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SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY DZIOK-SRIVASTAVA OPERATOR

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In this paper we introduce a new class of harmonic univalent functions defined by the Dziok-Srivastava operator. Coefficient estimates, extreme points, distortion bounds and convex combination for functions belonging to this class are obtained and also for a class preserving the integral operator.

1. Introduction

A continuous complex-valued function $f = u + iv$ is defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write

$$f = h + \overline{g}, \tag{1}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [6]).

Denote by S_H the class of functions f of the form (1) that are harmonic univalent and sense-preserving in the unit disc $U = \{z : |z| < 1\}$ for which

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$f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ g(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad |b_1| < 1. \tag{2}$$

In [6] Clunie and Shell-Small investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses. Denote by V_H the subclass of S_H consisting of functions of the form $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} |a_n| z^n, g(z) = \sum_{n=1}^{\infty} |b_n| z^n, \qquad |b_1| < 1. \tag{3}$$

For positive real parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ ($\beta_j \in \mathbb{C} \backslash \mathbb{Z}_0^-$, with $\mathbb{Z}_0^- = 0, -1, -2, \ldots$ and $j = 1, 2, \ldots, s$), the generalized hypergeometric function ${}_qF_s$ is defined by

$${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_q)_n}{(\beta_1)_n \ldots (\beta_s)_n n!} z^n$$

$$(q \leq s+1; s, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \mathbb{N} = \{1, 2, \ldots\};\ z \in U),$$

where $(\theta)_n$ is the Pochhammer symbol defined in terms of the Gamma function Γ by

$$(\theta)_n = \frac{\Gamma(\theta + n)}{\Gamma(\theta)} = \begin{cases} 1 & (n = 0) \\ \theta(\theta+1)\ldots(\theta+n-1) & (n \in \mathbb{N}). \end{cases}$$

For the function $h(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots \beta_s; z) = z_q F_s(\alpha_1, \ldots, \alpha_q;\ \beta_1, \ldots, \beta_s;\ z)$, the Dziok-Srivastava linear operator (see [8] and [9]) $H_{q,s}(\alpha_1, \ldots, \alpha_q;\ \beta_1, \ldots$ $\ldots, \beta_s)$ is defined by the Hadamard product as follows:

$$\begin{aligned} H_{q,s}(\alpha_1, \ldots, \alpha_q;\ \beta_1, \ldots, \beta_s) f(z) &= h(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots \beta_s; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \Psi_n(\alpha_1)\, a_n z^n \quad (z \in U), \end{aligned} \tag{4}$$

where

$$\Psi_n(\alpha_1) = \frac{(\alpha_1)_{n-1} \ldots (\alpha_q)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_s)_{n-1}\ (n-1)!}. \tag{5}$$

For brevity, we write

$$H_{q,s}(\alpha_1, \ldots, \alpha_q;\ \beta_1, \ldots, \beta_s; z) f(z) = H_{q,s}(\alpha_1) f(z).$$

Al-Kharsani and Al-Khal [2] and Al-Khal [1] defined the modified Dziok-Srivastava operator of the harmonic function $f = h + \overline{g}$ given by (1) as

$$H_{q,s}(\alpha_1)f(z) = H_{q,s}(\alpha_1)h(z) + \overline{H_{q,s}(\alpha_1)g(z)}.$$

For $1 < \gamma \leq 2$ and for all $z \in U$, let $S_{H_{q,s}}([\alpha_1];\gamma)$ denote the family of harmonic functions $f(z) = h(z) + \overline{g(z)}$, where h and g are given by (2) and satisfying the analytic criterion

$$\Re\left\{\frac{H_{q,s}(\alpha_1)h(z) + \overline{H_{q,s}(\alpha_1)g(z)}}{z}\right\} < \gamma. \tag{6}$$

Let $\overline{S}_{H_{q,s}}([\alpha_1];\gamma)$ be the subclass of $S_{H_{q,s}}([\alpha_1];\gamma)$ consisting of functions $f = h + g$ such that h and g given by (3).

