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ON A CLASS OF CONTROLLED FUNCTIONAL DIFFERENTIAL INCLUSIONS

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The aim of this paper is to establish the existence of solutions and some properties of solutions set for a class of functional differential equations with causal operator under assumption that the equation satisfies the Carathéodory type condition. Also, an application for an optimal control problem is given.

1. Introduction

The study of differential equations with causal operators or a non anticipative operator has been rapidly developing in the last years and some results are assembled in very recent monographs [2, 15]. The term of causal operators is adopted from engineering literature and the theory of these operators has the powerful quality of unifying ordinary differential equations, integro - differential equations, differential equations with finite or infinite delay, Volterra integral equations, and neutral functional equations, to name a few.

We considered the class $\mathcal{S}$ of all infinite-dimensional nonlinear M- input u, M- output y systems (p,g,Q) given by the following controlled nonlinear

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functional equation

$$\begin{cases} y'(t) = g(p(t), (\widehat{Q}y)(t), u(t)), \\ y|_{[-\sigma,0]} = y^0 \in C([-\sigma,0], \mathbb{R}^M), \end{cases} \tag{1}$$

where $\sigma \geq 0$ quantifies the memory of the system, $p(\cdot)$ is a perturbation term, $\widehat{Q}$ is a nonlinear causal operator, and $\mathcal{R}$ is the class of reference signals assumed to be $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^M)$ the set of all locally absolutely continuous and bounded with essentially bounded derivative. The control problem can be formulated in terms of a performance funnel $\mathcal{F}_\lambda : t \to \{e \in \mathbb{R}^M; \lambda(t)||e|| < 1\}$, where $\lambda \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}_+)$ is a prescribed function with $\lambda(t) > 0$ for all $t > 0$ and $\liminf_{t\to\infty} \lambda(t) > 0$. The objective is an $(\mathcal{R}, \mathcal{S})-$ universal feedback control which, when applied to any system of the admissible class $\mathcal{S}$ with any reference signal of class $\mathcal{R}$, ensure that the tracking error $e := y - r$, $r \in \mathcal{R}$, evolves within the performance funnel $\mathcal{F}_\lambda$, provided that the initial data is such that $e(0) = y^0(0) - r(0) \in \mathcal{F}_\lambda(0)$. The tracking objective can be achieved by a simple time-varying error feedback of the form

$$u(t) = -k(t)e(t), k(t) = \beta(\lambda(t)||e(t)||), \tag{2}$$

where $\beta : [0,1) \to \mathbb{R}_+$ is any continuous, unbounded injection, and $k : \mathbb{R}_+ \to \mathbb{R}_+$ given function. For more details see the papers [12, 20].
Writing $e'(t) = g(p(t), ((\widehat{Q}(e+r))(t), -\beta(\lambda(z(t))||e(t))||e(t)) - r'(t), z'(t) = 1$, and

$$f(t,x,w) := g(p(t),(e,z),w) = g(p(t),w,-\beta(\lambda(z)||e||)e) - r'(t), 1)$$

then we see that analysis of the behavior of a system $(p, g, Q) \in \mathcal{S}$ under control (2) constitutes a study of an initial-value problem of the form

$$x'(t) = f(t,x(t),(Qx)(t)),\ x|_{[-\sigma,0]} = x^0 \in C([-\sigma,0], \mathbb{R}^N), \tag{3}$$

where $N = M+1$, $x(t) := (e(t), z(t))$, $x^0 = (y^0 - r|_{[-\sigma,0]}, 0)$, and Q is an operator defined on $C([-\sigma,0], \mathbb{R}^N)$ by

$$(Qx)(t) = (\widehat{Q}(e,z))(t) := ((\widehat{Q}(e+r))(t).$$

In the context of the general class $\mathcal{S}$, we will investigate a nonsmooth generalization of (3) in which the nonsmooth counterpart of (3) takes the form $x'(t) \in F(t,x(t),(Qx)(t))$, with appropriate initial data, where F is a multifunction.

The aim of this paper is to give an existence result for the following initial value problem

$$x'(t) \in F(t,x(t),(Qx)(t)),\ x|_{[-\sigma,0]} = x^0 \in C([-\sigma,0], \mathbb{R}^N). \tag{4}$$

Also, we study the properties of the set of the solutions for (4), and give an application for an optimal control problem.

