\lambda}\frac{2z}{(1-z)^2},\end{aligned} \tag{19}$$

then

$$(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu} \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ *is the best dominant of (19).*

Theorem 3.4. *Let p be univalent in E, with $p(0)=1$ and $p(z)\neq 0$ for all $z\in E$. Let γ, $\mu\in\mathbb{C}^*$ and v, $\eta\in\mathbb{C}$, with $v+\eta\neq 0$. Let $f\in\sum$ and suppose that f and p satisfy the following conditions:*

$$\left[(v+\eta)z\left\{vH_{\lambda+1,q,s}(\alpha_1)f(z)+\eta H_{\lambda,q,s}(\alpha_1)f(z)\right\}\right]\neq 0,\ z\in E, \tag{20}$$

and

$$\Re\left(1+\frac{zp''(z)}{p'(z)}-\frac{zp'(z)}{p(z)}\right)>0,\ z\in E. \tag{21}$$

If

$$\begin{aligned}&1+\gamma\mu\left[-\frac{vz(H_{\lambda+1,q,s}(\alpha_1)f(z)'+\eta z\left(H_{\lambda,q,s}(\alpha_1)f(z)\right)'}{v\left(H_{\lambda+1,q,s}(\alpha_1)f(z)\right)+\eta\left(H_{\lambda,q,s}(\alpha_1)f(z)\right)}-1\right]\\&\prec 1+\gamma\frac{zp'(z)}{p(z)},\end{aligned} \tag{22}$$

then

$$\left[(v+\eta)z\left\{vH_{\lambda+1,q,s}(\alpha_1)f(z)+\eta H_{\lambda,q,s}(\alpha_1)f(z)\right\}\right]^{-\mu}\prec p(z),$$

and p is the best dominant of (22). The power is the principal one.

Proof. We begin by setting

$$\left[(v+\eta)z\left\{vH_{\lambda+1,q,s}(\alpha_1)f(z)+\eta H_{\lambda,q,s}(\alpha_1)f(z)\right\}\right]^{-\mu}=h(z),\ z\in E, \tag{23}$$

where $h(z)$ is analytic in E with $h(0)=1$. Differentiating Equation (23) logarithmically with respect to z, we have

$$\mu\left[-\frac{vz(H_{\lambda+1,q,s}(\alpha_1)f(z)'+\eta z\left(H_{\lambda,q,s}(\alpha_1)f(z)\right)'}{v\left(H_{\lambda+1,q,s}(\alpha_1)f(z)\right)+\eta\left(H_{\lambda,q,s}(\alpha_1)f(z)\right)}-1\right]=\frac{zh'(z)}{h(z)}.$$

To prove our result we use Lemma 2.2, we suppose that

$$\theta(w)=1 \text{ and } \varphi(w)=\frac{\gamma}{w},$$

then θ is analytic in $\mathbb{C}$ and $\varphi(w)\neq 0$ is analytic in $\mathbb{C}^*$. Also, if we let

$$Q(z)=zp'(z)\varphi(p(z))=\gamma\frac{zp'(z)}{p(z)},$$

and

$$g(z)=\theta(p(z))+Q(z)=1+\gamma\frac{zp'(z)}{p(z)},$$

then, since $Q(0) = 0$ and $Q'(0) \neq 0$, the assumption (21) would yield that Q is a starlike function in E. From (21), we have

$$\Re \frac{zg'(z)}{Q(z)} = \Re\left(1 + \frac{zp''(z)}{p'(z)} - \frac{zp'(z)}{p(z)}\right) > 0,\ z \in E,$$

and by using Lemma 2.2, we deduce that the subordination (22) implies that $h(z) \prec p(z)$, and the function p is the best dominant of (22).This completes the proof of our theorem. □

In particular, $v = 0$, $\eta = \gamma = 1$ and $p(z) = \frac{1+Az}{1+Bz}$ in the above Theorem 3.4, it is easy to see that the assumption (21) holds whenever $-1 \leq A < B \leq 1$, which leads to the next result:

Corollary 3.5. *Let $-1 \leq A < B \leq 1$ and $\mu \in \mathbb{C}^*$. Let $f \in \sum$, and suppose that $zH_{\lambda,q,s}(\alpha_1)f(z) \neq 0$ for $z \in E$. And assume (20). If*

$$1 + \mu\left[-\frac{z(H_{\lambda,q,s}(\alpha_1)f(z))'}{H_{\lambda,q,s}(\alpha_1)f(z))} - 1\right] \prec 1 + \frac{(A-B)z}{(1+Az)(1+Bz)}, \tag{24}$$

then

$$\left(zH_{\lambda,q,s}(\alpha_1)f(z)\right)^{-\mu} \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant of (24). The power is the principal one.

