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PERIODIC SOLUTIONS FOR A SECOND ORDER NONLINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATION WITH VARIABLE DELAY

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In this paper we study the existence of periodic solutions of the second order nonlinear neutral differential equation with functional delay

$$\begin{aligned}&\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)h(x(t))\\&=\frac{d}{dt}g(t,x(t-\tau(t)))+f(t,x(t),x(t-\tau(t))).\end{aligned}$$

We invert the given equation to obtain an integral, but equivalent, equation from which we define a fixed point mapping written as a sum of a large contraction and a compact map. We show that, under suitable conditions, such maps fit very nicely into the framework of Krasnoselskii-Burton's fixed point theorem so that the existence of periodic solutions is concluded.

1. Introduction

In recent years, there have been a few papers written on the existence of periodic solutions, nontrivial periodic solutions and positive periodic solutions for

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several classes of functional differential equations with delays, which arise from a number of mathematical ecological models, economical and control models, physiological and population models and other models, see [1]–[26] and the references therein.

In this paper, we are interested in the analysis of qualitative theory of periodic solutions of delay differential equations. Motivated by the papers [1]-[9], [12]-[14], [19]-[21], [24]-[26] and the references therein, we focus on the existence of periodic solutions for the second order nonlinear neutral differential equation

$$\begin{aligned}&\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)h(x(t))\\&=\frac{d}{dt}g(t,x(t-\tau(t)))+f(t,x(t),x(t-\tau(t))),\end{aligned}\tag{1}$$

where p, q are positive continuous real-valued functions. The function $g:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is differentiable, $h:\mathbb{R}\to\mathbb{R}$ and $f:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous with respect to its arguments.

To reach our desired end we have to transform (1) into an equivalent integral equation that does not change the basic structure and properties of the given one and from which we construct a fixed point mapping. In so many cases such a transformation tends to be a hard task. Nevertheless, getting an appropriate mapping is fundamental to the method employed in this paper. The transformation of (1) yields a mapping which splits in the sum of a large contraction and a compact map suitable for applying Krasnoselskii-Burton's fixed point theorem in such a way that the existence of periodic solutions is concluded.

The organization of this paper is as follows. In Section 2, we introduce some notations and lemmas, and state some preliminary results needed in later sections, then we give the Green's function of (1), which plays an important role in this paper. Also, we present the inversion of (1) and Krasnoselskii-Burton's fixed point theorem. In Section 3, we present our main result on existence.

2. Preliminaries

For $T>0$, let P_T be the set of all continuous scalar functions x, periodic in t of period T. Then $(P_T,\|.\|)$ is a Banach space with the supremum norm

$$\|x\|:=\sup_{t\in\mathbb{R}}|x(t)|=\sup_{t\in[0,T]}|x(t)|.$$

Since we are searching for periodic solutions for equation (1), it is natural to assume that

$$p(t+T)=p(t),\ q(t+T)=q(t),\ \tau(t+T)=\tau(t),\tag{2}$$

where τ is a continuous scalar function, and $\tau(t) \geq \tau^* > 0$. Also, we assume

$$\int_0^T p(s)\,ds > 0, \int_0^T q(s)\,ds > 0. \tag{3}$$

Functions $g(t,x)$ and $f(t,x,y)$ are periodic in t with period T. They are also supposed to be globally Lipschitz continuous in x and in x and y, respectively. That is,

$$g(t+T,x) = g(t,x), \; f(t+T,x,y) = f(t,x,y), \tag{4}$$

and there are positive constants k_1, k_2, k_3 such that

$$|g(t,x) - g(t,y)| \leq k_1 \|x-y\|, \tag{5}$$

and

$$|f(t,x,y) - f(t,z,w)| \leq k_2 \|x-z\| + k_3 \|y-w\|. \tag{6}$$

Lemma 2.1 ([20])**.** *Suppose that (2) and (3) hold and*

$$\frac{R_1 \left[\exp\left(\int_0^T p(u)\,du\right) - 1\right]}{Q_1 T} \geq 1, \tag{7}$$

where

$$R_1 = \max_{t\in[0,T]} \left| \int_t^{t+T} \frac{\exp\left(\int_t^s p(u)\,du\right)}{\exp\left(\int_0^T p(u)\,du\right) - 1} q(s)\,ds \right|,$$
$$Q_1 = \left(1 + \exp\left(\int_0^T p(u)\,du\right)\right)^2 R_1^2.$$

