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A REMARK ON λ-INTERTWINING HYPONORMAL OPERATORS.

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We extend a result concerning λ-commuting normal operators with empty point spectrum. More precisely, we prove that for a hyponormal operator T with empty point spectrum for which there exists a Hilbert-Schmidt operator K such that $TK = \lambda KT^* + \mu K$ for some $\lambda, \mu \in \mathbb{C}$, implies $K = 0$.

Let $\mathcal{H}$ be a complex, separable, infinite dimensional Hilbert space, and let $L(\mathcal{H})$ denote the algebra of all linear bounded operators on $\mathcal{H}$, let $\mathbb{K}$ denote the two-sided ideal of all compact operators on $\mathcal{H}$, and let $\mathcal{C}_2(\mathcal{H})$ denote the Hilbert-Schmidt class. An operator T in $L(\mathcal{H})$ is called normal if $T^*T = TT^*$, hyponormal if $T^*T \geq TT^*$, class $\mathcal{A}$ if $|T|^2 \leq |T^2|$, (where $|T| = (T^*T)^{\frac{1}{2}}$), and paranormal if $||Tx||^2 \leq ||T^2x|| \cdot ||x||$ for all $x \in \mathcal{H}$. It is obvious or easy to see that each of these classes is included into the next one in the order that they were enumerated. The "λ-commuting" property has its history related to the Invariant Subspace Problem of operators on Hilbert space. It was proved by V. Lomonosov [6] that any nontrivial operator T (i.e. that is not a scalar multiple of the identity) that commutes with a nonzero operator K has a nontrivial hyperinvariant subspace. Subsequently, the result was improved by Lomonosov

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and many other authors to operators that λ-commute, that is $TK = \lambda KT$, see for instance [3], [4], [7].

In [5] it was proved the following result concerning λ-commuting normal operators.

Theorem 1.1 ([5])**.** *If $T \in L(\mathcal{H})$ is a normal operator with empty point spectrum and K is a compact operator such that $TK = \lambda KT$ for some complex number λ, then $K = 0$.*

In [1], the authors attempted to extend several results involving λ-commuting operators.

Theorem 1.2 ([1])**.** *If $T \in L(\mathcal{H})$ is a paranormal operator with empty point spectrum and K is a finite rank operator such that $TK = \lambda KT$ for some complex number λ, then $K = 0$.*

In [1] it was also provided an example that shows that certain generalization of the above result is impossible. Namely, let l^2 be the Hilbert space consisting of all square-sumable complex sequences, let T be the unilateral shift, let $|\lambda| > 1$, and let K be the diagonal operator with entries $(\frac{1}{|\lambda|}, \frac{1}{|\lambda|^2}, \dots)$. One can easily verify that the unilateral shift is hyponormal, therefore paranormal, K is trace class, therefore compact, and $TK = \lambda KT$.

Recall that a nontrivial invariant subspace for an operator $T \in L(\mathcal{H})$ is a closed subspace $\mathcal{M}$, $(0) \neq \mathcal{M} \neq \mathcal{H}$, such that $T\mathcal{M} \subseteq \mathcal{M}$.

Theorem 1.3. *If $T_1, T_2^* \in L(\mathcal{H})$ are hyponormal operators such that T_1 and T_2 have no nontrivial invariant subspace and K is a Hilbert-Schmidt operator satisfying $T_1K = \lambda KT_2 + \mu K$ for some complex numbers λ and μ, then $K = 0$.*

Proof. The idea of the proof was first used by Furuta in [2], and for completeness we include some calculations. It is well known that $\mathcal{C}_2(\mathcal{H})$ is a Hilbert space with the scalar product

$$\langle X, Y \rangle = tr(XY^*).$$

For $S_1,\ S_2 \in L(\mathcal{H})$, let $\Delta_{S_1,S_2} : \mathcal{C}_2(\mathcal{H}) \to \mathcal{C}_2(\mathcal{H})$ be the operator defined by

$$\Delta_{S_1,S_2}(X) = S_1X - XS_2.$$

An elementary calculation shows that the adjoint operator of Δ_{S_1,S_2} is

$$(\Delta_{S_1,S_2})^*(X) = S_1^*X - XS_2^*,$$

that is $(\Delta_{S_1,S_2})^* = \Delta_{S_1^*,S_2^*}$, and consequently, its self-commutator

$$[(\Delta_{S_1,S_2})^*, \Delta_{S_1,S_2}](X) = [S_1^*, S_1]X - X[S_2^*, S_2],$$

where $[S^*,S]$ denotes the self-commutator of S, i.e., $S^*S - SS^*$.

Let $\Delta_0 = \Delta_{T_1,\lambda T_2}$ with T_1, T_2 as in the hypothesis of the theorem and $\lambda \in \mathbb{C}$. The calculations above imply

$$\begin{aligned}\langle [\Delta_0^*,\Delta_0](X), X\rangle = tr([\Delta_0^*,\Delta_0](X)\cdot X^*) &= tr(D_1XX^*) - |\lambda|^2\cdot tr(X(D_2)X^*)\\ &= tr(X^*D_1X) - |\lambda|^2 tr(XD_2X^*),\end{aligned}$$

where $D_i = [T_i^*, T_i], \ \ i = 1,2$.

Since $D_1 \geq 0$ and $D_2 \leq 0$, $tr(X^*D_1X) \geq 0$ and $tr(XD_2X^*) \leq 0$ and consequently the operator Δ_0 is a hyponormal operator on $\mathcal{C}_2(\mathcal{H})$.

Thus, if $K \in \mathcal{C}_2(\mathcal{H})$ such that $T_1K = \lambda KT_2 + \mu K$, that is $(\Delta_0 - \mu)K = 0$, then

$$(\Delta_0 - \mu)^*K = 0,$$

or equivalently $T_1^*K = \overline{\lambda}KT_2^* + \overline{\mu}K$, which further implies that $K^*(T_1 - \mu) = \lambda T_2K^*$. Multiplying this last equality to right side by K and using again

$$(T_1 - \mu)K = \lambda KT_2,$$

one obtains

$$\lambda T_2K^*K = K^*(T_1 - \mu)K = K^*\lambda KT_2 = \lambda K^*KT_2.$$

If $\lambda = 0$ and $K \neq 0$, then $T_1K = \mu K$, which contradicts the hypothesis that T_1 has no nontrivial invariant subspace. If $\lambda \neq 0$, then the above equality implies that T_2 commutes with $|K|^2$ which contradicts the hypothesis that T_2 has no nontrivial invariant subspace except when $K = 0$. □

Using similar circle of ideas as in the above proof, one can prove the following.

Corollary 1.4. *If $T \in L(\mathcal{H})$ is hyponormal operators such that T has empty point spectrum and K is a Hilbert-Schmidt operator satisfying*

$$TK = \lambda KT^* + \mu K$$

for some complex numbers λ and μ, then $K = 0$.

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