2. Preliminaries

Let $\mathbb{R}^N$ be the N-dimensional Euclidian space with norm $||\cdot||$. For $x \in \mathbb{R}^N$ and $r > 0$ let $B_r(x) := \{y \in \mathbb{R}^N; ||y-x|| < r\}$ be the open ball centered at x with radius r, and let $B_r[x]$ be its closure. By $K_c(\mathbb{R}^N)$ we will denote the set of all nonempty compact convex subsets of $\mathbb{R}^N$. If $I = [0,b) \subset \mathbb{R}$, $b \in (0,\infty]$, then we denote by $\mathcal{C}(I,\mathbb{R}^N)$ the Banach space of continuous functions from I into $\mathbb{R}^N$. If σ is a positive number then we put $\mathcal{C}_\sigma := \mathcal{C}([-\sigma,0],\mathbb{R}^N)$ and let $L^\infty_{loc}(I,\mathbb{R}^N)$ denote the space of measurable locally essentially bounded functions $x(\cdot) : I \to \mathbb{R}^N$.

Definition 2.1. Let $\sigma \geq 0$. An operator $Q : \mathcal{C}([-\sigma,b),\mathbb{R}^N) \to L^\infty_{loc}([0,b),\mathbb{R}^M)$ is a causal operator if the following property holds:

(Q) *for each* $\tau \in [0,b)$ *and for all* $x(\cdot), y(\cdot) \in \mathcal{C}([-\sigma,b),\mathbb{R}^N)$, *with* $x(t) = y(t)$ *for every* $t \in [-\sigma,\tau]$, *we have* $(Qx)(t) = (Qy)(t)$ *for a.e.* $t \in [0,\tau]$.

For concrete applications and examples which serve to illustrate that the class of causal operators is very large, we refer to the monographs [2–4, 9] and [15].

We consider the initial-valued problem with causal operator

$$x'(t) \in F(t,x(t),(Qx)(t)), \; x|_{[-\sigma,0]} = \varphi \in \mathcal{C}_\sigma, \tag{5}$$

under the following assumptions:

(h_1) Q is continuous;

(h_2) for each $r > 0$ and each $\tau \in (0,b)$, there exists $M > 0$ such that, for all $x(\cdot) \in \mathcal{C}([-\sigma,b),\mathbb{R}^N)$ with $\sup_{-\sigma \leq t \leq \tau} ||x(t)|| \leq r$, we have $||(Qx)(t)|| \leq M$ for a.e. $t \in [0,\tau]$;

(h_3) $F : [-\sigma,b) \times \mathbb{R}^N \times \mathbb{R}^M \to K_c(\mathbb{R}^N)$ is a Carathéodory function, that is:

(a) for a.e. $t \in [-\sigma,b)$, $F(t,\cdot,\cdot)$ is upper-semicontinuous,

(b) for each fixed $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$, $F(\cdot,x,y)$ has a measurable selection,

(c) for every bounded $B \subset \mathbb{R}^N \times \mathbb{R}^M$, there exists $\mu(\cdot) \in L^1_{loc}([-\sigma,b),\mathbb{R}_+)$ such that

$$|F(t,x,y)| := \sup\{||z||; z \in F(t,x,y)\} \leq \mu(t)$$

for a.e. $t \in [-\sigma,b)$ and all $(x,y) \in B$.

By a solution of (5) on $[-\sigma,T]$, we mean a function $x(\cdot) \in \mathcal{C}([-\sigma,T],\mathbb{R}^N)$, with $T \in (0,b]$ and $x|_{[-\sigma,0]} = \varphi$, such that $x|_{[0,T]}$ is absolutely continuous and satisfies (5) for a.e. $t \in [0,T]$.

We remark that, $x(\cdot) \in \mathcal{C}([-\sigma,T],\mathbb{R}^N)$ is a solution for (5) on $[-\sigma,T]$, if and only if $x|_{-\sigma,0]} = \varphi$ and

$$x(t) \in \varphi(0) + \int_0^t F(s,x(s),(Qx)(s))ds \text{ for } t \in (0,T].$$

The existence of solutions for the Cauchy problem (3) has been studied in [12], when $Q : \mathcal{C}([-\sigma,b),\mathbb{R}^N) \to L^\infty_{loc}([0,b),\mathbb{R}^M)$ is a locally Lipschitz operator. The existence of solutions for the following Cauchy problem

$$x'(t) = (Qx)(t), \; x(0) = x_0,$$

has been studied in [6], in the case when $Q : \mathcal{C}([0,b),E) \to \mathcal{C}([0,b),E)$ is a locally Lipschitz operator and E is a Banach space. Also, for other results see [7, 13, 16, 19].

