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PROJECTIVE CURVES, HYPERPLANE SECTIONS AND ASSOCIATED WEBS

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An integral and non-degenerate curve $C \subset \mathbb{P}^r$ is said to be ordinary (Gruson, Hantout and Lehmann) if the general hyperplane section $H \cap C$ of H is of maximal rank in H. Let $g'(r,d)$ be the maximal integer such that for every $g \in \{0,\dots,g'(r,d)\}$ there is a smooth ordinary curve $C \subset \mathbb{P}^r$ with degree d and genus g. Here we discuss the relevance of old papers to get a lower bound for $g'(r,d)$. We prove that arithmetically Gorenstein curves $C \subset \mathbb{P}^r$ are ordinary only if either $r = 2$ or $d = r+1$ and $\omega_C \cong \mathcal{O}_C$. We prove that general low genus curves are ordinary.

1. Introduction

Let $C \subset \mathbb{P}^r$ be an integral and non-degenerate curve. Set $d := \deg(C)$ and let $k_0(d,r)$ be the only positive integer such that $\binom{r+k_0(d,r)-1}{r-1} \leq d < \binom{r+k_0(d,r)}{r-1}$. In [12] and [18] C is said to be *ordinary* if for a general hyperplane $H \subset \mathbb{P}^r$ the set $C \cap H$ has maximal rank in H, i.e. for all $t \in \mathbb{Z}$ either $h^0(H,\mathcal{I}_{C\cap H}(t)) = 0$ or $h^1(H,\mathcal{I}_{C\cap H}(t)) = 0$, i.e. $h^0(H,\mathcal{I}_{C\cap H}(t)) = 0$ if $t \leq k_0(d,r)$ and $h^1(H,\mathcal{I}_{C\cap H}(t)) = 0$ for all $t > k_0(d,r)$, i.e. $h^0(H,\mathcal{I}_{C\cap H}(k_0(d,r))) = 0$ and $h^0(H,\mathcal{I}_{C\cap H}(t)) = \binom{r+t-1}{r-1} - d$ for all $t > k_0(r,d)$, i.e. (by the Castelnuovo-Mumford lemma) if

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$h^0(H,\mathcal{I}_{C\cap H}(k_0(d,r))) = 0$ and $h^0(H,\mathcal{I}_{C\cap H}(k_0(d,r)+1)) = \binom{r+k_0(d,r)}{r-1} - d$. Set $\pi'(r,d) := k_0(d,r)d - \binom{r+k_0(d,r)}{r} + 1$. Gruson, Hantout and Lehmann proved that $p_a(C) \le \pi'(r,d)$ if C is ordinary ([18, Théorème 1]). Let $\pi''(r,d)$ be the maximal integer g such that for all $q \in \{0,\dots,g\}$ there is an ordinary curve $C \subset \mathbb{P}^r$ with degree d and arithmetic genus q. Let $g'(r,d)$ be the maximal integer g such that for all $q \in \{0,\dots,g\}$ there is an ordinary smooth curve $C \subset \mathbb{P}^r$ with degree d and genus q. For any smooth curve $C \subset \mathbb{P}^r$ let N_C denote the normal bundle of C. Let $C \subset \mathbb{P}^r$ be a smooth and non-degenerate curve such that $h^1(C,N_C(-1)) = 0$. There is an ordinary curve C' near C (and in particular with $\deg(C') = \deg(C)$ and $p_a(C') = p_a(C)$([32, Théorème 1.5] , [25, §II.3]; see Remark 3.3 for more details). Moreover we may take C' smooth, too. As an obvious corollary we get that for any degree $d \ge r$ a general degree d smooth rational curve of $\mathbb{P}^r$ is ordinary (see Remark 3.3). Let $a(d,r)$ be the maximal integer ≥ 0 such that for all $0 \le g \le a(d,r)$ there is a smooth, connected and non-degenerate curve $C \subset \mathbb{P}^r$ with $h^1(C,N_C(-1)) = 0$. It is known that $a(d,r) \ge 2d/(r-2) + o(d)$ if $r \ge 4$ ([6, Théorème 5 (2)]). This bound is asymptotically sharp (Remark 3.4). Upper bounds and lower bounds for the integer $a(d,3)$ are known and they asymptotically agree, i.e. $a(d,3) = (\sqrt{8}/3)d^{3/2} + o(d^{3/2})$ ([32] (quoting unpublished results due to Ellingsrud and Hirschowitz), [16, Theorems 4.10 and 5.6], [25, II.3.7], [7]).

As far as we know the best result in $\mathbb{P}^3$ are the unpublished [34, Theorem 6.1] (which covers all the range A) and [16, Theorem 5.6] (which covers more than half of the range A). Fix integers $d \ge 3$ and $k > 0$ such that $\binom{k+2}{2} \le d < \binom{k+3}{2}$, i.e. such that $k_0(d,3) = k$. Fix an integer g. The pair (d,g) is said to be in the range A ([34], eq. (0.1.1), [35], eq. (0.1.1), [16], [7]) if

$$0 \le g \le dk + 1 - \binom{k+3}{3}, \tag{1}$$

i.e. if $0 \le g \le \pi'(d,3)$. By [34, Theorem 6.1] or [16, Theorem 5.6] for all (d,g) in the range A there is a smooth and connected curve $C \subset \mathbb{P}^r$ such that $\deg(C) = d$, $p_a(C) = g$ and $h^1(C,N_C(-1)) = 0$. Since such a curve C is ordinary (Remark 3.3), [34, Theorem 6.1] and [16, Theorem 5.6] close the problem of the existence for pairs (degree, genus) for integral (or for smooth) ordinary curves in $\mathbb{P}^3$.