We note that for suitable choices of q and s, we obtain the following subclasses:

1) Putting $q = 2, s = 1, \alpha_1 = a\,(a > 0), \alpha_2 = 1$ *and* $\beta_1 = c\,(c > 0)$ in (6), the class $\overline{S}_{H_{2,1}}([a,1;c];\gamma)$ reduces to the class $\mathcal{L}_H(a,c;\gamma)$

$$= \left\{f \in S_H : \Re\left\{\frac{L(a,c)h(z) + \overline{L(a,c)g(z)}}{z}\right\} < \gamma,\ 1 < \gamma \leq 2, a, c > 0,\right.$$
$$\left. z \in U\right\},$$

where $L(a,c)$ is the modified Carlson-Shaffer operator (see [3]), defined as follows:

$$L(a,c)f(z) = L(a,c)h(z) + \overline{L(a,c)g(z)};$$

2) Putting $q = 2, s = 1, \alpha_1 = \lambda + 1(\lambda > -1)$ *and* $\alpha_2 = \beta_1 = 1$ in (6), the class $\overline{S}_{H_{2,1}}([\lambda + 1];\gamma)$ reduces to the class $\overline{W}_H(\lambda;\gamma)$

$$= \left\{f \in S_H : \Re\left\{\frac{D^\lambda h(z) + \overline{D^\lambda g(z)}}{z}\right\} < \gamma,\ 1 < \gamma \leq 2, \lambda > -1, z \in U\right\},$$

where D^λ is the modified Ruscheweyh derivative operator (see [13]), defined as follows:

$$D^\lambda f(z) = D^\lambda h(z) + \overline{D^\lambda g(z)};$$

3) Putting $q = 2, s = 1, \alpha_1 = v + 1\,(v > -1), \alpha_2 = 1$ *and* $\beta_1 = v + 2$ in (6), the class $\overline{S}_{H_{2,1}}([v+1,1;v+2];\gamma)$ reduces to the class $\overline{\zeta}_H(v;\gamma)$

$$= \left\{f \in S_H : \Re\left\{\frac{J_v h(z) + \overline{J_v g(z)}}{z}\right\} < \gamma,\ 1 < \gamma \leq 2, v > -1, z \in U\right\},$$

where J_v is the modified generalized Bernardi-Libera-Livingston operator (see [10]), defined as follows:

$$J_v f(z) = J_v h(z) + \overline{J_v g(z)};$$

4) Putting $q = 2, s = 1, \alpha_1 = 2, \alpha_2 = 1$ *and* $\beta_1 = 2 - \mu \, (\mu \neq 2, 3, \ldots)$ in (6), the class $\overline{S}_{H_{2,1}}([2, 1; 2 - \mu]; \gamma)$ reduces to the class $\mathcal{F}_H(\mu; \gamma)$

$$= \left\{ f \in S_H : \Re \left\{ \frac{\Omega_z^{\mu} h(z) + \overline{\Omega_z^{\mu} g(z)}}{z} \right\} < \gamma, \; 1 < \gamma \leq 2, \mu \neq 2, 3, \ldots, z \in U \right\},$$

where Ω_z^{μ} is the modified Srivastava-Owa fractional derivative operator (see [12]), defined as follows:

$$\Omega_z^{\mu} f(z) = \Omega_z^{\mu} h(z) + \overline{\Omega_z^{\mu} g(z)};$$

5) Putting $q = 2, s = 1, \alpha_1 = \mu \, (\mu > 0), \alpha_2 = 1$ *and* $\beta_1 = \lambda + 1 \, (\lambda > -1)$ in (6), the class $\overline{S}_{H_{2,1}}([\mu, 1; \lambda + 1]; \gamma)$ reduces to the class $\overline{\pounds}_H(\mu, \lambda; \gamma) =$

$$\left\{ f \in S_H : \Re \left\{ \frac{I_{\mu,\lambda} h(z) + \overline{I_{\mu,\lambda} g(z)}}{z} \right\} < \gamma, \; 1 < \gamma \leq 2, \mu > 0, \lambda > -1, z \in U \right\},$$

where $I_{\lambda,\mu}$ is the modified Choi-Saigo-Srivastava operator (see [5]), defined as follows:

$$I_{\mu,\lambda} f(z) = I_{\mu,\lambda} h(z) + \overline{I_{\mu,\lambda} g(z)};$$

