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INTEGRAL TRANSFORMS OF THE k-GENERALIZED MITTAG-LEFFLER FUNCTION $E^{\gamma,\tau}_{k,\alpha,\beta}(z)$

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In this paper we derive the Euler transform, Laplace transform, Whittaker transform and fractional Fourier transform of order α, $0 < \alpha \leq 1$, of the generalized k-Mittag-Leffler function $E^{\gamma}_{k,\alpha,\beta}(z)$. The results obtained are of general character and provide extension of the results given by Saxena [9].

1. Introduction

The k-Pochhammer symbol was introduced in [1] in the form:

$$(x)_{n,k} = x(x+k)(x+2k)...(x+(n-1)k), \tag{1}$$

$$(x)_{(n+r)q,k} = (x)_{rq,k}(x+qrk)_{nq,k}, \tag{2}$$

where $x \in \mathbb{C}, k \in \mathbb{R}$ and $n \in \mathbb{N}$.

Proposition 1.1. *Let $\gamma \in \mathbb{C}$, $k,s \in \mathbb{R}$, then the following identity holds*

$$\Gamma_s(\gamma) = \left(\frac{s}{k}\right)^{\frac{\gamma}{s}-1} \Gamma_k\left(\frac{k\gamma}{s}\right), \tag{3}$$

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and in particular

$$\Gamma_k(\gamma) = (k)^{\frac{\gamma}{k}-1}\,\Gamma\left(\frac{\gamma}{k}\right), \tag{4}$$

Proposition 1.2. *Let $\gamma \in \mathbb{C}$ and $k, s \in \mathbb{R}$ and $n \in \mathbb{N}$, then the following identity holds*

$$(\gamma)_{nq,s} = \left(\frac{s}{k}\right)^{nq}\left(\frac{k\gamma}{s}\right)_{nq,k}, \tag{5}$$

and for particular case

$$(\gamma)_{nq,k} = (k)^{nq}\left(\frac{\gamma}{k}\right)_{nq}, \tag{6}$$

Note 1.3. More details of k-Pochammer symbol, k-special function and fractional Fourier transform one can refer to the papers by Romero et al [7, 8].

Definition 1.4. Let $k \in \mathbb{R}$; $\alpha, \beta, \gamma \in \mathbb{C}$; $Re(\alpha) > 0$, $Re(\beta) > 0$ and $\tau \in \mathbb{C}$, then the generalized k-Mittag-Leffler function, denoted by $E^{\gamma,\tau}_{k,\alpha,\beta}(z)$, is defined as

$$E^{\gamma,\tau}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}\, z^n}{\Gamma_k(n\alpha+\beta)n!}, \tag{7}$$

where $(x)_\tau, (x, \tau \in \mathbb{C})$ denotes the Pochhammer symbol with $(1)_n = n!$ for $n \in \mathbb{N} = \mathbb{N} \cup \{0\}$, defined in terms of gamma function as (also see [13, p. 199])

$$(x)_\tau = \frac{\Gamma(x+\tau)}{\Gamma(x)} = \begin{cases} 1 & (\tau = 0;\ x \in \mathbb{C}\backslash\{0\}) \\ x(x+1)\ldots(x+\tau-1) & (\tau = n \in N;\ x \in \mathbb{C}). \end{cases}$$

Particular cases of $E^{\gamma,\tau}_{k,\alpha,\beta}(z)$

(i) For $\tau = q$, equation (7) yields generalized k-Mittag-Leffler function, defined by

$$E^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}\, z^n}{\Gamma_k(n\alpha+\beta)n!} = E^{\gamma}_{k,\alpha,\beta}(z) \tag{8}$$

(ii) For $k = 1$, equation (8) yields generalized Mittag-Leffler function, defined by [11]

$$E^{\gamma,q}_{1,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq}\, z^n}{\Gamma(n\alpha+\beta)n!} = E^{\gamma,q}_{\alpha,\beta}(z) \tag{9}$$

(iii) For $q = 1$, equation (8) gives the Mittag-Leffler function defined by Doorego and Cerutti [2]

$$E^{\gamma,1}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}\, z^n}{\Gamma_k(n\alpha+\beta)n!} = E^{\gamma}_{k,\alpha,\beta}(z) \tag{10}$$

Note 1.5. A detailed account of Mittag-Leffler function and their applications can be found in the survey paper by Haubold, Mathai and Saxena [4], Saxena [9] and Saxena et al [10].