Letting $v = 0$, $\eta = 1$, $\alpha_i = \beta_i (i = 1, 2, \ldots, s)$, $\gamma = \frac{1}{ab}$, $a, b \in \mathbb{C}^*$, $\mu = a$, and $p(z) = \frac{1}{(1-z)^{2ab}}$ in Theorem 3.4, then combining this together with Lemma 2.6 we obtain the next result.

Corollary 3.6. *Let $a, b \in \mathbb{C}^*$ such that $|2ab - 1| \leq 1$ or $|2ab + 1| \leq 1$. Let $f \in \sum$ and let $zf(z) \neq 0$ for all $z \in E$. And assume (20).*

If

$$1 + \frac{1}{b}\left(-1 - \frac{zf'(z)}{f(z)}\right) \prec \frac{1+z}{1-z}, \tag{25}$$

then

$$(zf(z))^{-a} \prec \frac{1}{(1-z)^{2ab}},$$

and $\frac{1}{(1-z)^{2ab}}$ is the best dominant of (25). The power is the principal one.

In particular, $v = 0$, $\eta = \gamma = 1$, $\alpha_i = \beta_i (i = 1, 2, \ldots, s)$ and $p(z) = (1 + Bz)^{\frac{\mu(A-B)}{B}}$, $-1 \leq B < A \leq 1$, $B \neq 0$ in Theorem 3.4, and using Lemma 2.6, we obtain the next result.

Corollary 3.7. *Let $-1 \leq B < A \leq 1$, with $B \neq 0$, and suppose that $\left|\frac{\mu(A-B)}{B-1}\right| \leq 1$ or $\left|\frac{\mu(A-B)}{B+1}\right| \leq 1$. Let $f \in \sum$ such that $zf(z) \neq 0$ for all $z \in E$, and let $\mu \in \mathbb{C}^*$. And assume (20).*

If

$$1+\mu\left(-1-\frac{zf'(z)}{f(z)}\right) \prec \frac{1+[B+\beta(A-B)]z}{1+Bz}, \tag{26}$$

then

$$(zf(z))^{-\mu} \prec (1+Bz)^{\frac{\mu(A-B)}{B}},$$

and $(1+Bz)^{\frac{\mu(A-B)}{B}}$ is the best dominant of (26). Here the power is the principal one.

By taking $v=0$, $\mu=a$, $\eta=1$, $\alpha_i=\beta_i (i=1,2,\ldots,s)$, $\gamma=\frac{e^{i\lambda}}{ab\cos\lambda}$, a, $b \in \mathbb{C}^*$ and $|\lambda|<\pi/2$, and $q(z)=\frac{1}{(1-z)^{2ab\cos\lambda e^{-i\lambda}}}$ in Theorem 3.4, we obtain the following result.

Corollary 3.8. *Let $a, b \in \mathbb{C}^*$ and $|\lambda| < \frac{\pi}{2}$, and suppose that $\left|2ab\cos\lambda e^{-i\lambda}-1\right| \leq 1$ or $\left|2ab\cos\lambda e^{-i\lambda}+1\right| \leq 1$. Let $f \in \sum$ such that $zf(z) \neq 0$ for all $z \in E$. And assume (20).*

If

$$1+\frac{e^{i\lambda}}{b\cos\lambda}\left(-1-\frac{zf'(z)}{f(z)}\right) \prec \frac{1+z}{1-z}, \tag{27}$$

then

$$(zf(z))^{-a} \prec \frac{1}{(1-z)^{2ab\cos\lambda e^{-i\lambda}}},$$

and $\frac{1}{(1-z)^{2ab\cos\lambda e^{-i\lambda}}}$ is the best dominant of (27). The power is the principal one.

