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COEFFICIENT ESTIMATES OF BI-BAZILEVIČ FUNCTIONS DEFINED BY SRIVASTAVA-ATTIYA OPERATOR

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In this paper, we introduce and investigate two new subclasses of the function class Σ of bi-univalent functions defined in the open unit disk, which are associated with the generalized Srivastava-Attiya operator, satisfying subordinate conditions. Furthermore, we find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses. Several (known or new) consequences of the results are also pointed out.

1. Introduction, Definitions and Preliminaries

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

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Dedicated to my father Prof. P. M. Gangadhran,(1938-2011).

Further denote by $\mathcal{S}$, the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\alpha)$ of starlike functions of order $\alpha, (0 \leq \alpha < 1)$ in $\mathbb{U}$ and the class $\mathcal{K}(\alpha)$ of convex functions of order $\alpha(0 \leq \alpha < 1)$ in $\mathbb{U}$. It is well known (see [3, 4]) that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \quad (2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1).

An analytic function f is subordinate to an analytic function g, written by $f(z) \prec g(z)$, provided there is an analytic function w defined on $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. Ma and Minda [23] unified various subclasses of starlike and convex functions for which either of the quantity

$$\frac{z f'(z)}{f(z)} \text{ or } 1 + \frac{z f''(z)}{f'(z)}$$

is subordinate to a more general superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in the unit disk $\mathbb{U}$, $\phi(0) = 1, \phi'(0) > 0$, and ϕ maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $\frac{z f'(z)}{f(z)} \prec \phi(z)$. Similarly, the class of Ma-Minda convex functions of functions $f \in \mathcal{A}$ satisfying the subordination $1 + \frac{z f''(z)}{f'(z)} \prec \phi(z)$.

A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $\mathcal{S}^*_\Sigma(\phi)$ and $\mathcal{K}_\Sigma(\phi)$.In the sequel, it is assumed that ϕ is an analytic function with positive real part in the unit disk $\mathbb{U}$, satisfying $\phi(0) = 1, \phi'(0) > 0$, and $\phi(\mathbb{U})$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots , \quad (B_1 > 0). \quad (3)$$

The study of operators plays an important role in the geometric function theory and its related fields. Many differential and integral operators can be written in terms of convolution of certain analytic functions. It is observed that this formalism brings an ease in further mathematical exploration and also helps to understand the geometric properties of such operators better. The convolution or Hadamard product of two functions $f_1, f_2 \in \mathcal{A}$ is denoted by $f_1 * f_2$ and is defined as

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n, \tag{4}$$

where $f_1(z) = z + \sum\limits_{n=2}^{\infty} a_{n,1} z^n$ and $f_2(z) = z + \sum\limits_{n=2}^{\infty} a_{n,2} z^n$.

We recall here a general Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ defined in [33] by

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{5}$$

$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}, \text{when } |z| < 1; \Re(s) > 1 \text{ when } |z| = 1)$ where, as usual, $\mathbb{Z}_0^- := \mathbb{Z} \setminus \{\mathbb{N}\}$, $(\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ and $\mathbb{N} := \{1, 2, 3, \ldots\})$. Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ can be found in [7], and the references stated therein (see also [9, 20, 32]). Srivastava and Attiya [32] (also see [2, 15]) introduced and investigated the linear operator:

$$\mathcal{J}_b^{\mu} : \mathcal{A} \to \mathcal{A}$$

defined in terms of the Hadamard product by

$$\mathcal{J}_b^{\mu} f(z) = \mathcal{G}_b^{\mu} * f(z) \tag{6}$$

$(z \in \mathbb{U}; b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; \mu \in \mathbb{C}; f \in \mathcal{A})$, where, for convenience,

$$\mathcal{G}_b^{\mu}(z) := (1+b)^{\mu} [\Phi(z, \mu, b) - b^{-\mu}] \ \ (z \in \mathbb{U}). \tag{7}$$

We recall here the following relationships (given earlier by [32]) which follow easily by using (1), (6) and (7)

$$\mathcal{J}_b^{\mu} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b} \right)^{\mu} a_n z^n. \tag{8}$$