6) Putting $q = 2, s = 1, \alpha_1 = 2, \alpha_2 = 1$ *and* $\beta_1 = k + 1 \, (k > -1)$ in (6), the class $\overline{S}_{H_{2,1}}([2, 1; k + 1]; \gamma)$ reduces to the class $\overline{A}_H(k; \gamma)$

$$= \left\{ f \in S_H : \Re \left\{ \frac{I_k h(z) + \overline{I_k g(z)}}{z} \right\} < \gamma, \; 1 < \gamma \leq 2, k > -1, z \in U \right\},$$

where I_k is the modified Noor integral operator (see [11]), defined as follows:

$$I_k f(z) = I_k h(z) + \overline{I_k g(z)};$$

7) Putting $q = 2, s = 1, \alpha_1 = c \, (c > 0), \alpha_2 = \lambda + 1 \, (\lambda > -1)$ *and* $\beta_1 = a \, (a > 0)$ in (6), the class $\overline{S}_{H_{2,1}}([c, \lambda + 1; a]; \gamma)$ reduces to the class $\overline{\mathcal{F}}_H(c, a, \lambda; \gamma)$

$$= \left\{ f \in S_H : \Re \left\{ \frac{I^{\lambda}(a, c) h(z) + \overline{I^{\lambda}(a, c) g(z)}}{z} \right\} < \gamma, \; 1 < \gamma \leq 2, c > 0, \lambda > -1, a > 0, z \in U \right\},$$

where $I^{\lambda}(a,c)$ is the modified Cho-Kwon-Srivastava operator (see [4]), defined as follows:

$$I^{\lambda}(a,c)f(z)=I^{\lambda}(a,c)h(z)+\overline{I^{\lambda}(a,c)g(z)}.$$

2. Coefficient estimates

Unless otherwise mentioned, we shall assume in the reminder of this paper that, the parameters $\alpha_1,\ldots,\alpha_q$ and $\beta_1,\ldots,\beta_s$ are positive real numbers, $1<\gamma\le 2, z\in U$ *and* $\Psi_n(\alpha_1)$ is defined by (5).

Theorem 2.1. *Let $f=h+\overline{g}$ be such that $h(z)$ and $g(z)$ given by (2). Furthermore, let*

$$\sum_{n=2}^{\infty}\Psi_n(\alpha_1)|a_n|+\sum_{n=1}^{\infty}\Psi_n(\alpha_1)|b_n|\le\gamma-1. \tag{7}$$

Then $f(z)$ is sense-preserving, harmonic univalent in U and $f(z)\in S_{H_{q,s}}([\alpha_1];\gamma)$.

Proof. If $z_1\neq z_2$, then

$$\begin{aligned}\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\ge 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|=1-\left|\frac{\sum_{n=1}^{\infty}b_n(z_1^n-z_2^n)}{(z_1-z_2)+\sum_{n=2}^{\infty}a_n(z_1^n-z_2^n)}\right|\\ &>1-\frac{\sum_{n=1}^{\infty}n|b_n|}{1-\sum_{n=2}^{\infty}n|a_n|}\\ &\ge 1-\frac{\frac{\Psi_n(\alpha_1)}{\gamma-1}|b_n|}{\frac{\Psi_n(\alpha_1)}{\gamma-1}|a_n|}\ge 0,\end{aligned}$$

which proves univalence. Note that $f(z)$ is sense-preserving in U. This is because

$$\begin{aligned}\left|h'(z)\right| &\ge 1-\sum_{n=2}^{\infty}n|a_n||z|^{n-1}\\ &>1-\sum_{n=2}^{\infty}n|a_n|\ge\sum_{n=2}^{\infty}\frac{\Psi_n(\alpha_1)}{\gamma-1}|a_n|\\ &\ge\sum_{n=1}^{\infty}\frac{\Psi_n(\alpha_1)}{\gamma-1}|b_n|\ge\sum_{n=1}^{\infty}n|b_n|\\ &>\sum_{n=1}^{\infty}n|b_n|\left|z^{n-1}\right|\ge\left|g'(z)\right|.\end{aligned}$$