We feel that the picture is completely different if $r \ge 4$ and that for each $r \ge 4$ there are large families of integers d,g with $0 \le g \le \pi'(d,r)$ and such that there is no integral, non-degenerate and ordinary curve $C \subset \mathbb{P}^r$ with $\deg(C) = d$ and $p_a(C) = g$ (even allowing singular ordinary curves). We do not have explicit examples. Certainly, if g is very small with respect to d, then the pair (d,g) is realized by an ordinary curve of $\mathbb{P}^r$ (see Proposition 3.1 and Theorem 3.7). We raise the following question.

Question 1.1. Fix an integer $r \geq 4$. Is $\pi''(r,d) < \pi'(r,d)$ and $g'(r,d) < \pi'(r,d)$ if $d \gg r$?

If $r < d < (r+1)r/2$, then $k_0(d,r) = 1$ and hence $\pi'(r,d) = d - r$. The case $d < (r+1)r/2$ of Theorem 3.7 gives $g'(r,d) = \pi'(r,d)$ if $d < (r+1)r/2$.

For any integral projective curve $C \subset \mathbb{P}^r$ the index of speciality $e(C)$ of C is the maximal integer e such that $h^1(C, \mathcal{O}_C(e)) > 0$. We have $e(C) < 0$ if and only if $H^1(C, \mathcal{O}_C) = 0$. Since C is integral, we have $e(C) < 0$ if and only if C is a smooth rational curve. We immediately check that $e(C) = -2$ if C is a line, while $e(C) = -1$ if C is a smooth rational curve of degree ≥ 2. We have $e(C) = 0$ if and only if $p_a(C) > 0$ and $h^1(C, \mathcal{O}_C(1)) = 0$, i.e. the embedding of C is non-special. We recall that C is said to be *arithmetically Gorenstein* if it is arithmetically Cohen-Macaulay ([29, Definition 1.2.2]) and $\omega_C \cong \mathcal{O}_C(e)$ for some integer e (see [29, Proposition 4.1.1] for several equivalent definitions). Every complete intersection is arithmetically Gorenstein. If $r = 3$ a curve is arithmetically Gorenstein if and only if it is a complete intersection ([29, Example 4.1.11 (c)], [30, Example 1.1.28 (b)]). If $r \geq 4$ there are many arithmetically Gorenstein curves which are not complete intersections. There is a complete list of all degree, genera and minimal free resolutions of arithmetically Gorenstein curves in $\mathbb{P}^4$, their Hilbert scheme is well understood and we may construct them algorithmically, almost as in the case of complete intersection ([24], [13], [14], [22, Theorem 2.6], [28]). Unfortunately, this class is not helpful for finding ordinary curves. We prove the following result which extends [18, Théorèmes 5, 6], dealing with complete intersections. We say that a curve $C \subset \mathbb{P}^r$ is *linearly normal* if $h^1(\mathcal{I}_C(1)) = 0$, i.e. if the restriction map $\rho_C : H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \to H^0(C, \mathcal{O}_C(1))$ is surjective. Hence arithmetically Cohen-Macaulay curves are linearly normal. If C is non-degenerate, then C is linearly normal if and only if ρ_C is bijective.

Theorem 1.2. *The only arithmetically Gorenstein integral ordinary curves $C \subset \mathbb{P}^r$ are the plane curves ($r = 2$, any $\deg(C) \geq 2$) and the linearly normal curves $C \subset \mathbb{P}^r$, $r \geq 3$, with $\deg(C) = r+1$ and $p_a(C) > 0$. The latter curves have $p_a(C) = 1$ and $\omega_C \cong \mathcal{O}_C$ (hence $k_0(\deg(C), r) = 1$).*

If $r \geq 4$ the curves are not a complete intersection, because if $r \geq 4$ then $r+1$ is not the product of $r-1$ integers ≥ 2. See Example 2.1 for a description of all curves appearing in the statement of Theorem 1.2 when $r \geq 3$.

As remarked in [12, page 200] this concept raises several interesting questions in commutative algebra and projective geometry (sometimes solved for very different motivations).

(a) List all pairs (d, g) such that there is an ordinary integral curve $C \subset \mathbb{P}^4$ with degree d and arithmetic genus g. The same question for smooth curves.

(b) List all pairs $(d,g) \in \mathbb{N}^2$ such that there is an ordinary arithmetically Cohen-Macaulay curve (or an ordinary, smooth and arithmetically Cohen-Macaulay curve) $C \subset \mathbb{P}^4$ with degree d and genus g.

(c) Fix an ordinary curve $C \subset \mathbb{P}^r$, $r \geq 3$. Let $\mathbb{P}^{r\vee}$ denote the set of all hyperplanes of $\mathbb{P}^r$. Set $d := \deg(C)$ and $k := k_0(d,r)$. Let $\mathcal{B}(C)$ be the set of all $H \in \mathbb{P}^{r\vee}$ such that either $h^0(H, \mathcal{I}_{C\cap H}(k)) > 0$ or $h^1(\mathcal{I}_{C\cap H}(k+1)) > 0$. This is the set of all bad hyperplanes for C and, at least if C is smooth, it should be the exceptional set S of the web associated to C.

Concerning (b) we point out that [18, Théorème 4] gives all possible ordinary arithmetically Cohen-Macaulay space curves. Let $C \subset \mathbb{P}^r$, $r \geq 3$, be any ordinary arithmetically Cohen-Macaulay curve. Set $d := \deg(C)$ and $k := k_0(d,r)$. The Castelnuovo-Mumford lemma implies that the minimal free resolution $\mathbb{E}$ of $\mathcal{I}_C$ is very similar to the one listed in [18, Théorème 4]. There are non-negative integers a_i, b_i, $1 \leq i \leq r$, with the following property. $\mathbb{E}$ starts on the right with $\mathcal{O}_{\mathbb{P}^r}(k)^{a_1} \oplus \mathcal{O}_{\mathbb{P}^r}(k+1)^{b_1}$ and then it continues with only two degrees in each step, say $\mathcal{O}_{\mathbb{P}^r}(k+i-1)^{a_i} \oplus \mathcal{O}_{\mathbb{P}^r}(k+i)^{b_i}$ after $i-1$ steps. It seems to be very difficult to show that some string of integers a_i, b_i, $1 \leq i \leq r$, is realized by some curve C.

