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SOME PROPERTIES OF THE k-GAMMA FUNCTION

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We give completely monotonicity properties and inequalities for functions involving the Γ_k functions and their logarithmic derivatives ψ_k functions. We introduce a k-analogue of the Riemann Zeta function ζ_k as an integral and using Schwarz's and Holder's inequalities we obtain some inequalities relating ζ_k and Γ_k functions. The obtained results are the k-anologues of known results concerning functions involving the Gamma and psi functions.

1. Introduction

The Euler Gamma function $\Gamma(x)$ is defined [1] by $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ for $x \in C$ with $\Re x > 0$ and

$$\Gamma(x) = \lim_{n\to\infty} \frac{n! n^x}{x(x+1)\dots(x+n)}, x \in C \setminus Z^-. \tag{1}$$

The digamma (or psi) function is defined as the logarithmic derivative of Euler's Gamma function, that is $\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. The following integral and series representations are valid (see [1]):

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt, \tag{2}$$

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$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{n\geq 1} \frac{x}{n(n+x)}, x \neq 0, -1, -2, \cdots, \tag{3}$$

where $\gamma = 0.57721\cdots$ denotes Euler's constant.

For $k > 0$, the Γ_k function is defined [5] by

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, x \in C \setminus kZ^-, \tag{4}$$

where $(x)_{n,k} = x(x+k)(x+2k)...(x+(n-1)k)$.

The above definiton is a generalization of the definition of $\Gamma(x)$ function. For $x \in C$ with $\Re(x) > 0$, the function $\Gamma_k(x)$ is given by the integral [5]

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt. \tag{5}$$

and satisfies [6] the following properties:

(i) $\Gamma_k(x+k) = x\Gamma_k(x)$

(ii) $(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)}$

(iii) $\Gamma_k(k) = 1$

(iv) $\Gamma_k(x)$ is logarithmically convex, for $x \in R$

(v) $\Gamma_k(x) = a^{\frac{x}{k}} \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}a} dt$, for $a \in R$

(vi) $\frac{1}{\Gamma_k(x)} = xk^{-\frac{x}{k}} e^{\frac{x}{k}\gamma} \prod_{n=1}^{\infty} \left(\left(1+\frac{x}{nk}\right) e^{-\frac{x}{nk}}\right)$.

It is obvious that: $\Gamma_k \to \Gamma$ as $k \to 1$.

Let $\psi_k(x)$ be the k-analogue of the psi function, that is the logarithmic derivative of the Γ_k function:

$$\psi_k(x) = \frac{d}{dx} \ln \Gamma_k(x) = \frac{\Gamma_k'(x)}{\Gamma_k(x)}, \quad k > 0. \tag{6}$$

The function $\psi_k(x)$ has the following series representation (see [5,8,9])

$$\psi_k(x) = \frac{\ln k - \gamma}{k} - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{nk(x+nk)} \tag{7}$$

$$\psi_k^{(p)}(x) = (-1)^{p+1} p! \sum_{n=0}^{\infty} \frac{1}{(x+nk)^{p+1}}, \quad p \geq 1. \tag{8}$$

The motivation to introduce the function $\Gamma_k(x)$ is its connection with the symbol $(x)_{n,k}$ which appears in a variety of contexts see [5] and references there in. In the recent years there is a increasing interest about the function $\Gamma_k(x)$ see [5,6,8,9,11].

We recall the definition of completely and logarithmically completely monotonic functions, as well as two results given in [2,3] which we mention as Lemmas and are basic for the proof of our theorems.

A function $f(x)$ is said to be completely monotonic on an interval I, if it has derivatives of all orders on I and satisfies

$$(-1)^n f^{(n)}(x) \geq 0, (x \in I, n = 0, 1, 2, \ldots). \tag{9}$$

If the inequality (9) is strict, then $f(x)$ is said to be strictly completely monotonic on I. A theorem of Bernstein (see for example, [13]) states that $f(x)$ is completely monotonic if and only if $f(x) = \int_0^\infty e^{-xt} d\mu(t)$, where μ is a nonnegative measure on $[0,\infty)$ such that for all $x > 0$ the integral converges. A positive function $f(x)$ is said to be logarithmically completely monotonic on an interval I, if it satisfies

$$(-1)^n [\ln f(x)]^{(n)} \geq 0, (x \in I, n = 1, 2, \ldots). \tag{10}$$

If the inequality (10) is strict, then $f(x)$ is said to be strictly logarithmically completely monotonic.