The following definitions are also needed in the analysis that follows:

Definition 1.6. Euler Transform

The Euler transform of the function $f(z)$ is defined by

$$B\{f(z);a,b\} = \int_0^1 z^{a-1}(1-z)^{b-1} f(z)dz,\ a,b \in C,\ Re(a) > 0,\ Re(b) > 0. \quad (11)$$

Definition 1.7. Laplace Transform

The Laplace transform of the function $f(t)$, denoted by $F(s)$ is defined by the equation

$$F(s) = (Lf)(s) = L\{f(t);s\} = \int_0^\infty e^{-st} f(t)dt, \qquad Re(s) > 0. \quad (12)$$

which may be symbolically written as

$$F(s) = L\{f(t);s\} \text{ or } f(t) = L^{-1}\{F(s);t\}$$

provided that the function $f(t)$ is continuous for $t \geq 0$ and of exponential order as $t \to \infty$.

Definition 1.8. Let $u = u(t)$ be a function of the space $S(R)$, the Schwartzian space of the function that decay rapidly at infinity together with all derivatives.

The Fourier transform is given by the integral

$$\hat{u}(\omega) = \Im[u](\omega) = \int_R u(t)\exp(i\omega t)dt \quad (13)$$

and the inverse Fourier transform can be defined as

$$\Im^{-1}[\hat{u}](t) = \frac{1}{2\pi}\int_R \hat{u}(\omega)\exp(-i\omega t)d\omega \quad (14)$$

Definition 1.9. Lizorkin space

Let $V(\mathbb{R})$ be the set of functions

$$V(\mathbb{R}) = \left\{ \upsilon \in S(\mathbb{R}) : \upsilon^{(n)}(0) = 0, n = 0,1,2,\ldots \right\}. \quad (15)$$

The Lizorkin space of function $\phi(\mathbb{R})$ is defined as

$$\phi(\mathbb{R}) = \{\varphi \in S(\mathbb{R}) : \Im[\varphi] \in V(\mathbb{R})\}. \quad (16)$$

Definition 1.10. Let u be a function belonging to $\phi(\mathbb{R})$.
The fractional Fourier transform of the order α, $0 < \alpha \leq 1$ is defined by

$$\hat{u}_\alpha(\omega) = \Im_\alpha[u](\omega) = \int_R e^{i\omega^{1/\alpha}t} u(t)dt. \tag{17}$$

If put $\alpha = 1$, equation (17) reduces to the conventional Fourier transform and for $\omega > 0$, it reduces to the fractional Fourier Transform defined by Luchko et al [5].

Lemma 1.11. *Let u be a function of the space $\phi(\mathbb{R})$, let α be a real number, $0 < \alpha \leq 1$, then*

$$\Im_\alpha[u](\omega) = \Im[u](x), \qquad for\, x = \omega^{1/\alpha}. \tag{18}$$

The inversion Fractional Fourier transform of the order α, $0 < \alpha \leq 1$, $u \in \phi(R)$ is defined as

$$\Im_\alpha^{-1}\{\hat{u}_\alpha(\omega)\}(t) = \frac{1}{2\pi\alpha} \int_R e^{-i\omega^{1/\alpha}t} \hat{u}_\alpha(\omega)\omega^{\frac{1-\alpha}{\alpha}} d\omega. \tag{19}$$

The following result will be required in evaluating the integral (30).

$$\int_0^\infty e^{-1/2} t^{\upsilon-1} W_{\lambda,\mu}(t)dt = \frac{\Gamma(1/2+\mu+\upsilon)\Gamma(1/2-\mu+\upsilon)}{\Gamma(1-\lambda+\upsilon)}, \; Re(\upsilon \pm \mu) > -1/2 \tag{20}$$

where the Whittaker function $W_{\lambda,\mu}(z)$ is defined in [3] (see also Mathai et al [6]).

$$W_{\mu,\upsilon}(z) = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\lambda-\mu\right)} M_{\lambda,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}-\lambda+\mu\right)} M_{\lambda,-\mu}(z) \tag{21}$$

where $M_{\lambda,\mu}(z)$ is defined as

$$M_{\lambda,\mu}(z) = z^{1/2+\mu} e^{-1/2\,z}\, {}_1F_1\left(\frac{1}{2}+\mu-\lambda; 2\mu+1; z\right).$$