Concerning (c) we point out that all the hyperplane sections of integral arithmetically Cohen-Macaulay curves have the same postulation ([29, Corollary 1.3.5]). Hence one of them has maximal rank if and only if all hyperplane sections have maximal rank.

We work over an algebraically closed base field $\mathbb{K}$. In section 3 we assume $\mathrm{char}(\mathbb{K}) = 0$.

2. Proof of Theorem 1.2

Let $C \subset \mathbb{P}^r$, $r \geq 3$, be an integral and non-degenerate curve of degree $r+1$. If $p_a(C) = 0$, then C is a non-linearly normal smooth rational curve ([8, 4.7 (B)]). The case $p_a(C) > 0$ is described in [8, 4.7 (B)] (it has $p_a(C) = 1$), but we recall it as Example 2.1, because these curves are the ones arising in Theorem 1.2.

Example 2.1. Let Y be any integral projective curve with arithmetic genus 1. Equivalently, take as Y either a smooth elliptic curve or a singular rational curve with an ordinary node or an ordinary cusp as its only singularity. Since $p_a(C) = 1$, we have $h^0(Y, \omega_Y) = 1$. Since $\deg(\omega_Y) = 0$ and Y is integral, we get $\omega_Y \cong \mathcal{O}_Y$. Let L be any line bundle on Y such that $\deg(L) = r+1$. Since $\deg(L) > \deg(\omega_Y)$, we have $h^1(Y,L) = 0$. Since $\deg(L) = r+1$ and $p_a(C) = 1$, Riemann-Roch for singular curves gives $h^0(Y,L) = r+1$ ([31, page 130], [16, Definition 1.3], [21, Theorem 1.3]). For any degree 2 zero-dimensional scheme $Z \subset Y$, we have $h^1(Y, \mathcal{I}_Z \otimes L) = 0$, because $\deg(\mathcal{I}_Z \otimes L) = r-1 > \deg(\omega_Y)$. Hence L is very ample. Let $u : Y \to \mathbb{P}^r$ be the embedding induced by $H^0(Y, L)$. The curve $u(Y)$ is an integral, non-degenerate and linearly normal curve with arithmetic genus 1. Take another pair (Y', L') as above and call $u' : Y' \to \mathbb{P}^r$ the embedding associated to $H^0(Y', L')$. Since $u(Y)$ and $u'(Y')$ are linearly normal, they are projectively equivalent if and only if there is an isomorphism $f : Y' \to Y$ such that $L' \cong f^*(L)$. Now take any integral and non-degenerate curve $C \subset \mathbb{P}^r$ such that $\deg(C) = r+1$ and $p_a(C) > 0$. Since $p_a(C) = 1$ and C is linearly normal ([8, 4.7 (B)]), we are in the case just described with $Y := C$ and $L := \mathcal{O}_C(1)$.

For the classification of non-degenerate varieties $X \subset \mathbb{P}^r$ with $\deg(X) + \dim(X) = r+2$, see [24], [9], [10], [11]. For the classification of all curves $C \subset \mathbb{P}^r$ with $\deg(C) = r+2$, see [8].

For any integral and non-degenerate curve $C \subset \mathbb{P}^r$ let $a(C)$ be the maximal integer t such that $h^1(H, \mathcal{I}_{C \cap H}(t)) > 0$ for a general hyperplane $H \subset \mathbb{P}^r$.

Lemma 2.2. *Let $C \subset \mathbb{P}^r$, $r \geq 3$, be an integral, non-degenerate and arithmetically Cohen-Macaulay curve. Let $H \subset \mathbb{P}^r$ be a general hyperplane. We have $e(C) = a(C) - 1$ and $h^1(C, \mathcal{O}_C(e(C))) = h^1(H, \mathcal{I}_{C \cap H}(a(C)))$.*

Proof. Since $\dim(C) = 1$ and $r \geq 3$, the exact sequence

$$0 \to \mathcal{I}_C(t) \to \mathcal{O}_{\mathbb{P}^r}(t) \to \mathcal{O}_C(t) \to 0$$

gives $h^2(\mathcal{I}_C(t)) = h^1(C, \mathcal{O}_C(t))$ for all $t \in \mathbb{Z}$ (case $r \geq 4$) or for all $t \geq -3$ (case $r = 3$). Look at the exact sequence

$$0 \to \mathcal{I}_C(t-1) \to \mathcal{I}_C(t) \to \mathcal{I}_{C \cap H, H}(t) \to 0 \tag{2}$$

Since $h^1(\mathcal{I}_C(t)) = 0$, (2) gives the exact sequence

$$0 \to H^1(H, \mathcal{I}_{C \cap H, H}(t)) \to H^2(\mathcal{I}_C(t-1)) \to H^2(\mathcal{I}_C(t))$$

Since $h^1(C, \mathcal{O}_C(e(C))) > 0$ and $h^1(C, \mathcal{O}_C(e(C)+1)) = 0$, we get $e(C) = a(C) - 1$ and $h^1(C, \mathcal{O}_C(e(C))) = h^1(H, \mathcal{I}_{C \cap H}(a(C)))$. □

Lemma 2.3. *Let $C \subset \mathbb{P}^r$ be an integral and non-degenerate ordinary curve. Set $d := \deg(C)$ and $k := k_0(d, r)$. We have $\binom{r+k-1}{r-1} \leq d < \binom{r+k}{r-1}$. If $d = \binom{r+k-1}{r-1}$, then $a(C) = k-1$. If $d \neq \binom{r+k-1}{r-1}$, then $a(C) = k$.*

