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COUPLED COINCIDENCE POINT THEOREMS FOR MIXED MONOTONE NONLINEAR OPERATOR IN PARTIALLY ORDERED *G*-METRIC SPACES

SEYED H. RASOULI - M. BAHRAMPOUR

In this paper we present some coupled coincidence and coupled common fixed point theorems for mixed g-monotone mappings in partially ordered G-metric spaces.

1. Introduction

In a recent paper Bhaskar and Lakshmikantham [7] introduced mixed monotone operator and established coupled fixed point theorems for mixed monotone operators in partially ordered metric spaces. After their work, many authors have been studied about coupled fixed point [2,4,5,10,11,13,16,1718,19]. In [12] Lakashmikantham and Ciric introduced the concept of a mixed g-monotone mappings and proved coupled coincidence and coupled common fixed point theorems in partially ordered metric spaces. After this work some authors considered coupled coincidence and common fixed point theorems in their works [6,8]. Some authors generalized the concept of metric spaces. Mustafa and Sims [14] introduced the notion of G-metric. Some authors studied some fixed point theorems in partially ordered G-metric space [1,3,9,15].

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Mustafa and Simis [14] introduced following definition and obtained the following results.

Definition 1.1. ([14]) Let X be a non-empty set, $G : X \times X \times X \to \mathbb{R}_+$ be a function satisfying the following properties:
(G1) $G(x,y,z) = 0$ if $x = y = z$.
(G2) $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$.
(G3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$.
(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all three variables).
(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$, (rectangle inequality).
Then the function G is called a generalized metric, or, more specially, a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 1.2. ([14]) Let (X,G) be a G-metric space, and let (x_n) be a sequence of points of X. We say that (x_n) is G-convergent to $x \in X$ if
$\lim_{n,m\to\infty} G(x,x_n,x_m) = 0$, that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x,x_n,x_m) < \varepsilon$, for all $n,m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim x_n = x$.

Proposition 1.3. ([14]) *Let (X,G) be a G-metric space. The following are equivalent:*
(1) (x_n) is G-convergent to x.
(2) $G(x_n,x_n,x) \to 0$ as $n \to +\infty$.
(3) $G(x_n,x,x) \to 0$ as $n \to +\infty$.

Definition 1.4. ([14]) Let (X,G) be a G-metric space. A sequence (x_n) is called a G-Cauchy sequence if, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \varepsilon$ for all $m,n,l \geq N$, that is, $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to +\infty$.

Proposition 1.5. ([14]) *Let (X,G) be a G-metric space. Then the following are equivalent*
(1) the sequence (x_n) is G-Cauchy
(2) for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n,x_m,x_m) < \varepsilon$, for all $m,n \geq N$.

Proposition 1.6. ([14]) *Let (X,G) be a G-metric space. A mapping $f : X \to X$ is G-continuous at $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever (x_n) is G-convergent to x, $(f(x_n))$ is G-convergent to $f(x)$.*

Proposition 1.7. ([14]) *Let (X,G) be a G-metric space. Then, the function $G(x,y,z)$ is jointly continuous in all three of its variables.*

Proposition 1.8. ([14]) *Let (X,G) be a G-metric space, then for any $x,y,z,a \in X$ it follows*
(1) if $G(x,y,z) = 0$ then $x = y = z$,

(2) $G(x,y,z) \leq G(x,x,y) + G(x,x,z)$,
(3) $G(x,y,y) \leq 2G(y,x,x)$,
(4) $G(x,y,z) \leq G(x,a,z) + G(a,y,z)$.

Definition 1.9. ([14]) A G-metric space (X,G) is called G-complete if every G-Cauchy sequence is G-convergent in (X,G).

Definition 1.10. ([9]) Let (X,G) be a G-metric space. A mapping $F : X \times X \to X$ is said to be continuous if for any two G-convergent sequences (x_n) and (y_n) converging to x and y respectively, $\{F(x_n,y_n)\}$ is G-convergent to $F(x,y)$.

Bhaskar and Lakshmikantham in [7] introduced the concept of a mixed monotone property and following definitions.