The generalized Wright hypergeometric function ${}_p\Psi_q(z)$ is defined by Wright [14-16] (also see, [12, p.21]) in the following form :

$${}_p\Psi_q(z)\left[\begin{array}{l}(a_1,A_1),\dots,(a_p,A_p)\\(b_1,B_1),\dots,(b_q,B_q)\end{array}; Z\right] = \sum_{n=0}^\infty \left[\frac{\prod_{i=1}^p \Gamma(a_i+A_in)}{\prod_{j=1}^q \Gamma(b_j+B_jn)}\right] \frac{z^n}{n!} \tag{22}$$

where $a_i, b_j \in C$ and $A_i, B_j \in R (i = 1,\dots,p;\ j = 1,\dots,q)$ and the defining series (22) converges for

$$\sum_{j=1}^q B_j - \sum_{i=1}^p A_i > -1. \tag{23}$$

The object of this paper is to evaluate the Euler, Laplace, Whittaker and fractional Fourier transforms of the generalized k-Mittag-Leffler function $E_{k,\alpha,\beta}^{\gamma,\tau}(z)$, defined by (7). The results are obtained in a closed form in terms of the generalized Wright hypergeometric function, defined by (22). The results obtained in this paper are the generalization of the results given earlier by Saxena [9].

Theorem 1.12 (Euler Transform). *If $k \in R$; $\alpha,\beta,\gamma,a,b,\sigma \in C$; $Re(\alpha)>0$, $Re(\beta)>0$ and $\tau \in \mathbb{C}$, then*

$$\int_0^1 z^{a-1}(1-z)^{b-1}E^{\gamma,\tau}_{k,\alpha,\beta}(xz^{\sigma})dz = \frac{k^{1-\frac{\beta}{k}}\Gamma(b)}{\Gamma\left(\frac{\gamma}{k}\right)}\,{}_2\Psi_2\left[\begin{matrix}\left(\frac{\gamma}{k},\tau\right),(a,\sigma)\\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right),(a+b,\sigma)\end{matrix};k^{\tau-\frac{\alpha}{k}}x\right] \quad (24)$$

Proof. Using equation (7) and (11) and beta function formula, it gives

$$\int_0^1 z^{a-1}(1-z)^{b-1}E^{\gamma,\tau}_{k,\alpha,\beta}(xz^{\sigma})dz = \int_0^1 z^{a-1}(1-z)^{b-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}\,(xz^{\sigma})^n}{\Gamma_k(n\alpha+\beta)\,n!}dz$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}\,x^n}{\Gamma_k(n\alpha+\beta)\,n!}\int_0^1 z^{a-1}(1-z)^{b-1}z^{\sigma n}dz$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}\,x^n}{\Gamma_k(n\alpha+\beta)\,n!}\frac{\Gamma(b)\Gamma(\sigma n+a)}{\Gamma(\sigma n+a+b)}$$

using equation (4) and (6), it becomes

$$= \sum_{n=0}^{\infty}\frac{(k)^{n\tau}\left(\frac{\gamma}{k}\right)_{n\tau}\Gamma(b)\Gamma(\sigma n+a)}{(k)^{\frac{n\alpha+\beta}{k}-1}\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)\Gamma(\sigma n+a+b)}\frac{x^n}{n!}$$

$$= \frac{(k)^{1-\frac{\beta}{k}}\Gamma(b)}{\Gamma\left(\frac{\gamma}{k}\right)}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\gamma}{k}+n\tau\right)\Gamma(\sigma n+a)}{\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)\Gamma(\sigma n+a+b)}\frac{\left(k^{\tau-\frac{\alpha}{k}}x\right)^n}{n!}.$$

This completes the proof of the Theorem 1.12. □

Corollary 1.13. *For $\tau = q$, equation (24) reduces in the following form*

$$\int_0^1 z^{a-1}(1-z)^{b-1}E^{\gamma,q}_{k,\alpha,\beta}(xz^{\sigma})dz = \frac{k^{1-\frac{\beta}{k}}\Gamma(b)}{\Gamma\left(\frac{\gamma}{k}\right)}\,{}_2\Psi_2\left[\begin{matrix}\left(\frac{\gamma}{k},q\right),(a,\sigma)\\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right),(a+b,\sigma)\end{matrix};k^{q-\frac{\alpha}{k}}x\right] \quad (25)$$

Corollary 1.14. *For $k=1$, equation (25) reduces to*

$$\int_0^1 z^{a-1}(1-z)^{b-1}E^{\gamma,q}_{\alpha,\beta}(xz^{\sigma})dz = \frac{\Gamma(b)}{\Gamma(\gamma)}\,{}_2\Psi_2\left[\begin{matrix}(\gamma,q),(a,\sigma)\\ (\beta,\alpha),(a+b,\sigma)\end{matrix};x\right] \quad (26)$$

If we take $q=1$ in (26), it reduces to one given by Saxena [9].