Motivated essentially by the Srivastava-Attiya operator, Murugusundaramoorthy [24] introduced the generalized integral operator $\mathcal{J}_{\mu,b}^{m,k}$ given by

$$\mathcal{J}_{\mu,b}^{m,k} f(z) = z + \sum_{n=2}^{\infty} C_n^m(b, \mu, k) a_n z^n. \tag{9}$$

where

$$\Psi_n = C_n^m(b,\mu,k) = \left| \left(\frac{1+b}{n+b}\right)^{\mu} \frac{m!(n+k-2)!}{(k-2)!(n+m-1)!} \right| \tag{10}$$

and (throughout this paper unless otherwise mentioned) the parameters μ, b are constrained as $b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; \mu \in \mathbb{C}, k \geq 2$ and $m > -1$. It is of interest to note that $J_{\mu,b}^{1,2}$ is the Srivastava-Attiya operator and $J_{0,b}^{m,k}$ is the well-known Choi-Saigo- Srivastava operator (see [21]). Suitably specializing the parameters m, k, μ and b in $\mathcal{J}_{\mu,b}^{m,k} f(z)$ we can get various integral operators introduced by Alexander [1]and Bernardi[6]Libera and Livingston[17, 18]. Further we get the Jung-Kim-Srivastava integral operator [14] closely related to some multiplier transformation studied by Flett [10].

Several authors have discussed various subfamilies of Bazilevič functions [29] of type λ from various perspective. They discussed it from the perspective of convexity, inclusion theorem, radii of starlikeness, and convexity boundary rotational problem, subordination just to mention few. The most amazing thing is that, it is difficult to see any of this authors discussing the coefficient inequalities, and coefficient bounds of these subfamilies of Bazilevič function most especially when the parameter λ is greater than one (λ is real).

Recently there has been triggering interest to study bi-univalent function class Σ and obtained non-sharp coefficient estimates on the first two coefficients $|a_2|$ and $|a_3|$ of (1). But the coefficient problem for each of the following Taylor-Maclaurin coefficients:

$$|a_n| \qquad (n \in \mathbb{N} \setminus \{1,2\}; \;\; \mathbb{N} := \{1,2,3,\cdots\}$$

is still an open problem (see [3–5, 16, 25, 34]). Many researchers (see [12, 13, 22, 27, 30, 31, 35, 36]) have recently introduced and investigated several interesting subclasses of the bi-univalent function class Σ and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Motivated by the earlier work of Deniz [11], in the present paper we introduce new subfamilies of bi-Bazilevič functions of the function class Σ, involving generalized integral operator$\mathcal{J}_{\mu,b}^{m,k}$ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the new subclass of the function class Σ. Several related classes are also considered, and connection to earlier known (or new) results are made.

Definition 1.1. Let $h : \mathbb{U} \longrightarrow \mathbb{C}$ be a convex univalent function in $\mathbb{U}$ such that

$$h(0) = 1 \qquad \text{and} \qquad \Re\big(h(z)\big) > 0 \qquad (z \in \mathbb{U}).$$

Suppose also that the function $h(z)$ is given by

$$h(z) = 1 + \sum_{n=1}^{\infty} B_n z^n \qquad (z \in \mathbb{U}). \tag{11}$$

A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{B}_{\Sigma,\mu,b}^{m,k,\beta}(\lambda,h)$ if the following conditions are satisfied:

$$e^{i\beta}\left(\frac{z^{1-\lambda}(\mathcal{J}_{\mu,b}^{m,k}f(z))'}{[\mathcal{J}_{\mu,b}^{m,k}f(z)]^{1-\lambda}}\right) \prec h(z)\cos\beta + i\sin\beta \tag{12}$$

and

$$e^{i\beta}\left(\frac{w^{1-\lambda}(\mathcal{J}_{\mu,b}^{m,k}g(w))'}{[\mathcal{J}_{\mu,b}^{m,k}g(w)]^{1-\lambda}}\right) \prec h(w)\cos\beta + i\sin\beta \tag{13}$$

where $\beta \in (\frac{-\pi}{2},\frac{\pi}{2}); \lambda \geq 0; z,w \in \mathbb{U}$ and the function g is given by (2).