Now we will show that $f(z) \in S_{H_{q,s}}([\alpha_1];\gamma)$. We only need to show that if (7) holds then the condition (6) is satisfied. Using the fact that $Re\{w\} < \gamma$ if and only if $|1-w| < |w-(2\gamma-1)|$, it suffices to show that

$$\left| \frac{\frac{H_{q,s}(\alpha_1)h(z)+\overline{H_{q,s}(\alpha_1)g(z)}}{z} - 1}{\frac{H_{q,s}(\alpha_1)h(z)+\overline{H_{q,s}(\alpha_1)g(z)}}{z} - (2\gamma-1)} \right| < 1.$$

We have

$$\left| \frac{\frac{H_{q,s}(\alpha_1)h(z)+\overline{H_{q,s}(\alpha_1)g(z)}}{z} - 1}{\frac{H_{q,s}(\alpha_1)h(z)+\overline{H_{q,s}(\alpha_1)g(z)}}{z} - (2\gamma-1)} \right| = \left| \frac{\sum_{n=2}^{\infty} \Psi_n(\alpha_1) a_n z^{n-1} + \sum_{n=1}^{\infty} \Psi_n(\alpha_1) \overline{b_n z^{n-1}}}{2(\gamma-1) - \sum_{n=2}^{\infty} \Psi_n(\alpha_1) a_n z^{n-1} + \sum_{n=1}^{\infty} \Psi_n(\alpha_1) \overline{b_n z^{n-1}}} \right|$$

$$\leq \frac{\sum_{n=2}^{\infty} \Psi_n(\alpha_1)|a_n||z|^{n-1} + \sum_{n=1}^{\infty} \Psi_n(\alpha_1)|b_n||z|^{n-1}}{2(\gamma-1) - \sum_{n=2}^{\infty} \Psi_n(\alpha_1)|a_n||z|^{n-1} - \sum_{n=1}^{\infty} \Psi_n(\alpha_1)|b_n||z|^{n-1}}$$

$$< \frac{\sum_{n=2}^{\infty} \Psi_n(\alpha_1)|a_n| + \sum_{n=1}^{\infty} \Psi_n(\alpha_1)|b_n|}{2(\gamma-1) - \sum_{n=2}^{\infty} \Psi_n(\alpha_1)|a_n| - \sum_{n=1}^{\infty} \Psi_n(\alpha_1)|b_n|},$$

which is bounded above by 1 by using (7). This completes the proof of Theorem 2.1. □

The harmonic univalent functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} \frac{\gamma-1}{\Psi_n(\alpha_1)} x_n z^n + \sum_{n=1}^{\infty} \frac{\gamma-1}{\Psi_n(\alpha_1)} \overline{y_n z^n}, \tag{8}$$

where $\sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 1$, show that the coefficient bound given by (7) is sharp. It is worthy to note that the function of the form (8) belongs to the class $S_{H_{q,s}}([\alpha_1];\gamma)$ for all $\sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| \leq 1$ because coefficient inequality (7) holds.

Theorem 2.2. *A function $f(z)$ of the form* (3) *is in the class $\overline{S}_{H_{q,s}}([\alpha_1];\gamma)$ if and only if*

$$\sum_{n=2}^{\infty} \Psi_n(\alpha_1)|a_n| + \sum_{n=1}^{\infty} \Psi_n(\alpha_1)|b_n| \leq \gamma - 1. \tag{9}$$

Proof. Since $\overline{S}_{H_{q,s}}([\alpha_1];\gamma) \subset S_{H_{q,s}}([\alpha_1];\gamma)$, we only need to prove the "only if" part of this theorem. To this end , for functions $f(z)$ of the form (3), we notice that the condition

$$\Re\left\{\frac{H_{q,s}(\alpha_1)h(z)+\overline{H_{q,s}(\alpha_1)g(z)}}{z}\right\}<\gamma,$$

i.e.

$$\Re\left\{1+\sum_{n=2}^{\infty}\Psi_n(\alpha_1)\,|a_n|\,z^{n-1}+\sum_{n=1}^{\infty}\Psi_n(\alpha_1)\,|b_n|\,\overline{z^{n-1}}\right\}<\gamma.$$

Letting $z \to 1^-$ along the real axis, we obtain the inequality (9). This completes the proof of Theorem 2. □

Remark 2.3. Putting $q = 2, s = 1$, $\alpha_1 = 2$ and $\alpha_2 = \beta_1 = 1$ in Theorems 2.1 and 2.2, we obtain the result obtained by Dixit and Porwal [7, Theorem 2.1].