Theorem 3.9. *Let p be univalent in E with $p(0)=1$, let $\mu, \gamma \in \mathbb{C}^*$, and let δ, v, $\eta \in \mathbb{C}$ with $v+\eta \neq 0$. Let $f \in \sum$ and suppose that f and p satisfy the following conditions:*

$$(v+\eta)z\left[vH_{\lambda+1,q,s}(\alpha_1)f(z)+\eta H_{\lambda,q,s}(\alpha_1)f(z)\right] \neq 0,\ z \in E, \tag{28}$$

and

$$\Re\left(1+\frac{zp''(z)}{p'(z)}\right) > \max\left\{0; -\Re\frac{\delta}{\gamma}\right\},\ z \in E. \tag{29}$$

If

$$\begin{aligned}\psi(z) \equiv {} & \left[(v+\eta)z\left\{vH_{\lambda+1,q,s}(\alpha_1)f(z))+\eta H_{\lambda,q,s}(\alpha_1)f(z)\right\}\right]^{-\mu} \\ & \left[\delta+\gamma\mu\left(-1-\frac{vz\left(H_{\lambda+1,q,s}(\alpha_1)f(z)\right)'+\eta z\left(H_{\lambda,q,s}(\alpha_1)f(z)\right)'}{v\left(H_{\lambda+1,q,s}(\alpha_1)f(z)\right)+\eta\left(H_{\lambda,q,s}(\alpha_1)f(z)\right)}\right)\right],\end{aligned} \tag{30}$$

and

$$\psi(z) \prec \delta p(z) + \gamma z p'(z), \tag{31}$$

then

$$\left[(v+\eta)z\left\{vH_{\lambda+1,q,s}(\alpha_1)f(z)) + \eta H_{\lambda,q,s}(\alpha_1)f(z)\right\}\right]^{-\mu} \prec p(z),$$

and p is the best dominant of (31). All the powers are the principal ones.

Proof. We begin by setting

$$\left[(v+\eta)z\left\{vH_{\lambda+1,q,s}(\alpha_1)f(z)) + \eta H_{\lambda,q,s}(\alpha_1)f(z)\right\}\right]^{-\mu} = h(z). \tag{32}$$

Then $h(z)$ is analytic in E with $h(0) = 1$. Logarithmic differentiating of (32) yields

$$\mu\left(-1 - \frac{vz\left(H_{\lambda+1,q,s}(\alpha_1)f(z)\right)' + \eta z\left(H_{\lambda,q,s}(\alpha_1)f(z)\right)'}{v\left(H_{\lambda+1,q,s}(\alpha_1)f(z)\right) + \eta\left(H_{\lambda,q,s}(\alpha_1)f(z)\right)}\right) = \frac{zh'(z)}{h(z)},$$

and hence

$$\mu h(z)\left(-1 - \frac{vz\left(H_{\lambda+1,q,s}(\alpha_1)f(z)\right)' + \eta z\left(H_{\lambda,q,s}(\alpha_1)f(z)\right)'}{v\left(H_{\lambda+1,q,s}(\alpha_1)f(z)\right) + \eta\left(H_{\lambda,q,s}(\alpha_1)f(z)\right)}\right) = zh'(z).$$

Let us consider the functions:

$$\theta(w) = \delta w, \quad \varphi(w) = \gamma,\, w \in \mathbb{C},$$
$$Q(z) = zp'(z)\varphi(p(z)) = \gamma z p'(z),\, z \in E,$$

and

$$g(z) = \theta(p(z)) + Q(z) = \delta p(z) + \gamma z p'(z),\, z \in E.$$

From the assumption (29) we see that Q is starlike in E and, that

$$\Re\frac{zg'(z)}{Q(z)} = \Re\left(\frac{\delta}{\gamma} + 1 + \frac{zp''(z)}{p'(z)}\right) > 0,\ z \in E,$$

Now, using Lemma 2.2, the proof is completed. □

Taking $p(z) = \frac{(1+Az)}{(1+Bz)}$ in Corollary 3.7, where $-1 \le B < A \le 1$ and according to (17), the condition (29) becomes

$$\max\left\{0; -\Re\frac{\delta}{\gamma}\right\} \le \frac{1-|B|}{1+|B|}.$$

Hence, for the special case $v = 1 = \gamma$, $\eta = 0$, we obtain the next result:

Corollary 3.10. *Let $-1 \leq B < A \leq 1$ and let $\delta \in \mathbb{C}$ with*

$$\max\{0; -\Re\delta\} \leq \frac{1-|B|}{1+|B|}.$$

Let $f \in \sum$ and suppose that

$$zH_{\lambda,q,s}(\alpha_1)f(z) \neq 0,\ z \in E,$$

and let $\mu \in \mathbb{C}^*$. If

$$\begin{aligned}&\left[zH_{\lambda,q,s}(\alpha_1)f(z)\right]^{-\mu}\left[\delta+\mu\left(-\frac{z(H_{\lambda,q,s}(\alpha_1)f(z))'}{zH_{\lambda,q,s}(\alpha_1)f(z)}-1\right)\right]\\&\prec \delta\frac{1+Az}{1+Bz}+\frac{z(A-B)}{(1+Bz)^2},\end{aligned} \tag{33}$$

then

$$\left(zH_{\lambda,q,s}(\alpha_1)f(z)\right)^{-\mu} \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant of (33). All the powers are the principal ones.