Theorem 1.15 (Laplace Transform). *If $k \in R$; $\alpha,\beta,\gamma,a,b,\sigma \in C$; $Re(\alpha)>0$, $Re(\beta)>0$, $Re(s)>0$ and $\tau \in \mathbb{C}$, $\left|\frac{x}{s^{\sigma}}\right|<1$, then*

$$\int_0^{\infty} z^{a-1}e^{-sz}E^{\gamma,\tau}_{k,\alpha,\beta}(xz^{\sigma})dz = \frac{k^{1-\frac{\beta}{k}}s^{-a}}{\Gamma\left(\frac{\gamma}{k}\right)}\,{}_2\Psi_1\left[\begin{matrix}\left(\frac{\gamma}{k},\tau\right),(a,\sigma)\\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right),\end{matrix};\frac{k^{\tau-\frac{\alpha}{k}}x}{s^{\sigma}}\right] \quad (27)$$

Proof. Using equation (7) and (12) and gamma function formula, we obtain

$$\int_0^\infty z^{a-1}e^{-sz}E^{\gamma,\tau}_{k,\alpha,\beta}(xz^\sigma)dz = \int_0^\infty z^{a-1}e^{-sz}\sum_{n=0}^\infty \frac{(\gamma)_{n\tau,k}\,(xz^\sigma)^n}{\Gamma_k(n\alpha+\beta)\,n!}dz$$

$$= \sum_{n=0}^\infty \frac{(\gamma)_{n\tau,k}\,x^n}{\Gamma_k(n\alpha+\beta)\,n!}\frac{\Gamma(\sigma n+a)}{s^{\sigma n+a}}$$

using equation (4) and (6), it becomes

$$= s^{-a}\sum_{n=0}^\infty \frac{(k)^{n\tau}\left(\frac{\gamma}{k}\right)_{n\tau}\Gamma(\sigma n+a)}{(k)^{\frac{n\alpha+\beta}{k}-1}\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)n!}\left(\frac{x}{s^\sigma}\right)^n$$

$$= \frac{(k)^{1-\frac{\beta}{k}}s^{-a}}{\Gamma\left(\frac{\gamma}{k}\right)}\sum_{n=0}^\infty \frac{\Gamma\left(\frac{\gamma}{k}+n\tau\right)\Gamma(\sigma n+a)}{\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)n!}\left(\frac{k^{\tau-\frac{\alpha}{k}}x}{s^\sigma}\right)^n.$$

This completes the proof of the Theorem 1.15. □

Corollary 1.16. *For $\tau = q$, equation (27) reduces in the following form*

$$\int_0^\infty z^{a-1}e^{-sz}E^{\gamma,q}_{k,\alpha,\beta}(xz^\sigma)dz = \frac{k^{1-\frac{\beta}{k}}s^{-a}}{\Gamma\left(\frac{\gamma}{k}\right)}\,{}_2\Psi_1\left[\begin{matrix}\left(\frac{\gamma}{k},q\right),(a,\sigma)\\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right),\end{matrix};\frac{k^{q-\frac{\alpha}{k}}x}{s^\sigma}\right] \tag{28}$$

Corollary 1.17. *For $k = 1$, equation (28) reduces in the following form*

$$\int_0^\infty z^{a-1}e^{-sz}E^{\gamma,q}_{\alpha,\beta}(xz^\sigma)dz = \frac{s^{-a}}{\Gamma(\gamma)}\,{}_2\Psi_1\left[\begin{matrix}(\gamma,q),(a,\sigma)\\ (\beta,\alpha),\end{matrix};\frac{x}{s^\sigma}\right] \tag{29}$$

when $q = 1$ in (29), it yields a result given by Saxena [9].