3. Distortion theorem

Theorem 3.1. *Let the function $f(z)$ given in* (3) *belong to the class $\overline{S}_{H_{q,s}}([\alpha_1];\gamma)$. Then for $|z| = r < 1$, we have*

$$(1-|b_1|)\,r-\frac{\gamma-1-|b_1|}{\Psi_2(\alpha_1)}r^2 \le |f(z)| \le (1+|b_1|)\,r+\frac{\gamma-1-|b_1|}{\Psi_2(\alpha_1)}r^2 \tag{10}$$

for $|b_1| \le \gamma-1$. The results are sharp with equality for the functions $f(z)$ defined by

$$f(z)=z+b_1\overline{z}+\frac{\gamma-1-b_1}{\Psi_2(\alpha_1)}\overline{z}^2 \tag{11}$$

and

$$f(z)=z-b_1\overline{z}-\frac{\gamma-1-b_1}{\Psi_2(\alpha_1)}z^2. \tag{12}$$

Proof. We only prove the right-hand inequality. The proof for the left-hand inequality is similar and will be omitted. Since

$$f(z)=z+\sum_{n=2}^{\infty}\Psi_n(\alpha_1)\,a_nz^n+\sum_{n=1}^{\infty}\Psi_n(\alpha_1)\,\overline{b_nz^n},$$

then

$$|f(z)| \le (1+|b_1|)r+\sum_{n=2}^{\infty}(|a_n|+|b_n|)\,r^n \le (1+|b_1|)r+\sum_{n=2}^{\infty}(|a_n|+|b_n|)\,r^2 =$$

$$=(1+|b_1|)r+\frac{\gamma-1}{\Psi_2(\alpha_1)}\sum_{n=2}^{\infty}\frac{\Psi_2(\alpha_1)}{\gamma-1}(|a_n|+|b_n|)\,r^2$$

$$\leq \quad (1+|b_1|)r+\frac{\gamma-1}{\Psi_2(\alpha_1)}\sum_{n=2}^{\infty}\frac{\Psi_n(\alpha_1)}{\gamma-1}(|a_n|+|b_n|)r^2$$
$$\leq \quad (1+|b_1|)r+\frac{\gamma-1-|b_1|}{\Psi_2(\alpha_1)}r^2.$$

The functions $f(z)$ given by (11) and (12), respectively, for $|b_1|\leq\gamma-1$ show that the bounds given in Theorem 3.1 are sharp. □

4. Extreme points

Theorem 4.1. *Let $f(z)$ be given by* (3). *Then $f(z)\in\overline{S}_{H_{q,s}}([\alpha_1];\gamma)$ if and only if*

$$f(z)=\sum_{n=1}^{\infty}(\mu_n h_n(z)+\eta_n g_n(z)), \tag{13}$$

where $h_1(z)=z$,

$$h_n(z)=z+\frac{\gamma-1}{\Psi_n(\alpha_1)}z^n \quad (n=2,3,\dots) \tag{14}$$

and

$$g_n(z)=z+\frac{\gamma-1}{\Psi_n(\alpha_1)}\overline{z^n} \quad (n=1,2,\dots), \tag{15}$$

$\mu_n\geq 0,\eta_n\geq 0,$ $\sum_{n=1}^{\infty}(\mu_n+\eta_n)=1$. *In particular, the extreme points of the class $\overline{S}_{H_{q,s}}([\alpha_1];\gamma)$ are $\{h_n\}$ and $\{g_n\}$, respectively.*

Proof. Suppose that

$$f(z)=\sum_{n=1}^{\infty}(\mu_n h_n(z)+\eta_n g_n(z))=z+\sum_{n=2}^{\infty}\frac{\gamma-1}{\Psi_n(\alpha_1)}\mu_n z^n+\sum_{n=1}^{\infty}\frac{\gamma-1}{\Psi_n(\alpha_1)}\eta_n\overline{z^n}.$$

Then

$$\sum_{n=2}^{\infty}\frac{\Psi_n(\alpha_1)}{\gamma-1}\left(\frac{\gamma-1}{\Psi_n(\alpha_1)}\mu_n\right)+\sum_{n=1}^{\infty}\frac{\Psi_n(\alpha_1)}{\gamma-1}\left(\frac{\gamma-1}{\Psi_n(\alpha_1)}\eta_n\right)$$
$$=\sum_{n=2}^{\infty}\mu_n+\sum_{n=1}^{\infty}\eta_n=1-\mu_1\leq 1$$

and so $f(z)\in\overline{S}_{H_{q,s}}([\alpha_1];\gamma)$.