Proof. The Castelnuovo-Mumford lemma applied to the set $C \cap H \subset H$ gives $a(C) \leq k$ and that strict inequality holds if $d = \binom{r+k-1}{r-1}$. Since $h^1(H, \mathcal{I}_{H \cap C}(k)) = d - \binom{r+k-1}{r-1}$ and $h^1(H, \mathcal{I}_{H \cap C}(k-1)) = d - \binom{r+k-2}{r-1} > 0$, we get the lemma. □

Proof of Theorem 1.2. Since the case $r=2$ is obvious, we may assume $r\geq 3$. Let $A\subset \mathbb{P}^r$ be an integral, non-degenerate curve of degree $r+1$. We have $k_0(r+1,r)=1$. Assume that A is linearly normal. Hence $p_a(A)=1$ and $\omega_A\cong \mathcal{O}_A$ (Example 2.1). Fix any hyperplane $H\subset \mathbb{P}^r$ transversal to A. Fix any set $B\subset Y\cap H$ with $\sharp(B)=r$. Since $\omega_A\cong \mathcal{O}_A$ and $\deg(\mathcal{O}_A(1)(-B))=r+1-r>\deg(\omega_A)$, we have $h^1(A,\mathcal{O}_A(1)(-B))=0$. Since $p_a(A)=1$, Riemann-Roch for singular curves gives $h^0(A,\mathcal{O}_A(1)(-B))=\deg(\mathcal{O}_A(1)(-B))=1$ ([31, page 130], [16, Definition 1.3], [21, Theorem 1.3]). Hence H is the unique hyperplane containing B, i.e. any $B\subset Y\cap H$ with $\sharp(B)=r$ spans H. Hence $h^1(H,\mathcal{I}_{A\cap H}(2))=0$ ([19, Lemma 3.2]). Since A is integral, $h^0(A,\mathcal{O}_A)=1$. Hence $h^1(\mathcal{I}_A)=0$. Hence the case $t=1$ of (2) gives $h^0(H,\mathcal{I}_{A\cap H}(1))=0$. Hence A is ordinary. Since A is linearly normal, we have $h^1(\mathcal{I}_A(1))=0$. Since $h^1(H,\mathcal{I}_{A\cap H}(t))=0$ for all $t\geq 2$ and $h^2(\mathcal{I}_C(t-1))=h^0(C,\mathcal{O}_C(t-1))=0$ for all $t\geq 2$, using (2) and induction on t we get that A is arithmetically Cohen-Macaulay. Hence we checked the "if part".

Now we prove the "only if" part. Let $C\subset \mathbb{P}^r$, $r\geq 3$, be an integral, non-degenerate, ordinary and arithmetically Gorenstein curve. Since C is ordinary and arithmetically Cohen-Macaulay, then $p_a(C)=\pi'(d,r)$ ([18, Theorem 1]). Since $\mathcal{O}_C(e(C))\cong \omega_C$ and $h^1(C,\omega_C)=1$, we have $h^1(H,\mathcal{I}_{C\cap H}(a(C)))=1$ (Lemma 2.2). Set $d:=\deg(C)$ and $k:=k_0(d,r)$. We have $\binom{r+k-1}{r-1}\leq d<\binom{r+k}{r-1}$. First assume $d=\binom{r+k-1}{r-1}$. Lemma 2.3 gives $a(C)=k-1$. We have $h^1(H,\mathcal{I}_{C\cap H}(k-1))=d-\binom{r+k-2}{r-1}=\binom{r+k-2}{r-2}$. Since $r\geq 3$ and $k\geq 1$, we have $\binom{r+k-2}{r-2}\geq 2$, contradicting Lemma 2.2. Now assume $d\neq \binom{r+k-1}{r-1}$. Lemma 2.3 gives $a(C)=k$. Since $h^0(H,\mathcal{I}_{C\cap H}(k))=0$, we have $h^1(H,\mathcal{I}_{C\cap H}(k))=d-\binom{r+k-1}{r-1}$. Lemma 2.2 gives $d=\binom{r+k-1}{r-1}+1$ and $e(C)=k-1$. Hence $\omega_C\cong \mathcal{O}_C(k-1)$. Hence $\deg(\omega_C)=(k-1)d$. Since $\deg(\omega_C)=2p_a(C)-2=2\pi'(d,r)-2$ and $\pi'(d,r)=1+dk-\binom{r+k}{r}$, we get

$$(k+1)(1+\binom{r+k-1}{r-1})=2\binom{r+k}{r} \tag{3}$$

Hence $(k+1)\binom{r+k-1}{r-1}<2\binom{r+k}{r}$, i.e. $(k+1)r<2(r+k)$. Since $r\geq 3$, we get that either $k=1$ or $k=2$ and $r\leq 5$. If $k=1$, then $d=r+1$. We analyzed this case. Now assume $k=2$. From (3) we get $3+3(r+1)r/2=(r+2)(r+1)$, false if $r=3,4,5$. □

3. Low genera

In this section we assume $\mathrm{char}(\mathbb{K})=0$, except in Proposition 3.2.

The following result was classically known with another language.

Proposition 3.1. *Let $C \subset \mathbb{P}^r$ be any integral and non-degenerate curve such that* $\deg(C) \leq 2r-1$. *Then C is ordinary.*

Proof. Set $d := \deg(C)$. If $d = r$, then C is a rational normal curve. Each rational normal curve C is ordinary, because any transversal hyperplane section $H \cap C$ of C is given by r points spanning H. Hence we may assume $d > r$. We have $k_0(d,r) = 1$. Let $H \subset \mathbb{P}^r$ be a general hyperplane section. We recall that a finite set $S \subset H$ is said to be in linearly general position if every $E \subseteq S$ with $\sharp(E) \leq r-1$ is linearly independent. Since $\mathrm{char}(\mathbb{K}) = 0$, the set $C \cap H$ is in linearly general position ([2, page 109]). Since $d \leq 2(r-1)+1$, we have $h^1(H, \mathcal{I}_{C\cap H}(2)) = 0$ ([19, Lemma 3.2]). □

Proposition 3.1 is sharp, because no canonically embedded curve $C \subset \mathbb{P}^{g-1}$, $g \geq 4$ (it has $p_a(C) = g-1$ and degree $2g-2$) is ordinary by the following result (alternatively, either apply Theorem 1.2 and a theorem of Max Noether for Gorenstein curves ([28]) or use that $k_0(2r,r) = 1$ and hence $\pi'(g-1, 2g-2) = g-1$).