By taking $\gamma = \eta = 1$, $v = 0$, $\alpha_i = \beta_i (i = 1,2,\ldots,s)$ and $p(z) = \frac{1+z}{1-z}$ in Corollary 3.7, we obtain the next result.

Corollary 3.11. *Let $f \in \sum$ such that $zf(z) \neq 0$ for all $z \in E$, and let $\mu \in \mathbb{C}^*$. If*

$$[zf(z)]^{-\mu}\left[\delta+\mu\left(-1-\frac{z(f'(z))}{f(z)}\right)\right] \prec \delta\frac{1+z}{1-z}+\frac{2z}{(1-z)^2}, \tag{34}$$

then

$$[zf(z)]^{-\mu} \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is the best dominant of (34). All the powers are the principal ones.

4. Superordination and Sandwich results

Theorem 4.1. *Let p be convex in E with $p(0) = 1$, let $0 < \mu < 1$, $\alpha \in \mathbb{C}^*$ with $\Re\alpha > 0$. Let $f \in \sum$ be such that $zH_{\lambda,q,s}(\alpha_1)f(z) \neq 0$ and suppose that $\left(zH_{\lambda,q,s}(\alpha_1)f(z)\right)^{-\mu} \in \mathcal{H}[p(0),1] \cap \mathcal{Q}$. If the function*

$$(1+\alpha)(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu} - \alpha z(H_{\lambda+1,q,s}(\alpha_1)f(z))(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu-1}$$

is univalent in the unit disc E, and

$$
\begin{aligned}
& p(z)+\frac{\alpha}{\mu\lambda} z p'(z) \\
& \prec (1+\alpha)(z(H_{\lambda,q,s}(\alpha_1)f(z))^{-\mu} \\
& -\alpha z(H_{\lambda+1,q,s}(\alpha_1)f(z))(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu-1},
\end{aligned} \tag{35}
$$

then

$$
p(z) \prec (zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu},
$$

and p is the best subordinant of (35).

Proof. Setting

$$
(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu} = h(z),\ z \in E.
$$

Then $h(z)$ is analytic in E with $h(0) = 1$.

A simple computation together with (8) shows that

$$
\begin{aligned}
& h(z)+\frac{\alpha}{\mu\lambda} z h'(z) \\
& = (1+\alpha)(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu} - \alpha z(H_{\lambda+1,q,s}(\alpha_1)f(z))(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu-1},
\end{aligned}
$$

and now, by using Lemma 2.5, we obtain the desired result. □

Taking $p(z) = \frac{1+Az}{1+Bz}$ in Theorem 4.1, where $-1 \le B < A \le 1$, we obtain the next result.

Corollary 4.2. *Let p be convex in E with $p(0) = 1$, let $0 < \mu < 1$, $\alpha \in \mathbb{C}^*$ with $\Re\alpha > 0$. Let $f \in \sum$ be such that $zH_{\lambda,q,s}(\alpha_1)f(z) \neq 0$ and suppose that $\left(zH_{\lambda,q,s}(\alpha_1)f(z)\right)^{-\mu} \in \mathcal{H}[p(0),1] \cap \mathcal{Q}$. If the function*

$$
(1+\alpha)(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu} - \alpha z(H_{\lambda+1,q,s}(\alpha_1)f(z))(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu-1}
$$

is univalent in the unit disc E, and

$$
\begin{aligned}
& \frac{1+Az}{1+Bz}+\frac{\alpha(A-B)z}{\mu\lambda(1+Bz)^2} \\
& \prec (1+\alpha)(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu} \\
& -\alpha z(H_{\lambda+1,q,s}(\alpha_1)f(z))(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu-1},
\end{aligned} \tag{36}
$$

then

$$
\frac{1+Az}{1+Bz} \prec (z(H_{\lambda,q,s}(\alpha_1)f(z))^{-\mu},
$$

and $\frac{1+Az}{1+Bz}$ is the best subordinant of (36), where $-1 \le B < A \le 1$.

Using the same techniques as in Theorem 20, and then applying Lemma 2.4, we have the following theorem.