Definition 1.11. ([7]) Let $(X,\leq)$ be a partially ordered set and $F : X \times X \to X$. We say that F has the mixed monotone property if $F(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is , for any $x,y \in X$,

$$x_1,x_2 \in X, \quad x_1 \leq x_2 \Rightarrow F(x_1,y) \leq F(x_2,y)$$

and

$$y_1,y_2 \in X, \quad y_1 \leq y_2 \Rightarrow F(x,y_1) \geq F(x,y_2).$$

Definition 1.12. ([7]) An element $(x,y) \in X \times X$ is said to be a coupled fixed point of the mapping F if

$$F(x,y) = x \quad and \quad F(y,x) = y.$$

Lakshmikantham and Ciric in [12] introduced the notion of mixed g- monotone and coupled coincidence point.

Definition 1.13. ([12]) Let $(X,\leq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. The map F is said to have mixed g-monotone property if F is monotone g-non-decreasing in the first argument and is monotone g-non-increasing in its second argument, that is, for any $x,y \in X$

$$x_1,x_2 \in X, \quad g(x_1) \leq g(x_2) \Rightarrow F(x_1,y) \leq F(x_2,y)$$

and

$$y_1,y_2 \in X, \quad g(y_1) \leq g(y_2) \Rightarrow F(x,y_1) \geq F(x,y_2).$$

Definition 1.14. ([12]) An element $(x,y) \in X \times X$ is said to be a coupled coincidence point of the mapping F if

$$F(x,y) = g(x) \quad and \quad F(y,x) = g(y).$$

Definition 1.15. ([12]) Let X be a non-empty set and $F : X \times X \to X$ and $g : X \to X$. We say F and g are commutative if

$$g(F(x,y)) = F(g(x),g(y)).$$

Aydi and et al. in [1] considered the following assumptions and they proved the following theorem. Let Φ denote the set of functions $\phi : [0,\infty) \to \phi : [0,\infty)$ satisfying
(a) $\phi^{-1}(0) = 0$,
(b) $\phi(t) < t$ for all $t > 0$,
(c) $\lim_{r\to t^+} \phi(r) < t$ for all $t > 0$.

Theorem 1.16. ([1]) *Let $(X,\leq)$ be a partially ordered set and G be a G-metric on X such that (X,G) is a complete G-metric space. Suppose that there exist $\phi \in \Phi$, $F : X \times X \to X$ and $g : X \to X$ such that*

$$G(F(x,y),F(u,v),F(w,z)) \leq \phi(\frac{G(g(x),g(u),g(w)) + G(g(y),g(v),g(z))}{2})$$

for all $x,y,u,v,w,z \in X$ with $g(w) \leq g(u) \leq g(x)$ and $g(y) \geq g(v) \geq g(z)$. Suppose also that F is continuous and has the mixed g-monotone property, $F(X \times X) \subseteq g(X)$ and g is continuous and commutes with F. If there exist $x_0,y_0 \in X$ such that $g(x_0) \leq F(x_0,y_0)$ and $F(y_0,x_0) \leq gy_0$, then F and g have a coupled coincidence point, that is, there exists $(x,y) \in X \times X$ such that $g(x) = F(x,y)$ and $g(y) = F(y,x)$.

In this paper we obtain some coupled coincidence and common fixed point theorems. Our work generalize and extend the result of Aydi and et al. [1].

2. Main result

Theorem 2.1. *Let $(X,\leq)$ be a partially ordered set and G be a G-metric on X such that (X,G) is a complete G-metric space. Suppose that $g : X \to X$ and $F : X \times X \to X$ be a mapping having the mixed g-monotone property on X and there exists $\phi \in \Phi$ such that*

$$G(F(x,y),F(u,v),F(z,t)) + G(F(y,x),F(v,u),F(t,z)) \leq 2\phi(\frac{G(g(x),g(u),g(z)) + G(g(y),g(v),g(t))}{2}),$$

for all $x,y,u,v,z,t \in X$ with $g(x) \geq g(u) \geq g(z)$ and $g(y) \leq g(v) \leq g(t)$. Suppose also that F is continuous, $F(X \times X) \subseteq g(X)$ and g is continuous and commutes with F. If there exist $x_0, y_0 \in X$ with

$$g(x_0) \leq F(x_0,y_0) \quad \text{and} \quad g(y_0) \geq F(y_0,x_0)$$

then F and g have a coupled coincidence point.