Example 1.2. If we set $h(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$, then the class $\mathcal{B}_{\Sigma,\mu,b}^{m,k,\beta}(\lambda,h) \equiv \mathcal{B}_{\Sigma,\mu,b}^{m,k,\beta}(\lambda,A,B)$ which is defined as $f \in \Sigma$,

$$e^{i\beta}\left(\frac{z^{1-\lambda}(\mathcal{J}_{\mu,b}^{m,k}f(z))'}{[\mathcal{J}_{\mu,b}^{m,k}f(z)]^{1-\lambda}}\right) \prec \frac{1+Az}{1+Bz}\cos\beta + i\sin\beta,$$

and

$$e^{i\beta}\left(\frac{w^{1-\lambda}(\mathcal{J}_{\mu,b}^{m,k}g(w))'}{[\mathcal{J}_{\mu,b}^{m,k}g(w)]^{1-\lambda}}\right) \prec \frac{1+Aw}{1+Bw}\cos\beta + i\sin\beta,$$

Example 1.3. If we set $h(z) = \frac{1+(1-2\alpha)z}{1-z},\ 0 \leq \alpha < 1$ then the class $\mathcal{B}_{\Sigma,\mu,b}^{m,k,\beta}(\lambda,h) \equiv \mathcal{B}_{\Sigma,\mu,b}^{m,k,\beta}(\lambda,\alpha)$ which is defined as $f \in \Sigma$,

$$\Re\left[e^{i\beta}\left(\frac{z^{1-\lambda}(\mathcal{J}_{\mu,b}^{m,k}f(z))'}{[\mathcal{J}_{\mu,b}^{m,k}f(z)]^{1-\lambda}}\right)\right] > \alpha\cos\beta$$

and

$$\Re\left[e^{i\beta}\left(\frac{w^{1-\lambda}(\mathcal{J}_{\mu,b}^{m,k}g(w))'}{[\mathcal{J}_{\mu,b}^{m,k}g(w)]^{1-\lambda}}\right)\right] > \alpha\cos\beta$$

On specializing the parameters λ one can state the various new subclasses of Σ as illustrated in the following examples.

Example 1.4. For $\lambda = 0$ and a function $f \in \Sigma$, given by (1) is said to be in the class $\mathcal{B}_{\Sigma,\mu,b}^{m,k,\beta}(h)$ if the following conditions are satisfied:

$$e^{i\beta}\left(\frac{z(\mathcal{J}_{\mu,b}^{m,k}f(z))'}{\mathcal{J}_{\mu,b}^{m,k}f(z)}\right) \prec h(z)\cos\beta + i\sin\beta \tag{14}$$

and

$$e^{i\beta}\left(\frac{w(\mathcal{J}^{m,k}_{\mu,b}g(w))'}{\mathcal{J}^{m,k}_{\mu,b}g(w)}\right) \prec h(w)\cos\beta + i\sin\beta \tag{15}$$

where $\beta \in (\frac{-\pi}{2}, \frac{\pi}{2}); z, w \in \mathbb{U}$ and the function g is given by(2).

Example 1.5. For $\lambda = 1$ and a function $f \in \Sigma$, given by (1) is said to be in the class $\mathcal{G}^{m,k,\beta}_{\Sigma,\mu,b}(h)$ if the following conditions are satisfied:

$$e^{i\beta}\left(\mathcal{J}^{m,k}_{\mu,b}f(z))'\right) \prec h(w)\cos\beta + i\sin\beta \tag{16}$$

and

$$e^{i\beta}\left((\mathcal{J}^{m,k}_{\mu,b}g(w))'\right) \prec h(w)\cos\beta + i\sin\beta \tag{17}$$

where $\beta \in (\frac{-\pi}{2}, \frac{\pi}{2}); z, w \in \mathbb{U}$ and the function g is given by (2).