3. Existence of solutions

In this section, we present an existence result of the solutions for Cauchy problem (5), under conditions $(h_1)-(h_3)$.

Theorem 3.1. *Assume that the conditions $(h_1)-(h_3)$ hold. Then, for every $\varphi \in \mathcal{C}_\sigma$, there exists a solution $x(\cdot) : [-\sigma,T] \to \mathbb{R}^N$ for Cauchy problem (5) on some interval $[-\sigma,T]$ with $T \in (0,b)$.*

Proof. Let $\delta > 0$ be any number and let $r := ||\varphi||_\sigma + \delta$. If $x^0(\cdot) \in \mathcal{C}([-\sigma,b),\mathbb{R}^N)$ denotes the function defined by

$$x^0(t) = \begin{cases} \varphi(t), & \text{for } t \in [-\sigma,0) \\ \varphi(0), & \text{for } t \in [0,b), \end{cases}$$

then $\sup_{0\leq t<b} ||x^0(t)|| \leq r$. Therefore, by (h_2) , we have $||(Qx^0)(t)|| \leq M$ for a.e. $t \in [0,b)$. Since F is a Carathéodory function, there exists $\mu(\cdot) \in L^1_{loc}([0,b),\mathbb{R}_+)$ such that

$$|F(t,x,y)| \leq \mu(t) \text{ for a.e. } t \in [0,b) \text{ and } (x,y) \in B_r(0) \times B_M(0).$$

We construct a sequence $\{x_m\}_{m\geq 1}$ of continuous functions from $[-\sigma, T]$ to $\mathbb{R}^N$ as follows. Let $m \in \mathbb{N}$. For $i = 1,2,...,m$, define $x_m^i : [-\sigma, \frac{iT}{m}] \to \mathbb{R}^N$ by recursive procedure. For $t \in [-\sigma, \frac{T}{m}]$ we define $x_m^1(t) = x^0(t)$. Since $x_m^1 : [0, \frac{T}{m}] \to \mathbb{R}^N$ and $Qx_m^1 : [0, \frac{T}{m}] \to \mathbb{R}^M$ are measurable functions then, by (h_3) and [14, Theorem 1.3.5], [5, Proposition 3.5], there exists a measurable function $\widetilde{g}_m^1(\cdot)$ such that $\widetilde{g}_m^1(t) \in F(t, x_m^1(t), (Qx_m^1)(t))$ for a.e. $t \in [0, \frac{T}{m}]$. For $t \in (\frac{T}{m}, \frac{2T}{m}]$ we define

$$x_m^2(t) = \begin{cases} x_m^1(t), & t \in [-\sigma, \frac{T}{m}] \\ \varphi(0) + \int_0^{t-T/m} \widetilde{g}_m^1(s)ds, & t \in (\frac{T}{m}, \frac{2T}{m}]. \end{cases}$$

Let us assume that $x_m^i(t)$ is defined on $[-\sigma, \frac{iT}{m}]$, $1 \leq i < m$. Then for $t \in (\frac{iT}{m}, \frac{(i+1)T}{m}]$ we define

$$x_m^{i+1}(t) = \begin{cases} x_m^i(t), & t \in [-\sigma, \frac{iT}{m}] \\ x_m^i(\frac{iT}{m}) + \int_{(i-1)T/m}^{t-T/m} \widetilde{g}_m^i(s)ds, & t \in (\frac{iT}{m}, \frac{(i+1)T}{m}], \end{cases}$$

where $\widetilde{g}_m^i(\cdot)$ is a measurable function such that $\widetilde{g}_m^i(t) \in F(t, x_m^i(t), (Qx_m^i)(t))$ for a.e. $t \in (\frac{(i-1)T}{m}, \frac{iT}{m}]$. In fact, if for $i = 1,2,...,m$, we define $g_m^i(\cdot) : [0,T] \to \mathbb{R}^N$ by $g_m^1(t) = \widetilde{g}_m^1(t)$ if $t \in [0, \frac{T}{m}]$, and