Lemma 1.1. *[3] Let f'' be completely monotonic on $(0,\infty)$, then for $0 \leq s \leq 1$, the functions*

$$x \mapsto \exp\left(-\left(f(x+1) - f(x+s) - (1-s)f'\left(x + \frac{1+s}{2}\right)\right)\right)$$

$$x \mapsto \exp\left(f(x+1) - f(x+s) - \frac{1-s}{2}(f'(x+1) + f'(x+s))\right)$$

are logarithmically completely monotonic on $(0,\infty)$.

Lemma 1.2. *[2] If h' is completely monotonic on $(0,\infty)$, then $\exp(-h)$ is also completely monotonic on $(0,\infty)$.*

We also introduce the definition of the k-Riemann zeta function as an integral:

Definition 1.3. We define the function ζ_k as

$$\zeta_k(s) = \frac{1}{\Gamma_k(s)} \int_0^\infty \frac{t^{s-k}}{e^t - 1} dt, \quad s > k. \tag{11}$$

Note that when k tends to 1 we obtain the known Riemann Zeta function $\zeta(s)$.

In this paper we prove the completely monotonicity or the logarithmically completely monotonicity of some functions involving the functions $\Gamma_k(x)$ as well as inequalities for $\Gamma_k(x)$ and the k-Riemann zeta function. The obtained results are the k-analogues of results given by other authors see [2,3,4,7,10,12].

2. Main results for k -Gamma functions

Theorem 2.1. *For $0 \le s \le 1$, the functions*

$$x \mapsto \frac{\Gamma_k(x+s)}{\Gamma_k(x+1)} \exp\left((1-s)\psi_k\left(x+\frac{1+s}{2}\right)\right)$$

and

$$x \mapsto \frac{\Gamma_k(x+1)}{\Gamma_k(x+s)} \exp\left(-\frac{1-s}{2}\left(\psi_k(x+1)+\psi_k(x+s)\right)\right)$$

are logarithmically completely monotonic on $(0,\infty)$.

Proof. Applying Lemma 1.1 to $f(x)=\ln\Gamma_k(x)$, and using the fact that $f''(x)=\psi_k'(x)$ is completely monotonic on $(0,\infty)$ (see [8,9]), we obtain the desired result. □

Remark 2.2. Theorem 2.1 is the analogue of Corollary 2.4 proved in [3] for the function $\Gamma_q(x)$.

Theorem 2.3. *For positive x and $0 \le s \le 1$,*

$$\exp\left(\frac{1-s}{2}\left(\psi_k(x+1)+\psi_k(x+s)\right)\right) \le \frac{\Gamma_k(x+1)}{\Gamma_k(x+s)} \le \exp\left((1-s)\psi_k\left(x+\frac{1+s}{2}\right)\right).$$

Proof. Let $f_k(x)=\frac{\Gamma_k(x+s)}{\Gamma_k(x+1)}\exp\left((1-s)\psi_k\left(x+\frac{1+s}{2}\right)\right)$ and

$$g_k(x)=\frac{\Gamma_k(x+1)}{\Gamma_k(x+s)}\exp\left(-\frac{1-s}{2}\left(\psi_k(x+1)+\psi_k(x+s)\right)\right).$$

We know [8] that $\Gamma_k(x)=k^{\frac{x}{k}-1}\Gamma(\frac{x}{k})$ and $\psi_k(x)=\frac{1}{k}\ln k+\psi(\frac{x}{k})$, where $\psi(\frac{x}{k})=\partial_x(\ln\Gamma(\frac{x}{k}))$. For $x>0$ and $0\le s\le 1$, using Stirling's formula we are able to show that $\lim_{x\to\infty} f_k(x)=\lim_{x\to\infty} g_k(x)=1$ so the functions $f_k(x), g_k(x)$ decrease with respect to x and using Theorem 2.1 we obtain the desired inequalities. □

Remark 2.4. Theorem 2.3 is the analogue of Theorem 3.5 proved in [3] for the function $\Gamma_q(x)$.

For the proof of the following theorem it is necessary the following lemma which is mentioned in [2].