It is of interest to note that for $k = 2$ and $m = 1$ with $\mu = 0$, $b = 0$, the class $\mathcal{B}^{m,k,\beta}_{\Sigma,\mu,b}(\lambda, h)$ reduces to the following new subclasses:

Example 1.6. For $\lambda = 0$ and $\beta \in (\frac{-\pi}{2}, \frac{\pi}{2})$ a function $f \in \Sigma$, given by (1) is said to be in the class $\mathcal{B}^{1,2,\beta}_{\Sigma,0,0}(0,h) \equiv \mathcal{S}^*_\Sigma(\beta, h)$ if the following conditions are satisfied:

$$e^{i\beta}\left(\frac{zf'(z)}{f(z)}\right) \prec h(z)\cos\beta + i\sin\beta \ \ \textit{and} \ \ e^{i\beta}\left(\frac{wg'(w)}{g(w)}\right) \prec h(w)\cos\beta + i\sin\beta,$$

where $z, w \in \mathbb{U}$ and the function g is given by (2).

Example 1.7. For $\lambda = 1$ and $\beta \in (\frac{-\pi}{2}, \frac{\pi}{2})$, a function $f \in \Sigma$, given by (1) is said to be in the class $\mathcal{B}^{1,2,\beta}_{\Sigma,0,0}(1,h) \equiv \mathcal{H}^*_\Sigma(\beta, h)$ if the following conditions are satisfied:

$$e^{i\beta}\left(f'(z)\right) \prec h(z)\cos\beta + i\sin\beta \ \ \textit{and} \ \ e^{i\beta}\left(g'(w)\right) \prec h(w)\cos\beta + i\sin\beta,$$

where $z, w \in \mathbb{U}$ and the function g is given by (2)

Remark 1.8. As mentioned in Example 1.2, 1.3 we state some new analogous subclasses $\mathcal{B}^{m,k,\beta}_{\Sigma,\mu,b}(A,B)$; $\mathcal{B}^{m,k,\beta}_{\Sigma,\mu,b}(\alpha)$; $\mathcal{G}^{m,k,\beta}_{\Sigma,\mu,b}(A,B)$; $\mathcal{G}^{m,k,\beta}_{\Sigma,\mu,b}(\alpha)$; $\mathcal{S}^*_\Sigma(\beta,A,B)$; $\mathcal{S}^*_\Sigma(\beta,\alpha)$; $\mathcal{H}^*_\Sigma(\beta,A,B)$ and $\mathcal{H}^*_\Sigma(\beta,\alpha)$ for the classes defined in Examples 1.4 to 1.7 respectively by setting $h(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, (or $h(z) = \frac{1+(1-2\alpha)z}{1-z}$, $0 \le \alpha < 1$).

In the following section we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the bi-bazilevic function class $\mathcal{B}^{m,k,\beta}_{\Sigma,\mu,b}(\lambda, h)$ of the function class Σ.

In order to derive our main results, we shall need the following lemmas:

Lemma 1.9 (see [26]). *If $p \in \mathcal{P}$, then $|p_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of all functions p analytic in $\mathbb{U}$ for which $\Re(p(z)) > 0$, where $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ for $z \in \mathbb{U}$.*

Lemma 1.10 (see [8, 28]). *Let the function $\phi(z)$ given by $\phi(z) = \sum\limits_{n=1}^{\infty} B_n z^n, (z \in \mathbb{U})$ be convex in U.Suppose that the function $h(z)$ given by $h(z) = \sum\limits_{n=1}^{\infty} h_n z^n$,is holomorphic in $\mathbb{U}$.If $h(z) \prec \phi(z), (z \in \mathbb{U})$ then $|h_n| \leq |B_1|, (n \in \mathbb{N})$.*

2. Coefficient Bounds for the Function Class $\mathcal{B}^{m,k,}_{\Sigma,\mu,b}(\beta, \lambda, h)$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the bi-bazelvic class $\mathcal{B}^{m,k}_{\Sigma,\mu,b}(\beta, \lambda, h)$.

Theorem 2.1. *Let the function $f(z)$ given by (1) be in the class $\mathcal{B}^{m,k}_{\Sigma,\mu,b}(\beta, \lambda, h)$. Then*

$$|a_2| \leqq \sqrt{\frac{2|B_1|\cos\beta}{[(\lambda-1)(\lambda+2)\Psi_2^2 + 2(\lambda+2)\Psi_3]}} \tag{18}$$

and

$$|a_3| \leqq \frac{|B_1|\cos\beta}{(\lambda+2)\Psi_3} + \frac{|B_1|^2\cos^2\beta}{(1+\lambda)^2\Psi_2^2}. \tag{19}$$