$$g_m^i(t) = \begin{cases} g_m^{i-1}(t), & t \in [0, \frac{(i-1)T}{m}] \\ \widetilde{g}_m^i(t), & t \in (\frac{(i-1)T}{m}, \frac{iT}{m}], \end{cases}$$

for $2 \leq i \leq m$, then $g_m^i(\cdot)$ are measurable functions such that

$$g_m^i(t) \in F(t, (Qx_m^i)(t)) \text{ for a. e. } t \in [0, \frac{iT}{m}] \text{ and } i = 1,2,...,m.$$

Moreover, we have that

$$x_m^i(t) = \begin{cases} x_m^{i-1}(t), & t \in [-\sigma, \frac{(i-1)T}{m}] \\ \varphi(0) + \int_0^{t-T/m} g_m^{i-1}(s)ds, & t \in (\frac{(i-1)T}{m}, \frac{iT}{m}], \end{cases}$$

for $i = 2,3,...,m$. Further, we observe that $\left\|x_m^1(t)\right\| \leq r$ for all $t \in [0, \frac{T}{m}]$ and so, by (h_2), $\left\|(Qx_m^1)(t)\right\| \leq M$ for a.e. $t \in [0, \frac{T}{m}]$. If for $i \in \{1,2,...,p\}$, $p < m$, we assume that $\left\|x_m^i(t)\right\| \leq r$ for all $t \in [0, \frac{iT}{m}]$ and $\left\|(Qx_m^i)(t)\right\| \leq M$ for a.e. $t \in [0, \frac{iT}{m}]$ then, since

$$||x_m^{p+1}(t) - \varphi(0)|| \leq \int_0^{t-T/m} ||g_m^{i-1}(s)||ds \leq \int_0^{t-T/m} \mu(s)ds < \delta,$$

we have

$$||x_m^{p+1}(t)|| \leq ||x_m^{p+1}(t) - \varphi(0)|| + ||\varphi(0)|| < r,$$

and so, by (h_2), $||(Qx_m^{p+1})(t)|| \leq M$ for a.e. $t \in [0, \frac{(p+1)T}{m}]$. Therefore, by induction on i, we obtain that $\left\|x_m^i(t)\right\| \leq r$ for all $t \in [0, \frac{iT}{m}]$ and $\left\|(Qx_m^i)(t)\right\| \leq M$ for a.e. $t \in [0, \frac{iT}{m}]$, $i = 1, 2, ..., m$. For notational convenience, we write $g_m = g_m^m$ and $x_m = x_m^m$. By causality of Q, the sequence $\{x_m\}_{m\geq 1}$ so constructed has the property that, for each $m \geq 1$,

$$x_m(t) = \begin{cases} x^0(t), & t \in [-\sigma, \frac{T}{m}] \\ \\ \varphi(0) + \int_0^{t-T/m} g_m(s)ds, & t \in (\frac{T}{m}, T]. \end{cases}$$

Moreover, for all $m \geq 1$, $||x_m(t)|| \leq r$ for all $t \in [-\sigma, T]$ and so $\{x_m\}_{m\geq 1}$ is uniformly bounded. Next, we show that the sequence $\{x_m\}_{m\geq 1}$ is equicontinuous. Since on the closed interval $[0, T]$ the function $t \mapsto \int_0^t \mu(s)ds$ is uniformly continuous, then for any $\varepsilon > 0$ there exists $\eta > 0$ such that for all $t, s \in [0, T]$ with $|t - s| < \eta$ we have $\left|\int_s^t \mu(\tau)d\tau\right| < \varepsilon$. Let $m \geq 1$, $t, s \in [0, T]$ with $|t - s| < \eta$. Without loss of generality, we can assume that $s \leq t$. If $0 \leq s \leq t \leq \frac{T}{m}$, then $||x_m(t) - x_m(s)|| = 0$. If $0 \leq s \leq \frac{T}{m} \leq t$, then $t - \frac{T}{m} < \eta$ and so

$$||x_m(t) - x_m(s)|| = ||x_m(t) - \varphi(0)|| \leq \int_0^{t-T/m} \mu(s)ds < \varepsilon.$$