Lemma 2.5. *Let a_i and b_i ($i = 1,2,\dots,n$) be real numbers such that $0 < a_1 \leq \cdots \leq a_n$, $0 < b_1 \leq \cdots \leq b_n$ and $\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i$ for $k = 1,2,\dots,n$. If f is a decreasing and convex function on $\mathbb{R}$ then*

$$\sum_{i=1}^{n} f(b_i) \leq \sum_{i=1}^{n} f(a_i).$$

Theorem 2.6. *Let a_i and b_i ($i = 1,2,\dots,n$) be real numbers such that $0 < a_1 \leq \cdots \leq a_n$, $0 < b_1 \leq \cdots \leq b_n$ and $\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i$ for $k = 1,2,\dots,n$. Then the function*

$$x \mapsto \prod_{i=1}^{n} \frac{\Gamma_k(x+a_i)}{\Gamma_k(x+b_i)}$$

is completely monotonic on $(0,\infty)$.

Proof. Let $h(x) = \sum_{i=1}^{n}(\log\Gamma_k(x+b_i) - \log\Gamma_k(x+a_i))$. Then for $p \geq 0$ we have

$$\begin{aligned}
(-1)^p(h'(x))^{(p)} &= \sum_{i=1}^{n}(\psi_k^{(p)}(x+b_i) - \psi_k^{(p)}(x+a_i)) \\
&= (-1)^p \sum_{i=1}^{n}(-1)^{p+1}\sum_{n=0}^{\infty}\frac{p!}{(x+b_i+nk)^{p+1}} - (-1)^{p+1}\sum_{n=0}^{\infty}\frac{p!}{(x+a_i+nk)^{p+1}} \\
&= (-1)^{2p+1}p!\sum_{i=1}^{n}\sum_{n=0}^{\infty}\Big(\frac{1}{(x+b_i+nk)^{p+1}} - \frac{1}{(x+a_i+nk)^{p+1}}\Big).
\end{aligned}$$

Since the function $x \mapsto \frac{1}{x^{p+1}}$, $p \geq 0$ is decreasing and convex on $\mathbb{R}$, and using Lemma 2.5 we conclude that

$$\sum_{i=1}^{n}\left(\frac{1}{(x+b_i+nk)^{p+1}} - \frac{1}{(x+a_i+nk)^{p+1}}\right) \leq 0$$

and that implies that $(-1)^p(h'(x))^{(p)} \geq 0$ for $p \geq 0$. Hence h' is completely monotonic on $(0,\infty)$. By Lemma 1.2 we obtain that

$$\exp(-h(x)) = \prod_{i=1}^{n}\frac{\Gamma_k(x+a_i)}{\Gamma_k(x+b_i)}$$

is also completely monotonic on $(0,\infty)$. □

Remark 2.7. Theorem 2.6 is the k-analogue of Theorem 10 proved in [2].

Theorem 2.8. *The function* $f(x)=\frac{1}{[\Gamma_k(x+1)]^{\frac{1}{x}}}$ *is logarithmically completely monotonic on* $(k-1,\infty)$ *for* $k-1>0$.

Proof. Using Leibniz' rule

$$[u(x)v(x)]^{(n)}=\sum_{p=0}^{n}\binom{n}{p}u^{(p)}(x)v^{(n-p)}(x),$$

we obtain

$$\begin{aligned}[\ln f(x)]^{(n)}&=\sum_{p=0}^{n}\binom{n}{p}\Big(\frac{1}{x}\Big)^{(p)}\Big(-\ln\Gamma_k(x+1)\Big)^{(n-p)}\\&=-\frac{1}{x^{n+1}}\sum_{p=0}^{n}\binom{n}{p}(-1)^p p!x^{n-p}\psi_k^{(n-p-1)}(x+1)\\&\triangleq-\frac{1}{x^{n+1}}g(x)\end{aligned}$$