Proof. It follows from (12) and (13) that

$$e^{i\beta}\left(\frac{z^{1-\lambda}(\mathcal{J}^{m,k}_{\mu,b} f(z))'}{[\mathcal{J}^{m,k}_{\mu,b} f(z)]^{1-\lambda}}\right) = p(z)\cos\beta + i\sin\beta \tag{20}$$

and

$$e^{i\beta}\left(\frac{w^{1-\lambda}(\mathcal{J}^{m,k}_{\mu,b} g(w))'}{[\mathcal{J}^{m,k}_{\mu,b} g(w)]^{1-\lambda}}\right) = q(w)\cos\beta + i\sin\beta, \tag{21}$$

where the functions

$$p(z) \prec h(z) \qquad (z \in \mathbb{U}) \qquad \text{and} \qquad q(w) \prec h(w) \qquad (w \in \mathbb{U})$$

are in the above-defined class $\mathcal{P}$ and have the following forms:

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \tag{22}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + \cdots \tag{23}$$

respectively. Now, equating the coefficients in (20) and (21), we get

$$e^{i\beta}(1+\lambda)\Psi_2 a_2 = p_1 \cos\beta, \tag{24}$$

$$e^{i\beta}\left[\frac{(\lambda-1)(\lambda+2)}{2}\Psi_2^2 a_2^2 + (\lambda+2)\Psi_3 a_3\right] = p_2 \cos\beta \tag{25}$$

$$-e^{i\beta}(\lambda+1)\Psi_2 a_2 = q_1 \cos\beta \tag{26}$$

and

$$e^{i\beta}\left[\left(2(\lambda+2)\Psi_3 + \frac{(\lambda-1)(\lambda+2)}{2}\Psi_2^2\right)a_2^2 - (\lambda+2)\Psi_3 a_3\right] = q_2 \cos\beta \tag{27}$$

From (24) and (26), we find that

$$a_2 = \frac{p_1 \cos\beta e^{-i\beta}}{(\lambda+1)\Psi_2} = -\frac{q_1 \cos\beta e^{-i\beta}}{(\lambda+1)\Psi_2}, \tag{28}$$

which implies

$$p_1 = -q_1. \tag{29}$$

and

$$2(\lambda+1)^2\Psi_2^2 a_2^2 = (p_1^2+q_1^2)\cos^2\beta\ e^{-2i\beta}. \tag{30}$$

Adding (25) and (27), by using (28) and (29), we obtain

$$e^{i\beta}\left[(\lambda-1)(\lambda+2)\Psi_2^2 + 2(\lambda+2)\Psi_3\right]a_2^2 = (p_2+q_2)\cos\beta. \tag{31}$$

Thus,

$$a_2^2 = \frac{(p_2+q_2)\cos\beta}{\left[(\lambda-1)(\lambda+2)\Psi_2^2 + 2(\lambda+2)\Psi_3\right]}e^{-i\beta}. \tag{32}$$

Applying Lemma 1.9 for the coefficients p_2 and q_2, we immediately have

$$|a_2|^2 = \frac{2|B_1|\cos\beta}{\left[(\lambda-1)(\lambda+2)\Psi_2^2 + 2(\lambda+2)\Psi_3\right]}. \tag{33}$$

This gives the bound on $|a_2|$ as asserted in (18).

Next, in order to find the bound on $|a_3|$, by subtracting (27) from (25), we get

$$e^{i\beta}\left[2(\lambda+2)\Psi_3 a_3 - 2(\lambda+2)\Psi_3 a_2^2\right] = (p_2-q_2)\cos\beta. \tag{34}$$

It follows from (28), (29) and (34) that

$$a_3 = \frac{(p_2-q_2)\cos\beta\ e^{-i\beta}}{2(\lambda+2)\Psi_3} + \frac{(p_1^2+q_1^2)\cos^2\beta\ e^{-i2\beta}}{2(1+\lambda)^2\Psi_2^2}.$$

Applying Lemma 1.9 once again for the coefficients p_2 and q_2, we readily get

$$|a_3| \leqq \frac{|B_1|\cos\beta}{(\lambda+2)\Psi_3} + \frac{|B_1|^2\cos^2\beta}{(1+\lambda)^2\Psi_2^2}.$$