Then there are continuous T-periodic functions a and b such that $b(t) > 0$, $\int_0^T a(u)\,du > 0$ and

$$a(t) + b(t) = p(t), \; \frac{d}{dt}b(t) + a(t)b(t) = q(t), \text{ for } t \in \mathbb{R}.$$

Lemma 2.2 ([25])**.** *Suppose the conditions of Lemma 2.1 hold and $\phi \in P_T$. Then the equation*

$$\frac{d^2}{dt^2}x(t) + p(t)\frac{d}{dt}x(t) + q(t)x(t) = \phi(t),$$

has a T-periodic solution. Moreover, the periodic solution can be expressed by

$$x(t) = \int_t^{t+T} G(t,s)\,\phi(s)\,ds,$$

where

$$G(t,s)=\frac{\int_t^s \exp\left[\int_t^u b(v)\,dv+\int_u^s a(v)\,dv\right]du}{\left[\exp\left(\int_0^T a(u)\,du\right)-1\right]\left[\exp\left(\int_0^T b(u)\,du\right)-1\right]} + \frac{\int_s^{t+T} \exp\left[\int_t^u b(v)\,dv+\int_u^{s+T} a(v)\,dv\right]du}{\left[\exp\left(\int_0^T a(u)\,du\right)-1\right]\left[\exp\left(\int_0^T b(u)\,du\right)-1\right]}. \tag{8}$$

Corollary 2.3 ([25]). *Green's function G (8) satisfies the following properties*

$$G(t,t+T)=G(t,t),\ G(t+T,s+T)=G(t,s),$$
$$\frac{\partial}{\partial s}G(t,s)=a(s)\,G(t,s)-\frac{\exp\left(\int_t^s b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right)-1},$$
$$\frac{\partial}{\partial t}G(t,s)=-b(t)\,G(t,s)+\frac{\exp\left(\int_t^s a(v)\,dv\right)}{\exp\left(\int_0^T a(v)\,dv\right)-1}.$$

The following lemma is fundamental to our results.

Lemma 2.4. *Suppose (2)–(4) and (7) hold. If $x\in P_T$, then x is a solution of equation (1) if and only if*

$$x(t)=\int_t^{t+T} G(t,s)\,q(s)\,H(x(s))\,ds + \int_t^{t+T}\{g(s,x(s-\tau(s)))\,[E(t,s)-a(s)\,G(t,s)] + G(t,s)\,f(s,x(s),x(s-\tau(s)))\}\,ds, \tag{9}$$

where

$$E(t,s)=\frac{\exp\left(\int_t^s b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right)-1}, \tag{10}$$

and

$$H(x(s))=x(s)-h(x(s)). \tag{11}$$

Proof. Let $x\in P_T$ be a solution of (1). Rewrite (1) as

$$\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)\,x(t) = q(t)\,H(x(t))+\frac{d}{dt}g(t,x(t-\tau(t)))+f(t,x(t),x(t-\tau(t))).$$

From Lemma 2.2, we have

$$x(t)=\int_t^{t+T}G(t,s)\,q(s)\,H(x(s))\,ds$$
$$+\int_t^{t+T}G(t,s)\left[\frac{\partial}{\partial s}g(s,x(s-\tau(s)))+f(s,x(s),x(s-\tau(s)))\right]ds. \quad (12)$$

Performing an integration by parts, we have

$$\int_t^{t+T}G(t,s)\frac{\partial}{\partial s}g(s,x(s-\tau(s)))\,ds=-\int_t^{t+T}\left[\frac{\partial}{\partial s}G(t,s)\right]g(s,x(s-\tau(s)))\,ds$$
$$=\int_t^{t+T}g(s,x(s-\tau(s)))\left[E(t,s)-a(s)\,G(t,s)\right]ds, \quad (13)$$

where E is given by (10). We obtain (9) by substituting (13) in (12). Since each step is reversible, the converse follows easily. This completes the proof. □