If $0 \leq \frac{T}{m} \leq s \leq t$, then $|(t - \frac{T}{m}) - (s - \frac{T}{m})| = |t - s| < \eta$ and so

$$||x_m(t) - x_m(s)|| \leq \int_{s-T/m}^{t-T/m} \mu(s)ds < \varepsilon.$$

Recalling that $x_m|_{[-\sigma,0]} = \varphi$ for all $m \geq 1$, we conclude that the sequence $\{x_m\}_{m\geq 1}$ is equicontinuous. Hence by the Ascoli-Arzela theorem and extracting a subsequence if necessary, we may assume that the sequence $\{x_m\}_{m\geq 1}$ converge uniformly on $[0, T]$ to a function $x(\cdot)$ which is absolutely continuous and satisfies $x(t) = \varphi(t)$ for all $t \in [-\sigma, 0]$. Now, by (h_1), we have that $\lim_{m\to\infty} Qx_m = Qx$ in $L^\infty([0, T], \mathbb{R}^M)$ and so $\lim_{m\to\infty} (Qx_m)(t) = (Qx)(t)$ for a.e. $t \in [0, T]$. Since

$$x_m(t) \in \varphi(0) + \int_0^{t-T/m} F(s, x_m(s), (Qx_m)(s))ds$$

holds, we have by upper semicontinuity of $F(t, \cdot, \cdot)$ and Fatou's Lemma [1, Theorem 8.6.7] that

$$\begin{aligned} x(t) &\in \varphi(0)+\limsup_{m\to\infty}\int_0^{t-T/m} F(s,x_m(s),(Qx_m)(s))ds \\ &\subset \varphi(0)+\int_0^t \limsup_{m\to\infty} F(s,x_m(s),(Qx_m)(s))ds \\ &\quad +\limsup_{m\to\infty}\int_t^{t-T/m} F(s,x_m(s),(Qx_m)(s))ds \\ &\subset \varphi(0)+\int_0^t F(s,x_m(s),(Qx_m)(s))ds. \end{aligned}$$

Therefore, since $x(\cdot)$ is absolutely continuous on $[0,T]$, we conclude that $x'(t)\in F(t,x(t),(Qx)(t))$ a.e. on $[0,T]$. □

Theorem 3.2. *Assume that the conditions $(h_1)-(h_3)$ hold. Then, every solution of Cauchy problem (5) can be extended to a maximal solution $x(\cdot):[-\sigma,T)\to\mathbb{R}^N$. Moreover, if $x(\cdot)$ is bounded, then $T=b$.*

Proof. Let $x(\cdot):[-\sigma,T)\to\mathbb{R}^N$ be a solution of Cauchy problem (5) and let

$\mathcal{M}=\{(\tau,v);T\le\tau\le b,\ v(\cdot):[-\sigma,\tau)\to\mathbb{R}^N$ is a solution of (5) with $v|_{[-\sigma,0]}=\varphi\}$.

Then $\mathcal{M}$ is nonempty and we can define a partial order $\preccurlyeq$ on $\mathcal{M}$ by $(\tau_1,v_1)\preccurlyeq(\tau_2,v_2)$ if and only if $\tau_1\le\tau_2$ and $v_2(t)=v_1(t)$ for all $t\in[-\sigma,\tau_1)$. By Zorn' lemma, it follows that $\mathcal{M}$ contains at least one maximal element. Next, assume that $x(\cdot):[-\sigma,T)\to\mathbb{R}^N$ is a bounded maximal solution of (5) existing on $[-\sigma,T)$, $0<T<b$. Also, we suppose, by contradiction, that the value of $T<b$. Since $x(\cdot)$ is bounded and $x'(t)\in F(t,x(t),(Qx)(t))$ a.e. on $[0,T)$, then it follows that $x'(\cdot)$ is essentially bounded on $[0,T)$ Therefore, $x(\cdot)$ is uniformly continuous and so extend to a continuous function $x(\cdot):[-\sigma,T]\to\mathbb{R}^N$. Further on, we consider the Cauchy problem

$$\begin{cases} y'(t)\in F(t+T,y(t+T),(Qy(\cdot-T)(t+T)),0\le t<b-T \\ y|_{[-(\sigma+T),0]}=\psi \end{cases} \tag{6}$$