Proof. Let $x_0, y_0 \in X$ such that $g(x_0) \le F(x_0, y_0)$ and $g(y_0) \ge F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$. Again, since $F(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$. Continuing this process, we can define two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$g(x_{n+1}) = F(x_n, y_n) \quad \text{and} \quad g(y_{n+1}) = F(y_n, x_n).$$

Since F has the mixed g-monotone property, it is easily to seen that

$$g(x_n) = F(x_{n-1}, y_{n-1}) \le g(x_{n+1}) = F(x_n, y_n),$$

and

$$g(y_n) = F(y_{n-1}, x_{n-1}) \ge g(y_{n+1}) = F(y_n, x_n).$$

Let

$$t_n = G(g(x_{n+1}), g(x_{n+1}), g(x_n)) + G(g(y_{n+1}), g(y_{n+1}), g(y_n)).$$

Now, since

$$G(g(x_{n+1}), g(x_{n+1}), g(x_n)) = G(F(x_n, y_n), F(x_n, y_n), F(x_{n-1}, y_{n-1})), \tag{1}$$

and

$$G(g(y_{n+1}), g(y_{n+1}), g(y_n)) = G(F(y_n, x_n), F(y_n, x_n), F(y_{n-1}, x_{n-1})). \tag{2}$$

Adding (1) with (2), and using contractive condition, we obtain

$$\begin{aligned} t_n &= G(g(x_{n+1}), g(x_{n+1}), g(x_n)) + G(g(y_{n+1}), g(y_{n+1}), g(y_n)) \\ &\le 2\phi\Big(\frac{G(g(x_n), g(x_n), g(x_{n-1})) + G(g(y_n), g(y_n), g(y_{n-1}))}{2}\Big) \\ &= 2\phi\Big(\frac{t_{n-1}}{2}\Big). \end{aligned} \tag{3}$$

From the properties of ϕ we have $\phi(t) < t$ for all $t > 0$, then from (3) it follows that (t_n) is a monotone decreasing sequence. Therefore, there exists some $\delta \ge 0$ such that

$$\lim_{n\to\infty} t_n = \delta + .$$

We show that $\delta = 0$. Assume on the contrary, that is $\delta > 0$. Letting $n \to \infty$ in (3) and using the properties of ϕ, we obtain

$$\delta = \lim_{n\to\infty} t_n \le 2 \lim_{n\to\infty} \phi\Big(\frac{t_{n-1}}{2}\Big) = 2 \lim_{t_n \to \delta+} \phi\Big(\frac{t_{n-1}}{2}\Big) < \delta,$$

a contradiction. Thus $\delta = 0$ and hence

$$\lim_{n\to\infty} t_n = G(g(x_{n+1}), g(x_{n+1}), g(x_n)) + G(g(y_{n+1}), g(y_{n+1}), g(y_n)) = 0. \tag{4}$$

Next, we show that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences in the G-metric space (X,G). Suppose on the contrary. This means that at least one of $\{g(x_n)\}$ or $\{g(x_n)\}$ is not a Cauchy sequence. Then, there exists $\varepsilon > 0$ and sequences of natural numbers $(m(k))$ and $(n(k))$ such that $n(k) > m(k) > k$, for every natural number k and

$$r_k = G(g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)})) + G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)})) \geq \varepsilon. \tag{5}$$

Now corresponding to $m(k)$ we choose $n(k)$ to be the smallest for which (5) holds. Then

$$G(g(x_{n(k)-1}), g(x_{n(k)-1}), g(x_{m(k)})) + G(g(y_{n(k)-1}), g(y_{n(k)-1}), g(y_{m(k)})) < \varepsilon. \tag{6}$$