$$\begin{aligned}g^{'}(x)&=\sum_{p=0}^{n}\binom{n}{p}(-1)^p p!(n-p)x^{n-p-1}\psi_k^{(n-p-1)}(x+1)+\\&+\sum_{p=0}^{n}\binom{n}{p}(-1)^p p!x^{n-p}\psi_k^{(n-p)}(x+1)\\&=\sum_{p=0}^{n-1}\binom{n}{p}(-1)^p p!(n-p)x^{n-p-1}\psi_k^{(n-p-1)}(x+1)+\\&+\sum_{p=0}^{n}\binom{n}{p}(-1)^p p!x^{n-p}\psi_k^{(n-p)}(x+1)\\&=\sum_{p=0}^{n-1}\binom{n}{p}(-1)^p p!(n-p)x^{n-p-1}\psi_k^{(n-p-1)}(x+1)+\\&+x^n\psi_k^{(n)}(x+1)+\sum_{p=0}^{n-1}\binom{n}{p+1}(-1)^{p+1}(p+1)!x^{n-p-1}\psi_k^{(n-p-1)}(x+1)\\&=\sum_{p=0}^{n-1}\Big[\binom{n}{p}(n-p)-\binom{n}{p+1}(p+1)\Big](-1)^p p!x^{n-p-1}\psi_k^{(n-p-1)}(x+1)\\&+x^n\psi_k^{(n)}(x+1)=x^n\psi_k^{(n)}(x+1)\\&=x^n(-1)^{n+1}n!\sum_{p=0}^{\infty}\frac{1}{(x+pk)^{n+1}}\end{aligned}$$

We recall that the function $g(x)$ includes the function $\ln\Gamma_k(x+1)$ and its derivatives. Since $\Gamma_k(k)=1$ it is obvious that $g(k-1)=0$. So, if n is odd, then for $x>k-1$,

$$g^{'}(x)>0 \Rightarrow g(x)>g(k-1)=0 \Rightarrow (\ln f(x))^{(n)}<0 \Rightarrow (-1)^n(\ln f(x))^{(n)}>0.$$

If n is even, then for $x>k-1$,

$$g^{'}(x)<0 \Rightarrow g(x)<g(k-1)=0 \Rightarrow (\ln f(x))^{(n)}>0 \Rightarrow (-1)^n(\ln f(x))^{(n)}>0.$$

Hence,

$$(-1)^n(\ln f(x))^{(n)}>0$$

for all $x\in(k-1,\infty)$ and all integers $n\geq 1$, so the proof is complete. □

Remark 2.9. The above theorem is the k-analogue of Lemma 2.1 of [4] and the following theorem is k-analogue of Theorem 1.1 of [12].

Theorem 2.10. *Let s and t be two real numbers with $s\neq t, \alpha=\min\{s,t\}$ and $\beta\geq-\alpha$. For $x\in(-\alpha,\infty)$, we define*

$$h_{\beta,k}(x)=\begin{cases}\left[\frac{\Gamma_k(\beta+t)}{\Gamma_k(\beta+s)}\cdot\frac{\Gamma_k(x+s)}{\Gamma_k(x+t)}\right]^{\frac{1}{x-\beta}}, & x\neq\beta\\ \exp[\psi_k(\beta+s)-\psi_k(\beta+t)], & x=\beta\end{cases}$$

The function $h_{\beta,k}(x)$ is logarithmically completely monotonic on $(-\alpha,\infty)$.

Proof. It is assumed $s>t$ without loss the generality. For $x\neq\beta$, taking logarithm of the function $h_{\beta,k}(x)$ gives

$$\begin{aligned}\ln h_{\beta,q}(x)&=\frac{1}{x-\beta}\left[\ln\frac{\Gamma_k(\beta+t)}{\Gamma_k(\beta+s)}+\ln\frac{\Gamma_k(x+s)}{\Gamma_k(x+t)}\right]\\&=\frac{\ln\Gamma_k(x+s)-\ln\Gamma_k(\beta+s)}{x-\beta}-\frac{\ln\Gamma_k(x+t)-\ln\Gamma_k(\beta+t)}{x-\beta}\\&=\frac{1}{x-\beta}\int_\beta^x\psi_k(u+s)du-\frac{1}{x-\beta}\int_\beta^x\psi_k(u+t)du\\&=\frac{1}{x-\beta}\int_\beta^x[\psi_k(u+s)-\psi_k(u+t)]du=\end{aligned}$$

$$= \frac{1}{x-\beta} \int_{\beta}^{x} \int_{t}^{s} \psi_k^{'}(u+v)dvdu$$

$$\triangleq \frac{1}{x-\beta} \int_{\beta}^{x} \varphi_{k,s,t}(u)du \int_{0}^{1} \varphi_{k,s,t}((x-\beta)u+\beta)du$$

$$= \int_{0}^{1} \varphi_{k,s,t}((x-\beta)u+\beta)du.$$

Hence

$$[\ln h_{\beta,k}(x)]^{(p)} = \int_{0}^{1} u^p \varphi_{k,s,t}^{(p)}((x-\beta)u+\beta)du, \tag{12}$$

if $x=\beta$, formula (11) is also valid.