This completes the proof of Theorem 2.1. □

Putting $\lambda = 0$ in Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let the function $f(z)$ given by (1) be in the class $\mathcal{B}_{\Sigma,\mu,b}^{m,k,\beta}(h)$. Then*

$$|a_2| \leqq \sqrt{\frac{|B_1|\cos\beta}{2\Psi_3 - \Psi_2^2}} \tag{35}$$

and

$$|a_3| \leqq \frac{|B_1|\cos\beta}{2\Psi_3} + \frac{|B_1|^2\cos^2\beta}{\Psi_2^2}. \tag{36}$$

Putting $\lambda = 1$ in Theorem 2.1, we have the following corollary.

Corollary 2.3. *Let the function $f(z)$ given by (1) be in the class $\mathcal{G}_{\Sigma,\mu,b}^{m,k,\beta}(h)$. Then*

$$|a_2| \leqq \sqrt{\frac{|B_1|\cos\beta}{3\Psi_3}} \tag{37}$$

and

$$|a_3| \leqq \frac{|B_1|\cos\beta}{3\Psi_3} + \frac{|B_1|^2\cos^2\beta}{4\Psi_2^2}. \tag{38}$$

Taking $k = 2$ and $m = 1$ with $\mu = 0$, $b = 0$, in Corollary 2.2 and 2.3, we get the following corollaries.

Corollary 2.4. *Let the function $f(z)$ given by (1) be in the class $\mathcal{S}_\Sigma^*(\beta,h)$. Then*

$$|a_2| \leqq \sqrt{|B_1|\cos\beta} \tag{39}$$

and

$$|a_3| \leqq \frac{|B_1|\cos\beta}{2} + |B_1|^2\cos^2\beta. \tag{40}$$

Corollary 2.5. *Let the function $f(z)$ given by (1) be in the class $\mathcal{H}_\Sigma^*(\beta,h)$. Then*

$$|a_2| \leqq \sqrt{\frac{|B_1|\cos\beta}{3}} \tag{41}$$

and

$$|a_3| \leqq \frac{|B_1|\cos\beta}{3} + \frac{|B_1|^2\cos^2\beta}{4}. \tag{42}$$

3. Corollaries and Consequences

By setting $h(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$, in Theorem 2.1, we state the following Theorem .

Theorem 3.1. *Let the function $f(z)$ given by (1) be in the class $\mathcal{B}_{\Sigma,\mu,b}^{m,k,\beta}(\lambda, A, B)$. Then*

$$|a_2| \leqq \sqrt{\frac{2(A-B)\cos\beta}{\left[(\lambda-1)(\lambda+2)\Psi_2^2 + 2(\lambda+2)\Psi_3\right]}} \tag{43}$$

and

$$|a_3| \leqq \frac{(A-B)\cos\beta}{(\lambda+2)\Psi_3} + \frac{(A-B)^2\cos^2\beta}{(1+\lambda)^2\Psi_2^2}. \tag{44}$$

Putting $\lambda = 0$ in Theorem 3.1, we have the following corollary.

Corollary 3.2. *Let the function $f(z)$ given by (1) be in the class $\mathcal{B}_{\Sigma,\mu,b}^{m,k,\beta}(A, B)$. Then*

$$|a_2| \leqq \sqrt{\frac{(A-B)\cos\beta}{2\Psi_3 - \Psi_2^2}} \tag{45}$$

and

$$|a_3| \leqq \frac{(A-B)\cos\beta}{2\Psi_3} + \frac{(A-B)^2\cos^2\beta}{\Psi_2^2}. \tag{46}$$

Putting $\lambda = 1$ in Theorem 3.1, we have the following corollary.

Corollary 3.3. *Let the function $f(z)$ given by (1) be in the class $\mathcal{G}_{\Sigma,\mu,b}^{m,k,\beta}(A, B)$. Then*

$$|a_2| \leqq \sqrt{\frac{(A-B)\cos\beta}{3\Psi_3}} \tag{47}$$

and

$$|a_3| \leqq \frac{(A-B)\cos\beta}{3\Psi_3} + \frac{(A-B)^2\cos^2\beta}{4\Psi_2^2}. \tag{48}$$

Taking $k = 2$ and $m = 1$ with $\mu = 0$, $b = 0$, in Corollary 3.2 and 3.3, we get the following corollaries.