Theorem 1.18 (Whittaker Transform)**.** *If $k \in R$; $\alpha,\beta,\gamma,\rho,\delta \in C$; $Re(\alpha) > 0$, $Re(\beta) > 0$, $Re(\rho) > 0$, $Re(\rho\pm\mu) > -1/2$ and $\tau \in \mathbb{C}$, then*

$$\int_0^\infty t^{\rho-1}e^{-pt/2}W_{\lambda,\mu}(pt)E^{\gamma,\tau}_{k,\alpha,\beta}(wt^\delta)dt$$
$$= \frac{k^{1-\frac{\beta}{k}}p^{-p}}{\Gamma\left(\frac{\gamma}{k}\right)}\,{}_2\Psi_2\left[\begin{matrix}\left(\frac{\gamma}{k},\tau\right),(1/2\pm\mu+\rho,\delta)\\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right),(1-\lambda+\rho,\delta)\end{matrix};\frac{k^{\tau-\frac{\alpha}{k}}w}{p^\delta}\right] \tag{30}$$

Proof. By virtue of equation (7) and (20), it yields

$$\int_0^\infty t^{\rho-1}e^{-pt/2}W_{\lambda,\mu}(pt)E^{\gamma,\tau}_{k,\alpha,\beta}(wt^\delta)dt$$

if we set $pt = \upsilon$, then the above line is equal to

$$\int_0^\infty e^{-\upsilon/2}\left(\frac{\upsilon}{p}\right)^{\rho-1}W_{\lambda,\mu}(\upsilon)\sum_{n=0}^\infty\frac{(\gamma)_{n\tau,k}\,w^n}{\Gamma_k(n\alpha+\beta)\,n!}\left(\frac{\upsilon}{p}\right)^{\delta n}\frac{1}{p}\,d\upsilon$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}\, w^n}{\Gamma_k(n\alpha+\beta)\, n!} \int_0^{\infty} e^{-\upsilon/2} \left(\frac{\upsilon}{p}\right)^{\rho-1} \left(\frac{\upsilon}{p}\right)^{\delta n} W_{\lambda,\mu}(\upsilon) \frac{1}{p}\, d\upsilon$$

$$= p^{-\rho} \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)\, n!} \left(\frac{w}{p^{\delta}}\right)^n \int_0^{\infty} e^{-\upsilon/2} \upsilon^{\delta n+\rho-1} W_{\lambda,\mu}(\upsilon)\, d\upsilon$$

$$= p^{-\rho} \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}\, \Gamma(1/2+\mu+\delta n+\rho)\Gamma(1/2-\mu+\delta n+\rho)}{\Gamma_k(n\alpha+\beta)\Gamma(1-\lambda+\delta n+\rho)\, n!} \left(\frac{w}{p^{\delta}}\right)^n$$

using equation (4) and (6), it simplifies to

$$p^{-p} \sum_{n=0}^{\infty} \frac{(k)^{n\tau} \left(\frac{\gamma}{k}\right)_{n\tau} \Gamma(1/2+\mu+\delta n+\rho)\Gamma(1/2-\mu+\delta n+\rho)}{(k)^{\frac{n\alpha+\beta}{k}-1} \Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right) \Gamma(1-\lambda+\delta n+\rho)\, n!} \left(\frac{w}{p^{\delta}}\right)^n$$

$$= \frac{(k)^{1-\frac{\beta}{k}} p^{-p}}{\Gamma\left(\frac{\gamma}{k}\right)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\gamma}{k}+n\tau\right) \Gamma(1/2+\mu+\delta n+\rho)\Gamma(1/2-\mu+\delta n+\rho)}{\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right) \Gamma(1-\lambda+\delta n+\rho)\, n!} \left(\frac{k^{\tau-\frac{\alpha}{k}} w}{p^{\delta}}\right)^n.$$

This completes the proof of the Theorem 1.18. □

Corollary 1.19. *For $\tau = q$, equation (30) reduces in the following form*

$$\int_0^{\infty} t^{\rho-1} e^{-pt/2} W_{\lambda,\mu}(pt) E^{\gamma,q}_{k,\alpha,\beta}(wt^{\delta}) dt = \frac{k^{1-\frac{\beta}{k}} p^{-p}}{\Gamma\left(\frac{\gamma}{k}\right)} {}_2\Psi_2 \left[\begin{matrix} \left(\frac{\gamma}{k}, q\right), (1/2 \pm \mu+\rho, \delta) \\ \left(\frac{\beta}{k}, \frac{\alpha}{k}\right), (1-\lambda+\rho, \delta) \end{matrix} ; \frac{k^{q-\frac{\alpha}{k}} w}{p^{\delta}} \right] \quad (31)$$