Using (5) and (6) and the rectangle inequality, we obtain

$$\begin{aligned}
\varepsilon \leq r_k &\leq G(g(x_{n(k)}), g(x_{n(k)}), g(x_{n(k)-1})) + G(g(x_{n(k)-1}), g(x_{n(k)-1}), g(x_{m(k)})) \\
&+ G(g(y_{n(k)}), g(y_{n(k)}), g(y_{n(k)-1})) + G(g(y_{n(k)-1}), g(y_{n(k)-1}), g(y_{m(k)})) \\
&= G(g(x_{n(k)-1}), g(x_{n(k)-1}), g(x_{m(k)})) \\
&+ G(g(y_{n(k)-1}), g(y_{n(k)-1}), g(y_{m(k)})) + t_{n(k)-1} \; < \; \varepsilon + t_{n(k)-1}.
\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (4), we obtain

$$\lim_{k\to\infty} r_k = \varepsilon + . \tag{7}$$

Again, using rectangle inequality gives us

$$\begin{aligned}
r_k &= G(g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)})) + G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)})) \\
&\leq G(g(x_{n(k)}), g(x_{n(k)}), g(x_{n(k)+1})) + G(g(x_{n(k)+1}), g(x_{n(k)+1}), g(x_{m(k)+1})) \\
&+ G(g(x_{m(k)+1}), g(x_{m(k)+1}), g(x_{m(k)})) + G(g(y_{n(k)}), g(y_{n(k)}), g(y_{n(k)+1})) \\
&+ G(g(y_{n(k)+1}), g(y_{n(k)+1}), g(y_{m(k)+1})) + G(g(y_{m(k)+1}), g(y_{m(k)+1}), g(y_{m(k)})) \\
&= t_{m(k)} + G(g(x_{n(k)}), g(x_{n(k)}), g(x_{n(k)+1})) + G(g(y_{n(k)}), g(y_{n(k)}), g(y_{n(k)+1})) \\
&+ G(g(x_{n(k)+1}), g(x_{n(k)+1}), g(x_{m(k)+1})) + G(g(y_{n(k)+1}), g(y_{n(k)+1}), g(y_{m(k)+1}))
\end{aligned}$$

We have $G(x,x,y) \leq 2G(x,y,y)$ for any $x,y \in X$ (Proposition 5) and from (G2)-(G4) we obtain

$$\begin{aligned}
r_k &\leq t_{m(k)} + 2G(g(x_{n(k)}), g(x_{n(k)+1}), g(x_{n(k)+1})) \\
&+ 2G(g(y_{n(k)}), g(y_{n(k)+1}), g(y_{n(k)+1})) + G(g(x_{n(k)+1}), g(x_{n(k)+1}), g(x_{m(k)+1})) \\
&+ G(g(y_{n(k)+1}), g(y_{n(k)+1}), g(y_{m(k)+1})) = t_{m(k)} + 2t_{n(k)} \qquad (8) \\
&+ G(g(x_{n(k)+1}), g(x_{n(k)+1}), g(x_{m(k)+1})) + G(g(y_{n(k)+1}), g(y_{n(k)+1}), g(y_{m(k)+1}))
\end{aligned}$$

Now, since $n(k) > m(k)$, then $g(x_{n(k)}) \geq g(x_{m(k)})$ and $g(y_{n(k)}) \leq g(y_{m(k)})$ and using contractive condition gives us

$$\begin{aligned}
&G(g(x_{n(k)+1}), g(x_{n(k)+1}), g(x_{m(k)+1})) + G(g(y_{n(k)+1}), g(y_{n(k)+1}), g(y_{m(k)+1}))\\
&= G(F(x_{n(k)}, y_{n(k)}), F(x_{n(k)}, y_{n(k)}), F(x_{n(k)}), y_{n(k)})\\
&+ G(F(, y_{n(k)} x_{n(k)}), F(, y_{n(k)} x_{n(k)}), F(y_{n(k)}, x_{n(k)}))\\
&\leq 2\phi(\frac{G(g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)})) + G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)}))}{2})
\end{aligned} \tag{9}$$