Conversely, if $f(z) \in \overline{S}_{H_{q,s}}([\alpha_1];\gamma)$, then

$$|a_n| \leq \frac{\gamma - 1}{\Psi_n(\alpha_1)} \ (n \geq 2)$$

and

$$|b_n| \leq \frac{\gamma - 1}{\Psi_n(\alpha_1)} \ (n \geq 1).$$

Setting

$$\mu_n = \frac{\Psi_n(\alpha_1)}{\gamma - 1} \ |a_n| \quad (n = 2, 3, \dots)$$

and

$$\eta_n = \frac{\Psi_n(\alpha_1)}{\gamma - 1} \ |b_n| \quad (n = 1, 2, \dots).$$

Since $0 \leq \mu_n \leq 1$ $(n = 2, 3, \dots)$ and $0 \leq \eta_n \leq 1$ $(n = 1, 2, \dots)$, $\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n + \sum_{n=1}^{\infty} \eta_n \geq 0$, then, we can see that $f(z)$ can be expressed in the form (13). This completes the proof. □

5. Convolution and convex combination

For our next theorem, we need to define the convolution of two harmonic functions.
For harmonic functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \overline{z^n} \tag{16}$$

and

$$F(z) = z + \sum_{n=2}^{\infty} |A_n| z^n + \sum_{n=1}^{\infty} |B_n| \overline{z^n}, \tag{17}$$

the convolution of f and F is given by

$$(f * F)(z) = f(z) * F(z) = z + \sum_{n=2}^{\infty} |a_n A_n| z^n + \sum_{n=1}^{\infty} |b_n B_n| \overline{z^n}. \tag{18}$$

Using this definition, the next theorem shows that the class $\overline{S}_{H_{q,s}}([\alpha_1];\gamma)$ is closed under convolution.

Theorem 5.1. *For $1 < \gamma \leq \lambda \leq 2$, let $f(z) \in \overline{S}_{H_{q,s}}([\alpha_1];\lambda)$, where $f(z)$ is given by* (16) *and $F \in \overline{S}_{H_{q,s}}([\alpha_1];\gamma)$, where $F(z)$ is given by* (17)*. Then $f * F \in \overline{S}_{H_{q,s}}([\alpha_1];\lambda) \subset \overline{S}_{H_{q,s}}([\alpha_1];\gamma)$.*

Proof. We wish to show that the coefficients of $f * F$ satisfy the required condition given in Theorem 2.2. For $f \in \overline{S}_{H_{q,s}}([\alpha_1];\lambda)$ we note that $|a_n| \leq 1$ and $|b_n| \leq 1$. Now, for the convolution function $f * F$ we obtain

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{\Psi_n(\alpha_1)}{\lambda-1}|a_n A_n| z^n + \sum_{n=1}^{\infty} \frac{\Psi_n(\alpha_1)}{\lambda-1}|b_n B_n| \overline{z^n} \\
&\leq \sum_{n=2}^{\infty} \frac{\Psi_n(\alpha_1)}{\lambda-1}|A_n| z^n + \sum_{n=1}^{\infty} \frac{\Psi_n(\alpha_1)}{\lambda-1}|B_n| \overline{z^n} \\
&\leq \sum_{n=2}^{\infty} \frac{\Psi_n(\alpha_1)}{\gamma-1}|A_n| z^n + \sum_{n=1}^{\infty} \frac{\Psi_n(\alpha_1)}{\gamma-1}|B_n| \overline{z^n} \\
&\leq 1.
\end{aligned}$$

Therefore $f * F \in \overline{S}_{H_{q,s}}([\alpha_1];\lambda) \subset \overline{S}_{H_{q,s}}([\alpha_1];\gamma)$. □

Now we show that the class $\overline{S}_{H_{q,s}}([\alpha_1];\gamma)$ is closed under convex combinations of its members.