where $\psi(\cdot)\in\mathcal{C}_{\sigma+T}$ is defined by $\psi(s)=x(s+T)$, for all $s\in[-(\sigma+T),0]$. By Theorem 3.1, there exists a solution $v(\cdot):[-(\sigma+T),\tau)\to\mathbb{R}^N$ of Cauchy problem (6), where $\tau\in(0,b-T]$. It follows that $z(\cdot):[-\sigma,T+\tau]\to\mathbb{R}^N$, given by

$$z(t)=\begin{cases} x(t), & \text{for } t\in[-\sigma,T] \\ y(t-T), & \text{for } t\in[T,T+\tau], \end{cases}$$

Since, for a.e. $t \in [T, T+\tau]$, we have that

$$z'(t) = y'(t-T) \in F(t, x(t), (Qy(\cdot - T)(t)) = F(t, z(t), (Qz)(t))$$

it follows that $z(\cdot)$ is a solution of Cauchy problem (5) and a proper right extension of solution $x(\cdot)$. This contradicts the maximality of $x(\cdot)$. Therefore, $T = b$. □

Example 3.3. Let us consider the operator Q in the form

$$(Qx)(t) = \int_{t-\sigma}^{t} k(t,s)x(s)ds, \ 0 \leq t < b,$$

where $x(\cdot) \in \mathcal{C}([-\sigma, b), \mathbb{R}^N)$, and $k(t,s) : \Delta \to \mathbb{R}_+$, is a measurable kernel, integrable with respect to s for a.e. $t \in [0,b)$ with $t - \sigma \leq s < t < b$, such that

$$\underset{t \in [0,b)}{ess\sup} \int_{t-\sigma}^{t} |k(t,s)|ds \leq M < \infty.$$

Here $\Delta := \{(t,s); 0 \leq t < b, \ t-\sigma \leq s < t < b\}$. Then Q is a causal operator from $\mathcal{C}([-\sigma,b),\mathbb{R}^N)$ into $L^\infty([0,T],\mathbb{R}^N)$ and it satisfies the properties (h_1) and (h_2) (see [2]). Therefore, if the multifunction $F : [-\sigma, b) \times \mathbb{R}^N \times \mathbb{R}^N \to K_c(\mathbb{R}^N)$ satisfies the property (h_3), then the following integro-differential equation

$$x'(t) \in F\left(t, x(t), \int_{t-\sigma}^{t} k(t,s)x(s)ds\right), \ 0 \leq t < b,$$

has a maximally defined solution with maximal interval of existence $[0,b)$, $b > 0$.

4. Properties of the solutions set

In the following, for a fixed $\varphi \in \mathcal{C}_\sigma$ and a compact set $K \subset \mathbb{R}^N$, by $\mathcal{S}_T(\varphi, K)$ we denote the set of solutions $x(\cdot)$ of Cauchy problem (5) on $[-\sigma, T]$ with $T \in (0,b)$ and such that $x(t) \in K$ for each $t \in [-\sigma, T]$. By $\mathcal{A}_T(\varphi, K)$ we denote the attainable set; that is, $\mathcal{A}_T(\varphi, K) = \{x(T); x(\cdot) \in \mathcal{S}_T(\varphi, K)\}$.

Theorem 4.1. *Assume that the conditions $(h_1) - (h_3)$ hold. Then, for every $\varphi \in \mathcal{C}_\sigma$, $\mathcal{S}_T(\varphi, K)$ is compact set in $\mathcal{C}([-\sigma, T], \mathbb{R}^N)$.*

Proof. We consider a sequence $\{x_m\}_{n \geq 1}$ in $\mathcal{S}_T(\varphi, K)$ and we shall show that this sequence contains a subsequence which converges, uniformly on $[-\sigma, T]$, to a solution $x(\cdot) \in \mathcal{S}_T(\varphi, K)$. Since K is a bounded set, then there exists $r > 0$ such that $K \subset B_r(0)$. By (h_2), there exists $M > 0$ such that $||(Qx)(t)|| \leq M$ for every