Lemma 2.5 ([25])**.** *Let* $A=\int_0^T p(u)\,du$, $B=T^2\exp\left(\frac{1}{T}\int_0^T \ln(q(u))\,du\right)$. *If*

$$A^2\geq 4B, \quad (14)$$

then we have

$$\min\left\{\int_0^T a(u)\,du,\int_0^T b(u)\,du\right\}\geq\frac{1}{2}\left(A-\sqrt{A^2-4B}\right):=l,$$
$$\max\left\{\int_0^T a(u)\,du,\int_0^T b(u)\,du\right\}\leq\frac{1}{2}\left(A+\sqrt{A^2-4B}\right):=m.$$

Corollary 2.6 ([25])**.** *Functions G and E satisfy*

$$\frac{T}{(e^m-1)^2}\leq G(t,s)\leq\frac{T\exp\left(\int_0^T p(u)\,du\right)}{(e^l-1)^2},\ \ |E(t,s)|\leq\frac{e^m}{e^l-1}.$$

In the analysis, we employ a fixed point theorem in which the notion of a large contraction is required as one of the sufficient conditions. The following definition, due to Burton, can be found in [8], [9].

Definition 2.7 (Large Contraction)**.** Let $(\mathbb{M},d)$ be a metric space and consider $\mathcal{B}:\mathbb{M}\to\mathbb{M}$. Then $\mathcal{B}$ is said to be a *large contraction* if given $\phi,\varphi\in\mathbb{M}$ with $\phi\neq\varphi$ then $d(\mathcal{B}\phi,\mathcal{B}\varphi)\leq d(\phi,\varphi)$ and if for all $\varepsilon>0$, there exists a $\delta<1$ such that

$$[\phi,\varphi\in\mathbb{M},\ d(\phi,\varphi)\geq\varepsilon]\Rightarrow d(\mathcal{B}\phi,\mathcal{B}\varphi)\leq\delta d(\phi,\varphi).$$

The next theorem is also a result of T. A. Burton. This captivating theorem, which constitutes a basis for our main result, is a reformulated version of Krasnoselskii's fixed point theorem and has been used successfully in existence and stability in differential functional equations (see [[8], Theorem 3] and [9]).

Theorem 2.8 (Krasnoselskii-Burton)**.** *Let $\mathbb{M}$ be a closed bounded convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{M}$ into $\mathbb{M}$ such that*

(i) $x, y \in \mathbb{M}$, implies $\mathcal{A}x + \mathcal{B}y \in \mathbb{M}$,

(ii) $\mathcal{A}$ is compact and continuous,

(iii) $\mathcal{B}$ is a large contraction mapping.

Then there exists $z \in \mathbb{M}$ with $z = \mathcal{A}z + \mathcal{B}z$.

We will use this theorem to prove the existence of periodic solutions for equation (1).

3. Existence of periodic solutions

Obviously, if we want to apply Theorem 2.8, then we need to define a Banach space $\mathbb{B}$, a closed bounded convex subset $\mathbb{M}_L$ of $\mathbb{B}$ and construct two mappings, one is a large contraction and the other is compact. So, we let $(\mathbb{B}, \|.\|) = (P_T, \|.\|)$ and $\mathbb{M}_L = \{\varphi \in \mathbb{B} : \|\varphi\| \leq L\}$, where L is positive constant. We express equation (9) as

$$\varphi(t) = (\mathcal{B}\varphi)(t) + (\mathcal{A}\varphi)(t) := (\mathbb{C}\varphi)(t),$$

where $\mathcal{A}, \mathcal{B} : \mathbb{M}_L \to \mathbb{B}$ are defined by

$$\begin{aligned}(\mathcal{A}\varphi)(t) = \int_t^{t+T} &\{g(s, \varphi(s - \tau(s)))[E(t,s) - a(s)G(t,s)] \\ &+ G(t,s) f(s, \varphi(s), \varphi(s - \tau(s)))\} ds, \end{aligned} \tag{15}$$

and

$$(\mathcal{B}\varphi)(t) = \int_t^{t+T} G(t,s) q(s) H(\varphi(s)) ds. \tag{16}$$