Proposition 3.2. *Let $C \subset \mathbb{P}^r$, $r \geq 3$, be an integral and non-degenerate curve such that $d := \deg(C) < (r+1)r/2$ and $h^1(C, \mathcal{O}_C(1)) > 0$. Then C is not ordinary.*

Proof. Let $H \subset \mathbb{P}^r$ be a general hyperplane. Since $d = \sharp(C \cap H) < (r+1)r/2$, we have $h^0(H, \mathcal{I}_{C\cap H}(2)) > 0$. Hence it is sufficient to prove the inequality $h^1(H, \mathcal{I}_{C\cap H}(2)) > 0$. By (2) it is sufficient to prove that $h^2(\mathcal{I}_C(1)) > h^2(\mathcal{I}_C(2))$, i.e. $h^1(C, \mathcal{O}_C(1)) > h^1(C, \mathcal{O}_C(2))$, i.e. $h^0(C, \omega_C(-1)) > h^0(C, \omega_C(-2))$ (duality for the locally Cohen-Macaulay one-dimensional scheme C [1, 1.3, pages 5-6]). The last inequality is true, because $h^0(C, \omega_C(-1)) > 0$ and $\mathcal{O}_C(1)$ is very ample (e.g., we have either $h^0(C, \omega_C(-2)) = 0$ or $h^0(C, \omega_C(-1)) \geq h^0(C, \omega_C(-2)) + r$ by a lemma of Hopf ([15, page 544])). □

Remark 3.3. Let $C \subset \mathbb{P}^r$, $r \geq 3$, be an integral and non-degenerate curve. Set $d := \deg(C)$. Assume $h^1(C, N_C(-1)) = 0$, i.e. $h^0(C, N_C^\vee(1) \otimes \omega_C) = 0$ (duality). Hence $h^0(C, N_C^\vee \otimes \omega_C) = 0$, i.e. $h^1(C, N_C) = 0$. Since $h^1(C, N_C) = 0$, then C is a smooth point of the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^r)$ of $\mathbb{P}^r$, i.e. C belongs to a unique irreducible component, $\mathcal{S}$, of the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^r)$ of $\mathbb{P}^r$, $\dim(\mathcal{S}) = h^0(C, N_C)$ and $\mathcal{S}$ is smooth at C ([30, page 64]). Now we check that a general $D \in \mathcal{S}$ is ordinary. Let $H \subset \mathbb{P}^r$ be any hyperplane transversal to C. The set $\mathcal{V}$ of all sets $S \subset H$ with $\sharp(S) = d$ is a non-empty irreducible variety of dimension $d(r-1)$. Since $\mathcal{V}$ is irreducible, each non-empty open subset of $\mathcal{V}$ is dense in $\mathcal{V}$. Hence the intersection of finitely many non-empty open subsets of $\mathcal{V}$ is non-empty and dense in $\mathcal{V}$. The semicontinuity theorem for cohomology ([20, Theorem III.12.8]) gives the existence of a non-empty open subset $\mathcal{U}$ of $\mathcal{V}$

such that every $S \in \mathcal{U}$ has maximal rank in H. Let $\mathcal{S}' \subset \mathcal{S}$ be any open neighborhood of C formed by curves $D \in \mathcal{S}$ transversal to H. Hence $H \cap D$ is formed by d distinct points for all $D \in \mathcal{S}'$. Let $u : \mathcal{S}' \to \mathcal{V}$ be the map $D \mapsto D \cap H$. Since $h^1(C, N_C(-1)) = 0$, the map u is dominant ([32, Théorème 1.5]). Hence for a general $S \subset H$ such that $\sharp(S) = d$ there is $D \in \mathcal{S}'$ such that $D \cap H = S$. Since S is general, for each $t \in \mathbb{Z}$ either $h^0(H, \mathcal{I}_S(t)) = 0$ or $h^1(H, \mathcal{I}_S(t)) = 0$. Hence D is ordinary.

Remark 3.4. Let $C \subset \mathbb{P}^r$, $r \geq 4$, be a smooth and non-degenerate curve of genus g and degree d. We have $\chi(N_C(-1)) = 2d - (r-3)(g-1)$ (Riemann-Roch). Hence $g - 1 \leq 2d/(r-3)$ if $h^1(C, N_C(-1)) = 0$. For a fixed integer $r \geq 4$ when $d \gg 0$ $2d/(r-3)$ is linear in d, while $\pi'(r,d) \sim d^2/(2r-2)$ is quadratic in d. Hence for each fixed integer $r \geq 4$ we may have $h^1(C, N_C(-1)) = 0$ only for a quite small set of pairs (d, g). For fixed r the paper [6] asymptotically covers this small range. If $g = 0$, then $h^1(C, N_C(-1)) = 0$ for the following reason. The Euler's sequence of $T\mathbb{P}^r$ ([20, Theorem II.8.13]):

$$0 \to \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_{\mathbb{P}^r}(1)^{\oplus(r+1)} \to T\mathbb{P}^r \to 0$$

gives that $T\mathbb{P}^r(-1)|C$ is spanned by its global sections. Since C is smooth, $N_C(-1)$ is a quotient of $T\mathbb{P}^r(-1)|C$. Hence $N_C(-1)$ is spanned by its global sections. Hence there is an exact sequence of coherent sheaves on C:

$$0 \to \mathcal{F} \to \mathcal{O}_C^m \to N_C(-1) \to 0$$

Since C is smooth and rational, we have $h^1(C, \mathcal{O}_C) = 0$. Since $\dim(C) = 1$, we have $h^2(C, \mathcal{F}) = 0$. Hence $h^1(C, N_C(-1)) = 0$.