Theorem 4.3. *Let p be convex in E with $p(0) = 1$, let μ, $\gamma \in \mathbb{C}^*$, and let δ, v, $\eta \in \mathbb{C}$ with $v+\eta \neq 0$ and $\Re\frac{\delta}{\gamma} > 0$. Let $f \in \sum$ and suppose that f satisfies the following conditions:*

$$\left[(v+\eta)z\left\{vH_{\lambda+1,q,s}(\alpha_1)f(z)) + \eta H_{\lambda,q,s}(\alpha_1)f(z)\right\}\right] \neq 0,\ z \in E,$$

and

$$\left[(v+\eta)z\left\{vH_{\lambda+1,q,s}(\alpha_1)f(z)) + \eta H_{\lambda,q,s}(\alpha_1)f(z)\right\}\right]^{-\mu} \in \mathcal{H}[p(0),1] \cap \mathcal{Q}.$$

If the function ψ given by equation (30) is univalent in E, and

$$\delta q(z) + \gamma z q'(z) \prec \psi(z), \tag{37}$$

then

$$p(z) \prec \left[(v+\eta)z\left\{vH_{\lambda+1,q,s}(\alpha_1)f(z)) + \eta H_{\lambda,q,s}(\alpha_1)f(z)\right\}\right]^{-\mu},$$

and p is the best subordinate of (37). All the powers are the principal ones.

Note that by combining Theorem 3.1 with Theorem 4.1 and Corollary 3.10 with Theorem 4.3, we have, respectively, the following two sandwich theorems:

Theorem 4.4. *Let p_1 and p_2 be two convex functions in E with $p_1(0) = p_2(0) = 1$, let $0 < \mu < 1$, $\alpha \in \mathbb{C}^*$ with $\Re\alpha > 0$. Let $f \in \sum$ be such that $zH_{\lambda,q,s}(\alpha_1)f(z) \neq 0$ and suppose that $\left(zH_{\lambda,q,s}(\alpha_1)f(z)\right)^{-\mu} \in \mathcal{H}[p(0),1] \cap \mathcal{Q}$. If the function*

$$(1+\alpha)(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu} - \alpha z(H_{\lambda+1,q,s}(\alpha_1)f(z))(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu-1}$$

is univalent in the unit disc E, and

$$\begin{aligned} &p_1(z) + \frac{\alpha}{\mu\lambda} z p_1'(z) \\ &\prec (1+\alpha)(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu} - \alpha z(H_{\lambda+1,q,s}(\alpha_1)f(z))(zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu-1} \\ &\prec p_2(z) + \frac{\alpha}{\mu\lambda} z p_2'(z), \end{aligned} \tag{38}$$

then

$$p_1(z) \prec (zH_{\lambda,q,s}(\alpha_1)f(z))^{-\mu} \prec p_2(z),$$

and p_1 and p_2 are, respectively, the best subordinate and the best dominant of (38).

Theorem 4.5. *Let p_1 and p_2 be two convex functions in E with $p_1(0) = p_2(0) = 1$, let $\mu, \gamma \in \mathbb{C}^*$, and let δ, v, $\eta \in \mathbb{C}$ with $v+\eta \neq 0$ and $\Re\frac{\delta}{\gamma} > 0$. Let $f \in \sum$ satisfy the following conditions:*

$$\left[(v+\eta)z\left\{vH_{\lambda+1,q,s}(\alpha_1)f(z)) + \eta H_{\lambda,q,s}(\alpha_1)f(z)\right\}\right] \neq 0,\ z \in E,$$

and

$$\left[(v+\eta)z\left\{vH_{\lambda+1,q,s}(\alpha_1)f(z)) + \eta H_{\lambda,q,s}(\alpha_1)f(z)\right\}\right]^{-\mu} \in \mathcal{H}[p(0),1] \cap \mathcal{Q}.$$

If the function ψ given by (30) is univalent in E, and

$$\delta p_1(z) + \gamma z p_1'(z) \prec \psi(z) \prec \delta p_2(z) + \gamma z p_2'(z), \tag{39}$$

then

$$p_1(z) \prec \left[(v+\eta)z\left\{vH_{\lambda+1,q,s}(\alpha_1)f(z)) + \eta H_{\lambda,q,s}(\alpha_1)f(z)\right\}\right]^{-\mu} \prec p_2(z),$$

and p_1 and p_2 are, respectively, the best subordinate and the best dominant of (39). All the powers are the principal ones.

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