Since $\psi_k^{'}$ is completely monotonic (see [8,9]), $\varphi_{k,s,t}$ is completely monotonic on $(-t,\infty)$. This means that $(-1)^i[\varphi_{k,s,t}(x)]^{(i)} \geq 0$ holds on $(-t,\infty)$ for any nonnegative integer i.

Thus

$$(-1)^{(p)}[\ln h_{\beta,k}(x)]^{(p)} = \int_{0}^{1} u^p (-1)^p \varphi_{k,s,t}^{(p)}((x-\beta)u+\beta)du \geq 0$$

on $(-t,\infty)$ for $k \in \mathbb{N}$. The proof is complete. □

3. Main results for k-Riemann zeta function

Theorem 3.1. *Let $\zeta_k(s)$ be the k-Riemann zeta function defined by (11).Then the following inequality is valid*

$$(s+k)\cdot \frac{\zeta_k(s)}{\zeta_k(s+k)} \geq s\frac{\zeta_k(s+k)}{\zeta_k(s+2k)}, \quad s>k. \tag{13}$$

Proof. The proof of the theorem is based on the following consequence of Schwarz's inequality [10]: Let f,g be two nonnegative functions of a real variable and m,n real numbers such that integrals in (14) exist. Then

$$\int_a^b g(t)(f(t))^m dt \cdot \int_a^b g(t)(f(t))^n dt \geq \Big(\int_a^b g(t)(f(t))^{\frac{m+n}{2}} dt\Big)^2. \tag{14}$$

So, applying inequality (14) with $g(t)=\frac{1}{e^t-1}$, $f(t)=t$, $m=s-k$, $n=s+k$, $a=0$, $b+\infty$, we obtain

$$\int_0^\infty \frac{t^{s-k}}{e^t-1}dt \cdot \int_0^\infty \frac{t^{s+k}}{e^t-1}dt \geq \Big(\int_0^\infty \frac{t^s}{e^t-1}dt\Big)^2.$$

Further, using (11) we have

$$\zeta_k(s)\Gamma_k(s)\zeta_k(s+2k)\Gamma_k(s+2k) \geq (\zeta_k(s+k))^2(\Gamma_k(s+k))^2$$

and using the property $\Gamma_k(s+k) = s \cdot \Gamma_k(s)$ implies the desired result. □

Remark 3.2. For k tends to 1 we obtain Theorem 2.3 of [10].

Theorem 3.3. *Let $\zeta_k(u)$ be the k-Riemann zeta function. Then the inequality*

$$\frac{\Gamma_k\left(\frac{u}{p}+\frac{v}{q}\right)}{\Gamma_k^{\frac{1}{p}}(u)\cdot\Gamma_k^{\frac{1}{q}}(v)} \leq \frac{\zeta_k^{\frac{1}{p}}(u)\cdot\zeta_k^{\frac{1}{q}}(v)}{\zeta_k\left(\frac{u}{p}+\frac{v}{q}\right)}$$

holds, where $u > k$, $v > k$, $\frac{1}{p}+\frac{1}{q} = 1$ and $\frac{u}{p}+\frac{v}{q} > k$.

Proof. Using Holder's inequality for $p > 1$

$$\Big|\int_0^\infty f(t)\cdot g(t)dt\Big| \leq \Big(\int_0^\infty |f(t)|^p dt\Big)^{\frac{1}{p}}\Big(\int_0^\infty |g(t)|^q dt\Big)^{\frac{1}{q}}, \tag{15}$$

with $f(t) = \dfrac{t^{\frac{u-k}{p}}}{(e^t-1)^{\frac{1}{p}}}$ and $g(t) = \dfrac{t^{\frac{u-k}{q}}}{(e^t-1)^{\frac{1}{q}}}$. Using definition 1.3 we obtain the inequality

$$\Gamma_k\left(\frac{u}{p}+\frac{v}{q}\right)\cdot\zeta_k\left(\frac{u}{p}+\frac{v}{q}\right) \leq \Gamma_k^{\frac{1}{p}}(u)\cdot\Gamma_k^{\frac{1}{q}}(v)\cdot\zeta_k^{\frac{1}{p}}(u)\cdot\zeta_k^{\frac{1}{q}}(v),$$

which completes the proof. □

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