Lemma 3.5. *Let X be an integral projective variety with* $\dim(X) > 0$ *and let $\mathcal{L}$ be a line bundle on X.*

(a) Assume $h^0(X, \mathcal{L}) = 0$. Then $h^0(X, \mathcal{I}_P \otimes L) = 0$ and $h^0(X, \mathcal{I}_P \otimes L) = h^1(X, \mathcal{L}) + 1$ for all $P \in X$.

(b) Assume $h^0(X, \mathcal{L}) > 0$. Let $W \subseteq H^0(X, \mathcal{L})$ be any non-zero linear subspace. A general $P \in X$ is not in the base locus of W. Fix $P \in X$ which is not in the base locus of W. Then:

1. $\dim(W \cap H^0(X, \mathcal{I}_P \otimes \mathcal{L})) = \dim(W) - 1$.
2. $h^0(X, \mathcal{I}_P \otimes \mathcal{L}) = h^0(X, \mathcal{L}) - 1$ *and* $h^1(X, \mathcal{I}_P \otimes \mathcal{L}) = h^1(X, \mathcal{L})$.

Proof. Fix any $Q \in X$. Since $\dim(\{Q\}) = 0$ and $\mathcal{L}$ is a line bundle, we have $h^i(X, \mathcal{L}|\{Q\}) = 0$ for all $i > 0$ and $h^0(X, \mathcal{L}|\{Q\}) = 1$. Hence the exact sequence

$$0 \to \mathcal{I}_Q \otimes \mathcal{L} \to \mathcal{L} \to \mathcal{L}|\{Q\} \to 0$$

shows that either $h^0(X,\mathcal{I}_Q\otimes\mathcal{L}) = h^0(X,\mathcal{L})$ and $h^1(X,\mathcal{I}_Q\otimes\mathcal{L}) = h^1(X,\mathcal{L})+1$ or $h^0(X,\mathcal{I}_Q\otimes\mathcal{L}) = h^0(X,\mathcal{L})-1$ and $h^1(X,\mathcal{I}_Q\otimes\mathcal{L}) = h^1(X,\mathcal{L})$. Moreover, the first case occurs if and only if Q is a base point of $\mathcal{L}$. We get part (a) and the second half of part (b). We have $\dim(W\cap H^0(X,\mathcal{I}_Q\otimes\mathcal{L})) = \dim(W)-1$ if and only if Q is not in the base locus of $|W|$. Since $W\neq 0$, a general $P\in X$ is not in the base locus of $|W|$. Hence we get the first half of part (b). □

Lemma 3.6. *Let $S\subset\mathbb{P}^m$, $m\geq 2$, be a finite set such that for each $t\in\mathbb{Z}$ either $h^0(\mathcal{I}_S(t)) = 0$ or $h^1(\mathcal{I}_S(t)) = 0$. Fix an integer $x>0$. Let $A\subset\mathbb{P}^m$ be a general subset with cardinality x. Then for each $t\in\mathbb{Z}$ either $h^0(\mathcal{I}_{S\cup A}(t)) = 0$ or $h^1(\mathcal{I}_{S\cup A}(t)) = 0$.*

Proof. By induction on x we reduce to the case $x=1$. Since $h^1(\mathcal{I}_{S\cup A}(t)) = 0$ for each $t\geq\sharp(S\cup A)$, it is sufficient to check the condition for finitely many line bundles $\mathcal{O}_{\mathbb{P}^r}(t)$. Apply Lemma 3.5. □

Theorem 3.7. *Fix integers d,r,g such that $r\geq 3$, $g\geq 0$ and $d\geq g+r$. Let $C\subset\mathbb{P}^r$ be a general smooth curve $C\subset\mathbb{P}^r$ such that $\deg(C) = d$, $p_a(C) = g$, C is non-degenerate and $h^1(C,\mathcal{O}_C(1)) = 0$. Then C is ordinary.*

In the previous statement the word " general " makes sense, because the set $Z(d,g,r)$ of all non-degenerate smooth and non-special curves in $\mathbb{P}^r$ with degree d and genus g is irreducible ([19, page 62]).

Proof of Theorem 3.7. By Proposition 3.1 we may assume $d\geq 2r$. Set $k := k_0(d,r)$ and $\alpha := d-g-r$. Fix a hyperplane $H\subset\mathbb{P}^r$. Let $Z'(d,g,r)$ be the closure of $Z(d,g,r)$ in the Hilbert scheme of $\mathbb{P}^r$. Set $x := \lfloor r/2\rfloor$. Assume for the moment $x\leq g$. Fix a rational normal curve $D\subset\mathbb{P}^r$ transversal to H. Let $W(r,0)$ be the set of all curves $D\cup L_1\cup\cdots\cup L_x$ with L_i defined in the following way. Let L_1 be a secant line of D and with $D\cup L_1$ transversal to H. For each $i\in\{2,\dots,x\}$ define recursively the line L_i as any line meeting both D and L_{i-1} and such that $D\cup L_1\cup\cdots\cup L_i$ is a nodal curve of degree $r+i$ and arithmetic genus i transversal to H. Let $W(r,t)$ be the set of all nodal curves $Y = Y_1\cup R_1\cup\cdots\cup R_t$ of degree $r+x+t$ and arithmetic genus $x+t$ with $Y_1\in W(r,0)$, each R_j a line intersecting D and intersecting Y_1 quasi-transversally and at exactly two points and with Y transversal to H, while $R_j\cap R_h = \emptyset$ for all $j\neq h$. Hence each curve of $W(r,t)$ is nodal, connected, with arithmetic genus $x+t$ and degree $r+x+t$. We have $W(r,t)\subset Z'(r+x+t,x+t,r)$ ([33], [23, Corollary 4.2 and Remark 4.1.1]). By [5, Lemma 1.4], applied to the integers k and $k+1$ for any integer $t\geq 0$ there are $A_t\in W(r,t)$ and $B_t\in W(r,t)$ intersecting H transversally and with $h^0(H,\mathcal{I}_{A_t\cap H}(k)) = \max\{0,\binom{r+k-1}{r-1}-\deg(A_t)\}$, $h^0(H,\mathcal{I}_{B_t\cap H}(k+1)) = \max\{0,\binom{r+k}{r-1}-\deg(B_t)\}$. Let A (resp. B) be the union of A_{g-x} (resp. B_{g-x})