Inserting (9) in (8), we get

$$r_k \leq t_{m(k)} + 2t_{n(k)} + 2\phi(\frac{r_k}{2}).$$

Letting $k \to \infty$ in the last inequality and using (4), (7) and properties of ϕ, we have

$$\varepsilon \leq 2 \lim_{k\to\infty} \phi(\frac{r_k}{2}) = 2 \lim_{r_k \to \varepsilon+} \phi(\frac{r_k}{2}) < \varepsilon,$$

a contradiction. Thus, $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences in the complete G-metric space (X, G). Therefore, there are $x, y \in X$ such that $\{g(x_n)\}$ and $\{g(y_n)\}$ are respectively G-convergent to x and y, that is from proposition (1), we have

$$\lim_{n\to\infty} G(g(x_n), g(x_n), x) = \lim_{n\to\infty} G(g(x_n), x, x) = 0 \tag{10}$$

$$\lim_{n\to\infty} G(g(y_n), g(y_n), y) = \lim_{n\to\infty} G(g(y_n), y, y) = 0. \tag{11}$$

From proposition (3) and continuity of G, we obtain

$$\lim_{n\to\infty} G(g(g(x_n)), g(g(x_n)), g(x)) = \lim_{n\to\infty} G(g(g(x_n)), g(x), g(x)) = 0 \tag{12}$$

$$\lim_{n\to\infty} G(g(g(y_n)), g(g(y_n)), g(y)) = \lim_{n\to\infty} G(g(g(y_n)), g(y), g(y)) = 0. \tag{13}$$

On the other hand, since $g(x_{n+1}) = F(x_n, y_n)$ and $g(y_{n+1}) = F(y_n, x_n)$ and from commutativity of F and g, we have

$$g(g(x_{n+1})) = g(F(x_n, y_n)) = F(g(x_n), g(y_n)), \tag{14}$$

and

$$g(g(y_{n+1})) = g(F(y_n, x_n)) = F(g(y_n), g(x_n)). \tag{15}$$

We prove that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.
Since $\{g(x_n)\}$ and $\{g(y_n)\}$ are respectively G-convergent to x and y, by Definition (5) and (14), we have

$$\lim_{n\to\infty} F(g(x_n), g(y_n)) = F(x, y) = \lim_{n\to\infty} g(g(x_{n+1})).$$

Therefore, from (12) we have $g(x) = F(x,y)$.
Similarly, we can show that $g(y) = F(y,x)$, and this proves that (x,y) is a coupled coincidence point of F and g. □

The previous result is still valid for F not necessarily continuous. Instead, we require that underlying G-metric space X has an additional property. We discuss this in the following theorem.

Theorem 2.2. *If in Theorem 2.1 we omit the continuity of F and assume that X has the following property:*
if (x_n) is a non-decreasing sequence with $x_n \to x$, then $x_n \leq x$ for each $n \in \mathbb{N}$ and
if (y_n) is a non-increasing sequence with $y_n \to y$ then $y \leq y_n$ for each $n \in \mathbb{N}$ then F and G have a coupled coincidence point.

Proof. Following the proof of Theorem 2 we only have to show that $F(x,y) = g(x)$ and $F(y,x) = g(y)$. By our assumption, since $g(x_n)$ is non-decreasing and $g(y_n)$ is non-increasing in X, we have $g(x_n) \leq x$ and $g(y_n) \geq y$ for all $n \geq 0$. Using rectangle inequality and contractive condition, we obtain

$$\begin{aligned}
&G(F(x,y),g(x),g(x)) + G(F(y,x),g(y),g(y)) \\
&\leq G(F(x,y),g(g(x_{n+1})),g(g(x_{n+1}))) + G(g(g(x_{n+1})),g(x),g(x)) \\
&+ G(g(g(y_{n+1})),g(y),g(y)) + G(g(g(y_{n+1})),g(g(y_{n+1})),F(y,x)) \\
&\leq G(g(g(x_{n+1})),g(x),g(x)) + G(g(g(y_{n+1})),g(y),g(y)) \\
&+ 2\phi(\frac{G(g(x),g(g(x_n)),g(g(x_n))) + G(g(y),g(g(y_n)),g(g(y_n)))}{2}).
\end{aligned}$$