$x(\cdot) \in \mathcal{C}([-\sigma,T],E)$ with $\sup_{-\sigma\le t\le T}||x(t)|| < r$. Since F is a Carathéodory function, there exists $\mu(\cdot) \in L^1([0,T],\mathbb{R}_+)$ such that

$$|F(t,x,y)| \le \mu(t) \text{ for a.e. } t \in [0,T] \text{ and } (x,y) \in B_r(0) \times B_M(0).$$

Since $x_m|_{[-\sigma,0]} = \varphi$, we have that $x_m \to \varphi$ uniformly on $[-\sigma,0]$. On the other hand, since $x_m'(t) \in F(t,x_m(t),(Qx_m)(t))$ for a.e. $t \in [0,T]$ and all $m \ge 1$, we have that

$$x_m(t) - x_m(s) \in \int_s^t F(\tau,x_m(s),(Qx_m)(\tau))d\tau \text{ for } s,t \in [0,T]$$

and hence

$$||x_m(t) - x_m(s)|| \le \left|\int_s^t \mu(\tau)d\tau\right| \le r|t-s| \text{ for } s,t \in [0,T].$$

Therefore, $\{x_m\}_{m\ge1}$ is equicontinuous on $[0,T]$. Since, for every $m \ge 1$, x_m satisfies the same initial condition, $x_m(0) = \varphi(0)$, then we deduce that $\{x_m\}_{m\ge1}$ is uniformly bounded. Further, by the Ascoli-Arzela theorem and extracting a subsequence if necessary, we may assume that the sequence $\{x_m\}_{m\ge1}$ converges uniformly on $[0,T]$ to an absolutely continuous function x. Moreover, $x(t) \in K$ for any $t \in [0,T]$. If we extend $x(\cdot)$ to $[-\sigma,T]$ such that $x|_{[-\sigma,0]} = \varphi$ then clearly $x_m \to x$ uniformly on $[-\sigma,T]$. Moreover, we have that $x(t) \in K$ for each $t \in [-\sigma,T]$. Now, by (h_1), we have that $\lim\limits_{n\to\infty} Qx_m = Qx$ in $L^\infty([0,T],\mathbb{R}^M)$. Therefore, $\lim\limits_{m\to\infty}(Qx_m)(t) = (Qx)(t)$ for a.e. $t \in [0,T]$ and so, by the upper semicontinuity of $F(t,\cdot)$ and Fatou's Lemma [1, Theorem 8.6.7], we have that

$$\begin{aligned} x(t) - x(s) &\in \limsup_{n\to\infty} \int_s^t F(\tau,x_m(\tau),(Qx_m)(\tau))d\tau \\ &\subset \int_s^t \limsup_{m\to\infty} F(\tau,x_m(\tau),(Qx_m)(\tau))d\tau \\ &\subset \int_0^t F(\tau,x_m(\tau),(Qx_m)(\tau))d\tau, \end{aligned}$$

for $t,s \in [0,T]$. Therefore, we conclude that $x(\cdot)$ is absolutely continuous on $[0,T]$, $x'(t) \in F(t,x(t),(Qx)(t))$ for a.e. $t \in [0,T]$ and $u|_{[-\sigma,0]} = \varphi$. □

Theorem 4.2. *Assume that the conditions $(h_1) - (h_3)$ hold. Then, the multifunction $S_T(\cdot,K) : \mathcal{C}_\sigma \to \mathcal{C}([-\sigma,T],\mathbb{R}^N)$ is upper semicontinuous.*

Proof. Let $\mathcal{K}$ be a closed set in $\mathcal{C}([-\sigma,T],\mathbb{R}^N)$ and $\mathcal{B} = \{\varphi \in \mathcal{C}_\sigma; \mathcal{S}_T(\varphi,K) \cap \mathcal{K} \neq \emptyset\}$. We must show that $\mathcal{B}$ is closed in $\mathcal{C}_\sigma$. For this, let $\{\varphi_m\}_{m\ge1}$ be a sequence in $\mathcal{B}$ such that $\varphi_m \to \varphi$ on $[-\sigma,0]$. Further, for any $m \ge 1$, let $x_m(\cdot) \in$