and α general lines of $\mathbb{P}^r$ meeting D at one point. We have $A \in Z'(d,g,r)$ and $B \in Z'(d,g,r)$ ([33], [23, Corollary 4.2 and Remark 4.1.1], [5, Lemma 0.2]). Since $A \cap H$ is the union of $A_{g-x} \cap H$ and α general points of H and $d = \deg(A) \geq \binom{r+k-1}{r-1}$, we have $h^0(H, \mathcal{I}_{A\cap H}(k)) = 0$ (Lemma 3.6). By the semicontinuity theorem for cohomology ([20, Theorem III.12.8]) we get $h^0(H, \mathcal{I}_{C\cap H}(k)) = 0$ for a general $C \in Z(d,g,r)$. Since $B \cap H$ is the union of $B_{g-x} \cap H$ and α general points of H and $d < \binom{r+k-1}{k-1}$, we have $h^1(H, \mathcal{I}_{A\cap H}(k+1)) = 0$ (Lemma 3.6). By the semicontinuity theorem for cohomology we get $h^1(H, \mathcal{I}_{C\cap H}(k+1)) = 0$ for a general $C \in Z(d,g,r)$. Since $Z(d,g,r)$ is irreducible, we get that a general $C \in Z(d,g,r)$ is ordinary.

Now assume $x > g$. Take a general $E \in Z(g+r,g,r)$ and a general hyperplane $H \subset \mathbb{P}^r$. Proposition 3.1 gives $h^0(H, \mathcal{I}_{E\cap H}(1)) = 0$ and $h^1(H, \mathcal{I}_{E\cap H}(t)) = 0$ for all $t \geq 2$. Let $F \subset \mathbb{P}^r$ be a general union of E and α lines meeting E at a unique point. We saw that $F \in Z'(d,g,r)$. Since $F \cap H$ is a union of $E \cap H$ and α general points of H, Lemma 3.5 gives $h^0(H, \mathcal{I}_{F\cap H}(k)) = 0$ and $h^1(H, \mathcal{I}_{F\cap H}(k+1)) = 0$. By the semicontinuity theorem for cohomology we have $h^0(H, \mathcal{I}_{C\cap H}(k)) = 0$ and $h^1(H, \mathcal{I}_{C\cap H}(k+1)) = 0$ for a general $C \in Z(d,g,r)$. □

Proposition 3.8. *Fix integers r, g such that $r \geq 3$ and $0 \leq g \leq r(r-1)/2$. There is a smooth and non-degenerate ordinary curve $C \subset \mathbb{P}^r$ such that $p_a(C) = g$, $\deg(C) = g + r$, $h^1(C, \mathcal{O}_C(1)) = 0$ and $\mathcal{B}(C) = \emptyset$.*

Proof. We have $k_0(g+r,r) = 1$ if $g < r(r-1)/2$ and $k_0(r(r+1)/2, r) = 2$. There is a smooth and linearly normal curve $C \subset \mathbb{P}^r$ such that $\deg(C) = g + r$, $p_a(C) = g$, $h^1(C, \mathcal{O}_C(1)) = 0$ and with maximal rank, i.e. for all $t \in \mathbb{N}$ either $h^0(\mathcal{I}_C(t)) = 0$ or $h^1(\mathcal{I}_C(t)) = 0$ ([3] if $r = 4$, [4] if $r = 3$, [5] for all $r \geq 5$). Fix any hyperplane $H \subset \mathbb{P}^r$. Riemann-Roch gives $h^0(C, \mathcal{O}_C(2)) = 2d + 1 - g \leq \binom{r+2}{2}$ by hypothesis. Hence $h^1(\mathcal{I}_C(2)) = 0$. Since $h^1(C, \mathcal{O}_C(1)) = 0$, the Castelnuovo-Mumford lemma gives $h^1(\mathcal{I}_C(t)) = 0$ for all $t \geq 3$. Since $\deg(C) = g + r$ and $h^1(C, \mathcal{O}_C(1)) = 0$, Riemann-Roch gives $h^0(C, \mathcal{O}_C(1)) = r + 1$. Since C is non-degenerate, we get $h^1(\mathcal{I}_C(1)) = 0$. Since C is connected, we have $h^1(\mathcal{I}_C) = 0$. Hence C is arithmetically Cohen-Macaulay. Since $h^1(\mathcal{I}_C(t)) = 0$ for all $t \geq 0$, (1) gives $h^1(H, \mathcal{I}_{C\cap H}(t)) = 0$ for all $t \geq 1$. Hence C is ordinary with $\mathcal{B}(C) = \emptyset$ if $g \neq r(r+1)/2$. If $g = r(r+1)/2$ we may use that $h^0(H, \mathcal{I}_{C\cap H}(2)) = 0 = h^1(H, \mathcal{I}_{C\cap H}(2))$ and hence $h^1(H, \mathcal{I}_{C\cap H}(3)) = 0$ by the Castelnuovo-Mumford lemma. □

REFERENCES

[1] A. Altman - S. L. Kleiman, *Introduction to Grothendieck duality theory*, Lect. Notes in Math. 146, Springer, Heidelberg, 1970.