Taking $n \to \infty$ in the last inequality and using (12),(13) and the properties of ϕ we have

$$G(F(x,y),g(x),g(x)) + G(F(y,x),g(y),g(y)) = 0,$$

which implies that $G(F(x,y),g(x),g(x)) = G(F(y,x),g(y),g(y)) = 0$. Therefore $g(x) = F(x,y)$ and $g(y) = F(y,x)$. □

Corollary 2.3. *Let $(X,\leq)$ be a partially ordered set and G be a G-metric on X such that (X,G) is a complete G-metric space. Suppose that $g : X \to X$ and $F : X \times X \to X$ be a mapping having the mixed g-monotone property on X and there exists $k \in [0,1)$ such that*

$$\begin{aligned}
&G(F(x,y),F(u,v),F(z,t)) + G(F(y,x),F(v,u),F(t,z)) \\
&\leq k[G(g(x),g(u),g(z)) + G(g(y),g(v),g(t))],
\end{aligned}$$

for all $x,y,u,v,z,t \in X$ with $g(x) \geq g(u) \geq g(z)$ and $g(y) \leq g(v) \leq g(t)$.
Suppose also that $F(X \times X) \subseteq g(X)$ and g is continuous and commutes with F.
Suppose that either
(a) F is continuous or
(b) X has the following property:

(i) If (x_n) is a non-decreasing sequence with $x_n \to x$ then $x_n \leq x$, for all $n \in \mathbb{N}$.

(ii) If (y_n) is a non-increasing sequence with $y_n \to y$ then $y_n \geq y$, for all $n \in \mathbb{N}$.

If there exist $x_0, y_0 \in X$ with

$$g(x_0) \leq F(x_0,y_0) \quad \textit{and} \quad g(y_0) \geq F(y_0,x_0)$$

then F and g have a coupled coincidence point.

Proof. Taking $\phi(t) = kt$, with $0 \leq k < 1$ in Theorem 2 and Theorem 3, we obtain Corollary 1. □

3. Uniqueness

Remark 3.1. Notice that, since the contractivity condition in Theorem 1 is valid only for comparable elements, therefore Theorem 1 and Theorem 2 cannot guarantee the uniqueness of coupled fixed point.

Now we prove the existence and uniqueness theorem of coupled fixed point. Notice that if $(X,\leq)$ is a partially ordered set, we endow the product space $X \times X$ with the partial order relation given by

$$(u,v) \leq (x,y) \Leftrightarrow x \geq u \quad \text{and} \quad y \leq v$$

Theorem 3.2. *In addition to the hypothesis of Theorem 2, suppose that for all (x,y), $(u,v) \in X \times X$, there exists $(z,t) \in X \times X$ such that $(F(z,t),F(t,z))$ is comparable with $(F(x,y),F(y,x))$ and $(F(u,v),F(v,u))$. Then F and g have a unique coupled common fixed point, that is, there exists a unique $(x,y) \in X \times X$ such that*

$$x = g(x) = F(x,y) \quad \textit{and} \ y = g(y) = F(y,x).$$

Proof. Suppose that (x,y) and (u,v) are coupled coincidence point of F.
By assumption, there exists (z,t), an element of $X \times X$ such that $(F(z,t),F(t,z))$ is comparable with $(F(x,y),F(y,x))$ and $(F(u,v),F(v,u))$. Without restriction to the generality, we can assume that

$$(F(x,y),F(y,x)) \leq (F(z,t),F(t,z))$$

and

$$(F(u,v),F(v,u)) \leq (F(z,t),F(t,z)).$$

Put $z_0 = z$, $t_0 = t$, and choose $z_1, t_1 \in X$ such that $g(z_1) = F(z_0,t_0)$, $g(t_1) = F(t_1,z_1)$. Then, similarly as in proof of Theorem 2, we can inductively define sequences $(g(z_n))$ and $(g(t_n))$ as follows