$\mathcal{S}_T(\varphi_m, K) \cap \mathcal{K}$. Then, $x_m = \varphi_m$ on $[-\sigma, 0]$ for all $m \geq 1$, and $x_m(t) \in \varphi_m(0) + \int_0^t F(s, x_m(s), (Qx_m)(s))ds$ for all $t \in (0, T]$. As in proof of Theorem 4.1 we can show that $\{x_m\}_{n\geq 1}$ is equicontinuous and equibounded on $[0, T]$. Therefore, by the Ascoli-Arzela theorem and extracting a subsequence if necessary, we may assume that the sequence $\{x_m\}_{m\geq 1}$ converges uniformly on $[-\sigma, T]$ to a continuous function $x(\cdot) \in \mathcal{K}$. Since $x(t) = \lim_{m\to\infty} x_m(t) \in \varphi(0) + \int_0^t F(s, x_m(s), (Qx)(s))ds$ for all $t \in [0, T]$, we deduce that $x(\cdot) \in \mathcal{S}_T(\varphi, K) \cap \mathcal{K}$. This prove that $\mathcal{B}$ is closed and so $\varphi \mapsto \mathcal{S}_T(\varphi, K)$ is upper semicontinuous. □

Corollary 4.3. *Assume that the conditions $(h_1) - (h_3)$ hold. Then, for any $\varphi \in \mathcal{C}_\sigma$ and any $t \in [0, T]$ the attainable set $\mathcal{A}_t(\varphi, K)$ is compact in $\mathcal{C}([-\sigma, t], \mathbb{R}^N)$ and the multifunction $(t, \varphi) \mapsto \mathcal{A}_t(\varphi, K)$ is jointly upper semicontinuous.*

5. An optimal control problem

In this section, we consider the following control problem:

$$\begin{cases} x'(t) \in F(t, x(t), (Qx)(t)) & \text{for a.e. } t \in [0, T] \\ x|_{[-\sigma,0]} = \varphi \\ \textit{minimize } \xi(x(T)), \end{cases} \tag{7}$$

where $\xi : \mathbb{R}^N \to \mathbb{R}$ is a given function.

It is well known that the initial value problem (5) appears in the theory of control systems having equations of motion of the form

$$x'(t) = f(t, x(t), (Qx)(t), u(t)), \ \ x|_{[-\sigma,0]} = \varphi, \tag{8}$$

where the admissible control function u can be chosen as any measurable function with value at time t in a preassigned set $U(t, x(t)) \subset \mathbb{R}^k$. The problem is to determine an admissible control u^*, which causes the system (8) to follow an admissible trajectory $x^* = x(t, u^*)$ that minimizes the cost function $J(x, u) = \xi(x(T))$. Such controls u^* are called optimal controls. If we put

$$F(t, x(t), (Qx)(t)) = \{f(t, x(t), (Qx)(t), u(t)); x(t) \in U(t, x(t))\}$$

then, under suitable assumptions, a function $x(\cdot)$ is a solution of (5) if and only if $x(\cdot)$ is a solution of (8) for some admissible control u (see [4, 8, 10]).

Theorem 5.1. *Let $\mathcal{K}_0$ be a compact set in $\mathcal{C}_\sigma$ and let $\xi : \mathbb{R}^N \to \mathbb{R}$ be a lower semicontinuous function. If the conditions $(h_1) - (h_3)$ hold, then the control problem (7) has an optimal solution; that is, there exists $\varphi_0 \in \mathcal{K}_0$ and $x_0(\cdot) \in \mathcal{S}_T(\varphi_0, K)$ such that*

$$\xi(x_0(T)) = \inf\{\xi(x(T)); x(\cdot) \in \mathcal{S}_T(\varphi, K), \varphi \in \mathcal{K}_0\}.$$

Proof. From Corollary 4.3 we deduce that the attainable set $\mathcal{A}_T(\varphi, K)$ is upper semicontinuous. Then by [11, Corollary 1.2.20] the set

$$\mathcal{A}_T(\mathcal{K}_0) = \{x(T); x(\cdot) \in \mathcal{S}_T(\varphi, K), \varphi \in \mathcal{K}_0\} = \cup_{\varphi \in \mathcal{K}_0} \mathcal{A}_T(\varphi, K)$$

is compact in $\mathbb{R}^N$ and so, since ξ is lower semicontinuous, there exists $\varphi_0 \in \mathcal{K}_0$ such that $\xi(x_0(T)) = \inf\{\xi(x(T)); x(\cdot) \in \mathcal{S}_T(\varphi, K), \varphi \in \mathcal{K}_0\}$. □

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