Corollary 1.20. *For $k = 1$, equation (31) reduces in the following form*

$$\int_0^{\infty} t^{\rho-1} e^{-pt/2} W_{\lambda,\mu}(pt) E^{\gamma,q}_{\alpha,\beta}(wt^{\delta}) dt = \frac{p^{-p}}{\Gamma(\gamma)} {}_2\Psi_2 \left[\begin{matrix} (\gamma, q),\ (1/2 \pm \mu+\rho, \delta) \\ (\beta, \alpha),\ (1-\lambda+\rho, \delta) \end{matrix} ; \frac{w}{p^{\delta}} \right] \quad (32)$$

when $q = 1$, (32) reduces to one given by Saxena [9].

Fractional Fourier Transform (FFT) of generalized k-Mittag-Leffler Function

Theorem 1.21. *The FFT of order α of the generalized k-Mittag-Leffler function for $t < 0$ is given by*

$$\Im_{\alpha} \left[E^{\gamma,\tau}_{k,\alpha,\beta}(t) \right] (\omega) = \frac{(k)^{1-\frac{\beta}{k}}}{\Gamma\left(\frac{\gamma}{k}\right)} \sum_{n=0}^{\infty} \frac{(k)^{\left(\tau-\frac{\beta}{k}\right)n} \Gamma\left(\frac{\gamma}{k}+n\tau\right) (-1)^n (i)^{-n-1} \omega^{-(n+1)/\alpha}}{\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)} \quad (33)$$

Proof. Using equation (7) and (17) and gamma function formula, it gives

$$\Im_\alpha\left[E^{\gamma,\tau}_{k,\alpha,\beta}(t)\right](\omega) = \int_R e^{i\omega^{1/\alpha}t} E^{\gamma,\tau}_{k,\alpha,\beta}(t)dt$$

$$= \int_R e^{i\omega^{1/\alpha}t}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}\, t^n}{\Gamma_k(n\alpha+\beta)n!}dt$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)n!}\int_R e^{i\omega^{1/\alpha}t}t^n dt$$

if we set $i\omega^{1/\alpha}t = -\xi$, then

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)n!}\int_{-\infty}^{0} e^{-\xi}\left(\frac{-\xi}{i\omega^{1/\alpha}}\right)^n\left(\frac{-d\xi}{i\omega^{1/\alpha}}\right)$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)(n!)(i)^{n+1}\omega^{(n+1)/\alpha}(-1)^n}\int_0^{\infty} e^{-\xi}\xi^n d\xi$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}(i)^{-n-1}\omega^{-(n+1)/\alpha}(-1)^{-n}}{\Gamma_k(n\alpha+\beta)(n!)}\Gamma(n+1)$$

In view of equation (4) and (6), the above line

$$= \sum_{n=0}^{\infty}\frac{(k)^{n\tau}\left(\frac{\gamma}{k}\right)_{n\tau}(-1)^n(i)^{-n-1}\omega^{-(n+1)/\alpha}}{(k)^{\frac{n\alpha+\beta}{k}-1}\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)}.$$

This completes the proof of the Theorem 1.21. □

Corollary 1.22. *When $\tau = q$, equation (33) reduces in the following form*

$$\Im_\alpha\left[E^{\gamma,q}_{k,\alpha,\beta}(t)\right](\omega) = \frac{(k)^{1-\frac{\beta}{k}}}{\Gamma\left(\frac{\gamma}{k}\right)}\sum_{n=0}^{\infty}\frac{(k)^{\left(q-\frac{\beta}{k}\right)n}\Gamma\left(\frac{\gamma}{k}+nq\right)(-1)^n(i)^{-n-1}\omega^{-(n+1)/\alpha}}{\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)} \tag{34}$$

Corollary 1.23. *For $k = 1$, equation (34) reduces in the following form*

$$\Im_\alpha\left[E^{\gamma,q}_{\alpha,\beta}(t)\right](\omega) = \frac{1}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+nq)(-1)^n(i)^{-n-1}\omega^{-(n+1)/\alpha}}{\Gamma(\alpha n+\beta)}. \tag{35}$$

Note 1.24. Fractional Fourier transform is studied among others, by Luchko et al [5] and Romero et al [7, 8].

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