$$g(z_{n+1}) = F(z_n,t_n) \quad \text{and} \quad g(t_{n+1}) = F(t_n,z_n).$$

Further, it is easily to seen that $(g(x),g(y)$ and $(g(z_n),g(z_n))$ are comparable. Since $g(x) \leq g(z_1)$ and $g(y) \geq g(t_1)$, using this fact that F is mixed g-monotone mapping, we can show that $g(x) \leq g(z_n)$ and $g(y) \geq g(t_n)$. Similarly, $g(u) \leq g(z_n)$ and $g(v) \geq g(t_n)$.
Now, using contractive condition gives us

$$\begin{aligned}
&\frac{G(g(z_{n+1}),g(x),g(x)) + G(g(y),g(y),g(t_{n+1}))}{2} \\
&= \frac{G(F(z_n,t_n),F(x,y),F(x,y)) + G(F(y,x),F(y,x),F(t_n,z_n))}{2} \\
&\leq \phi(\frac{G(g(z_n),g(x),g(x)) + G(g(t_n),g(y),g(y))}{2}).
\end{aligned} \tag{16}$$

From the properties of ϕ we deduce that the sequences

$$\delta_n = \frac{G(g(z_n),g(x),g(x)) + G(g(t_n),g(y),g(y))}{2}$$

is decreasing and nonnegative, so

$$\lim_{n\to\infty} \delta_n = r,$$

for certain $r \geq 0$.
Letting $n \to \infty$ in (16) and using the properties of ϕ, we obtain

$$r \leq \lim_{n\to\infty} \delta_n = \lim_{\delta_n \to r+} \phi(\delta_{n-1}) = \phi(r) < r,$$

and consequently, $r = 0$.
Therefore

$$\lim_{n\to\infty} G(g(z_n),g(x),g(x)) = 0 \quad \text{and} \quad \lim_{n\to\infty} G(g(t_n),g(y),g(y)) = 0. \tag{17}$$

Similarly, we can prove that

$$\lim_{n\to\infty} G(g(z_n),g(u),g(u)) = 0 \quad \text{and} \quad \lim_{n\to\infty} G(g(t_n),g(v),g(v)) = 0. \tag{18}$$

Now, using (17), (18) and rectangle inequality, we obtain

$$G(g(x),g(u),g(u)) \leq G(g(x),g(z_n),g(z_n)) + G(g(z_n),g(u),g(u)),$$

and

$$G(g(y),g(v),g(v)) \leq G(g(y),g(t_n),g(t_n)) + G(g(t_n),g(v),g(v)).$$

Letting $n \to \infty$ in two above inequalities, we obtain

$$g(x) = g(u) \quad \text{and} \quad g(y) = g(v). \tag{19}$$

Now, since $g(x) = F(x,y)$ and $g(y) = F(y,x)$ and F is commutative with g, we have

$$g(g(x)) = g(F(x,y)) = F(g(x),g(y)) \text{ and } \\ g(g(y)) = g(F(y,x)) = F(g(y),g(x)). \tag{20}$$

Denote, $g(x) = x^*$ and $g(y) = y^*$, then from (20), we have

$$g(x^*) = F(x^*,y^*) \quad \text{and} \quad g(y^*) = F(y^*,x^*). \tag{21}$$

Thus, (x^*,y^*) is a coincidence point. Then, with $u = x^*$ and $v = y^*$, we get

$$g(x) = g(x^*) \quad \text{and} \quad g(y) = g(y^*),$$

that is

$$g(x^*) = x^* \quad \text{and} \quad g(y^*) = y^*. \tag{22}$$

From (21) and (22), we have

$$x^* = g(x^*) = F(x^*,y^*) \quad \text{and} \quad y^* = g(y^*) = F(y^*,x^*).$$

Then, (x^*,y^*) is a coupled common fixed point of F and g. To prove the uniqueness, assume (p,q) is another coupled common fixed point of F and g. Then by (19) and (22) we have

$$p = g(p) = g(x^*) = x^* \quad \text{and} \quad q = g(q) = g(y^*) = y^*.$$

$\square$

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SEYED H. RASOULI
Department of Mathematics
Faculty of Basic Science
Babol University of Technology, Babol, Iran
e-mail: s.h.rasouli@nit.ac.ir

M. BAHRAMPOUR
Department of Mathematics
Islamic Azad University Ghaemshahr branch, Iran
e-mail: md.bahrampour@gmail.com