Theorem 5.2. *The class $\overline{S}_{H_{q,s}}([\alpha_1];\gamma)$ is closed under convex combination.*

Proof. For $i = 1,2,3,\ldots$, let $f_i \in \overline{S}_{H_{q,s}}([\alpha_1];\gamma)$, where f_i is given by

$$f_i = z + \sum_{n=2}^{\infty} |a_{n_i}| z^n + \sum_{n=1}^{\infty} |b_{n_i}| \overline{z^n}.$$

Then by using Theorem 2.2, we have

$$\sum_{n=2}^{\infty} \frac{\Psi_n(\alpha_1)}{\gamma-1}|a_{n_i}| z^n + \sum_{n=1}^{\infty} \frac{\Psi_n(\alpha_1)}{\gamma-1}|b_{n_i}| \overline{z}^n \leq 1. \tag{19}$$

For $\sum\limits_{n=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i |a_{n_i}|\right) z^n + \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i |b_{n_i}|\right) \overline{z^n}. \tag{20}$$

Then by (19), we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{\Psi_n(\alpha_1)}{\gamma-1}\left(\sum_{i=1}^{\infty} t_i |a_{n_i}|\right) + \sum_{n=1}^{\infty} \frac{\Psi_n(\alpha_1)}{\gamma-1}\left(\sum_{i=1}^{\infty} t_i |b_{n_i}|\right) \\
&= \sum_{i=1}^{\infty} t_i \left(\sum_{n=2}^{\infty} \frac{\Psi_n(\alpha_1)}{\gamma-1}|a_{n_i}| + \sum_{n=1}^{\infty} \frac{\Psi_n(\alpha_1)}{\gamma-1}|b_{n_i}|\right) \\
&\leq \sum_{i=1}^{\infty} t_i = 1.
\end{aligned}$$

This is the condition required by (7) and so $\sum_{i=1}^{\infty} t_i f_i(z) \in \overline{S}_{H_{q,s}}([\alpha_1];\gamma)$. □

6. A family of integral operators

Theorem 6.1. *Let the function $f(z)$ defined by (1) be in the class $\overline{S}_{H_{q,s}}([\alpha_1];\gamma)$ and let c be a real number such that $c > -1$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}h(t)\,dt + \overline{\frac{c+1}{z^c}\int_0^z t^{c-1}g(t)\,dt} \;\; (c > -1) \tag{21}$$

also belongs to the class $\overline{S}_{H_{q,s}}([\alpha_1];\gamma)$.

Proof. Let the function $f(z)$ be defined by (1). Then from the representation (21) of $F(z)$, it follows that

$$F(z) = z + \sum_{n=2}^{\infty} d_n z^n + \sum_{n=1}^{\infty} \zeta_n \overline{z^n},$$

where

$$d_n = \left(\frac{c+1}{c+n}\right)|a_n| \text{ and } \zeta_n = \left(\frac{c+1}{c+n}\right)|b_n|.$$

Therefore, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} \Psi_n(\alpha_1)\,d_n + \sum_{n=1}^{\infty} \Psi_n(\alpha_1)\,\zeta_n \\
&= \sum_{n=2}^{\infty} \Psi_n(\alpha_1)\left(\frac{c+1}{c+n}\right)|a_n| + \sum_{n=1}^{\infty} \Psi_n(\alpha_1)\left(\frac{c+1}{c+n}\right)|b_n| \\
&\le \sum_{n=2}^{\infty} \Psi_n(\alpha_1)|a_n| + \sum_{n=1}^{\infty} \Psi_n(\alpha_1)|b_n| \le (1-\alpha),
\end{aligned}$$

since $f(z) \in \overline{S}_{H_{q,s}}([\alpha_1];\gamma)$. Hence, by Theorem 2.2, $F(z) \in \overline{S}_{H_{q,s}}([\alpha_1];\gamma)$. This completes the proof. □

Remark 6.2. Specializing the parameters $q;s;\alpha_1,\dots\alpha_q$ and $\beta_1,\dots,\beta_s$ in the above results, we obtain the corresponding results for the corresponding classes defined in the introduction.

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