[2] E. Arbarello - M. Cornalba - P. A. Griffiths - J. Harris, *Geometry of algebraic curves. Vol. I*, Springer, Berlin, 1985.

[3] E. Ballico - Ph. Ellia, *On postulation of curves in* $\mathbb{P}^4$, Math. Z. 188 (1985), 215–223.

[4] E. Ballico - Ph. Ellia, *The maximal rank conjecture for nonspecial curves in* $\mathbb{P}^3$, Invent. Math. 79 (3) (1985), 541–555.

[5] E. Ballico - Ph. Ellia, *The maximal rank conjecture for non-special curves in* $\mathbb{P}^n$, Math. Z. 196 (1987), 355–367.

[6] E. Ballico - Ph. Ellia, *Bonnes petites composantes des schémas de Hilbert de courbes lisses de* $\mathbb{P}^n$, C. R. Acad. Sci. Paris Sér. I Math. 306 (4) (1988), 187–190.

[7] E. Ballico - G. Bolondi - Ph. Ellia - R. M. Miró-Roig, *Curves of maximum genus in range A and stick-figures*, Trans. Amer. Math. Soc. 349 (11) (1997), 4589–4608.

[8] M. Bordmann - P. Schenzel, *Curves of degree* $r+2$ *in* $\mathbb{P}^r$*: cohomological, geometric, and homological aspects*, J. Algebra 242 (2) (2001), 577–623.

[9] M. Bordmann - P. Schenzel, *On varieties of almost minimal degree in small codimension*, J. Algebra 305 (2) (2006), 789–801.

[10] M. Bordmann - P. Schenzel, *Arithmetic properties of projective varieties of almost minimal degree*, J. Algebraic Geom. 16 (2) (2007), 347–400.

[11] M. Bordmann - P. Schenzel, *On varieties of almost minimal degree II: A rank-depth formula*, Proc. Amer. Math. Soc. 139 (6) (2011), 2025–2032.

[12] V. Cavalier - D. Lehmann, *Ordinary holomorphic webs of codimension one*, Ann. Sc. Norm. Super. Pisa, cl. Sci (5) 11 (2012), 197–214.

[13] E. De Negri - G. Valla, *The h-vector of a Gorenstein codimension three domain*, Nagoya Math. J. 138 (1995), 113–140.

[14] S. Diesel, *Irreducibility and dimension theorems for families of height 3 Gorenstein algebras*, Pacific J. Math. 172 (2) (1996), 365–397.

[15] D. Eisenbud, *Linear sections of determinantal varieties*, Amer. J. Math. 110 (3) (1988), 541–575.

[16] G. Fløystad, *Construction of space curves with good properties*, Math. Ann. 289 (1991), 33–54.

[17] T. Fujita, *Defining equations for certain types of polarized varieties*, Complex analysis and algebraic geometry, pp. 165–173, Iwanami Shoten, Tokyo, 1977.

[18] L. Gruson - Y. Hantout - D. Lehmann, *Courbes algébriques ordinaire and associated webs*, C. R. Acad. Sci. Paris, Ser. I 350 (2012), 513–518.

[19] J. Harris, *Curves in projective space*, With the collaboration of David Eisenbud, Séminaire de Mathématiques Supérieures 85, Les Presses de l'Université de Montréal, Montréal, Que., 1982.

[20] R. Hartshorne, *Algebraic Geometry*, Springer, 1977.

[21] R. Hartshorne, *Generalized divisors on Gorenstein curves and a theorem of Noether*, J. Math. Kyoto Univ. 26 (3) (1986), 375–386.

[22] R. Hartshorne, *Geometry of arithmetically Gorenstein curves in* $\mathbb{P}^4$, Collect. Math. 55 (1) (2004), 97–111.

[23] R. Hartshorne - A. Hirschowitz, *Smoothing algebraic space curves*, Algebraic Geometry, Sitges 1983, 98–131, Lecture Notes in Math. 1124, Springer, Berlin, 1985.

[24] J. Herzog - N. V. Trung - G. Valla, *On hyperplane sections of reduced irreducible varieties of low codimension*, J. Math. Kyoto Univ. 34 (1994), 47–72.

[25] A. Hirschowitz, *Sections planes et multisecantes pour les courbes gauches génériques principales*, Space curves (Rocca di Papa, 1985), 124–155, Lecture Notes in Math. 1266, Springer, Berlin, 1987.

[26] L. T. Hoa - J. Stückrad - W. Vogel, *Towards a structure theory for projective varieties of degree=codimension+*2, J. Pure Appl. Algebra 71 (2-3) (1991), 203–231.

[27] S. L. Kleiman - R. V. Martins, *The canonical model of a singular curve*, Geom. Dedicata 139 (2009), 139–166.

[28] J. O. Kleppe - R. M. Miró-Roig, *The dimension of the Hilbert scheme of Gorenstein codimension 3 subschemes*, J. Pure Appl. Algebra 127 (1) (1998), 73–82.

[29] J. Migliore, *Introduction to liaison theory and deficiency modules*, Progress in Math. 165, Birkhäuser, Boston, 1998.

[30] R. M. Miró-Roig, *Determinantal ideals*, Progress in Math. 264, Birkhäuser, Basel, 2008.

[31] P. E. Newstead, *Introduction to moduli problems and orbit spaces*, Springer, Bombay, 1978.

[32] D. Perrin, *Courbes passant par m points généraux de* $\mathbb{P}^3$, Bull. Soc. Math. France, Mem. 28/29, 1987.

[33] E. Sernesi, *On the existence of certain families of curves*, Invent. Math. 75 (1) (1984), 25–57.

[34] C. Walter, *The cohomology of the normal bundle of space curves*, preprint, 1989.

[35] C. Walter, *Curves in* $\mathbb{P}^3$ *with the expect monad*, J. Algebraic Geom. 4 (1995), 301–320.

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