Corollary 3.4. *Let the function $f(z)$ given by (1) be in the class $\mathcal{S}_{\Sigma}^{*}(\beta, A, B)$. Then*

$$|a_2| \leqq \sqrt{(A-B)\cos\beta} \tag{49}$$

and

$$|a_3| \leqq \frac{(A-B)\cos\beta}{2} + (A-B)^2\cos^2\beta. \tag{50}$$

Corollary 3.5. *Let the function $f(z)$ given by (1) be in the class $\mathcal{H}_{\Sigma}^{*}(\beta, A, B)$. Then*

$$|a_2| \leqq \sqrt{\frac{(A-B)\cos\beta}{3}} \tag{51}$$

and

$$|a_3| \leqq \frac{(A-B)\cos\beta}{3} + \frac{(A-B)^2\cos^2\beta}{4}. \tag{52}$$

Further, by setting $h(z) = \frac{1+(1-2\alpha)z}{1-z}$, $0 \leq \alpha < 1$ in Theorem 2.1 we get the following result.

Theorem 3.6. *Let the function $f(z)$ given by (1) be in the class $\mathcal{B}_{\Sigma,\mu,b}^{m,k,\beta}(\lambda, \alpha)$. Then*

$$|a_2| \leqq \sqrt{\frac{4(1-\alpha)\cos\beta}{[(\lambda-1)(\lambda+2)\Psi_2^2 + 2(\lambda+2)\Psi_3]}} \tag{53}$$

and

$$|a_3| \leqq \frac{2(1-\alpha)\cos\beta}{(\lambda+2)\Psi_3} + \frac{4(1-\alpha)^2\cos^2\beta}{(1+\lambda)^2\Psi_2^2}. \tag{54}$$

Putting $\lambda = 0$ in Theorem 3.6, we have the following corollary.

Corollary 3.7. *Let the function $f(z)$ given by (1) be in the class $\mathcal{B}_{\Sigma,\mu,b}^{m,k,\beta}(\alpha)$. Then*

$$|a_2| \leqq \sqrt{\frac{2(1-\alpha)\cos\beta}{2\Psi_3 - \Psi_2^2}} \tag{55}$$

and

$$|a_3| \leqq \frac{(1-\alpha)\cos\beta}{\Psi_3} + \frac{4(1-\alpha)^2\cos^2\beta}{\Psi_2^2}. \tag{56}$$

Putting $\lambda = 1$ in Theorem 3.6, we have the following corollary.

Corollary 3.8. *Let the function $f(z)$ given by (1) be in the class $\mathcal{G}_{\Sigma,\mu,b}^{m,k,\beta}(\alpha)$. Then*

$$|a_2| \leqq \sqrt{\frac{2(1-\alpha)\cos\beta}{3\Psi_3}} \tag{57}$$

and

$$|a_3| \leqq \frac{2(1-\alpha)\cos\beta}{3\Psi_3} + \frac{(1-\alpha)^2\cos^2\beta}{\Psi_2^2}. \tag{58}$$

Taking $k = 2$ and $m = 1$ with $\mu = 0$, $b = 0$, in Corollary 3.7 and 3.8, we get the following corollaries.

Corollary 3.9. *Let the function $f(z)$ given by (1) be in the class $\mathcal{S}^*_\Sigma(\beta,\alpha)$. Then*

$$|a_2| \leqq \sqrt{2(1-\alpha)\cos\beta} \tag{59}$$

and

$$|a_3| \leqq (1-\alpha)\cos\beta \; + 4(1-\alpha)^2\cos^2\beta. \tag{60}$$

Corollary 3.10. *Let the function $f(z)$ given by (1) be in the class $\mathcal{H}^*_\Sigma(\beta,\alpha)$. Then*

$$|a_2| \leqq \sqrt{\frac{2(1-\alpha)\cos\beta}{3}} \tag{61}$$

and

$$|a_3| \leqq \frac{2(1-\alpha)\cos\beta}{3} + (1-\alpha)^2\cos^2\beta. \tag{62}$$

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