To simplify notations, we introduce the following constants

$$\begin{aligned}&\alpha = \frac{T \exp\left(\int_0^T p(u) du\right)}{(e^l - 1)^2}, \ \beta = \frac{e^m}{e^l - 1}, \ \sigma = \max_{t \in [0,T]} |q(t)|, \ \lambda = \max_{t \in [0,T]} |b(t)|, \\ &\mu = \max_{t \in [0,T]} |a(t)|, \ \rho_1 = \max_{t \in [0,T]} |g(t,0)|, \ \rho_2 = \max_{t \in [0,T]} |f(t,0,0)|. \end{aligned} \tag{17}$$

We need the following assumptions

$$\alpha\sigma T \leq 1, \tag{18}$$

$$JT\left[\left(k_1 L+\rho_1\right)\left(\beta+\mu\alpha\right)+\alpha\left(\left(k_2+k_3\right)L+\rho_2\right)\right] \leq L, \tag{19}$$

$$\max\left(\left|H\left(-L\right)\right|,\left|H\left(L\right)\right|\right) \leq \frac{\left(J-1\right)L}{J}, \tag{20}$$

where J is constant with $J \geq 3$.

We begin with the following proposition (see [1]) and for convenience we present, below, its proof. Let L be a fixed number. In the next proposition we prove that, for a well chosen function h, the mapping H in (11) is a large contraction on $\mathbb{M}_L$. So, let us make the following assumptions on the function $h:\mathbb{R}\to\mathbb{R}$.

(H1) h is continuous on $U_L = [-L,L]$ and differentiable on $(-L,L)$.

(H2) h is strictly increasing on U_L.

(H3) $\sup\limits_{s\in(-L,L)} h'(s) \leq 1$.

Proposition 3.1. *Let $h:\mathbb{R}\to\mathbb{R}$ be a function satisfying (H1)–(H3). Then the mapping H in (11) is a large contraction on the set $\mathbb{M}_L$.*

Proof. Let $\phi,\varphi \in \mathbb{M}_L$ with $\phi \neq \varphi$. Then $\phi(t) \neq \varphi(t)$ for some $t \in \mathbb{R}$. Define the set

$$D(\phi,\varphi) := \{t \in \mathbb{R} : \phi(t) \neq \varphi(t)\}.$$

Note that $\varphi(t) \in U_L$ for all $t \in \mathbb{R}$ whenever $\varphi \in \mathbb{M}_L$. Since h is strictly increasing

$$\frac{h(\varphi(t))-h(\phi(t))}{\varphi(t)-\phi(t)} = \frac{h(\phi(t))-h(\varphi(t))}{\phi(t)-\varphi(t)} > 0, \tag{21}$$

holds for all $t \in D(\phi,\varphi)$. On the other hand, for all $t \in D(\phi,\varphi)$, we have

$$\begin{aligned}\left|(H\phi)(t)-(H\varphi)(t)\right| &= \left|\phi(t)-h(\phi(t))-\varphi(t)+h(\varphi(t))\right| \\ &= \left|\phi(t)-\varphi(t)\right|\left|1-\left(\frac{h(\phi(t))-h(\varphi(t))}{\phi(t)-\varphi(t)}\right)\right|.\end{aligned} \tag{22}$$

For each fixed $t \in D(\phi,\varphi)$, define the set $U_t \subset U_L$ by

$$U_t = \begin{cases} (\varphi(t),\phi(t)), & \text{if } \phi(t) > \varphi(t), \\ (\phi(t),\varphi(t)), & \text{if } \varphi(t) > \phi(t), \end{cases} \quad \text{for } t \in D(\phi,\varphi).$$

The mean value theorem implies that for each fixed $t \in D(\phi,\varphi)$ there exists a real number $c_t \in U_t$ such that

$$\frac{h(\phi(t))-h(\varphi(t))}{\phi(t)-\varphi(t)} = h'(c_t).$$

By (H2) and (H3), we have

$$1 \geq \sup_{t\in(-L,L)} h'(t) \geq \sup_{t\in U_t} h'(t) \geq h'(c_t) \geq \inf_{s\in U_t} h'(s) \geq \inf_{t\in(-L,L)} h'(t) \geq 0. \tag{23}$$

Consequently, by (21)–(23), we obtain

$$|(H\phi)(t)-(H\varphi)(t)| \leq \left|1-\inf_{u\in(-L,L)} h'(u)\right| |\phi(t)-\varphi(t)|, \tag{24}$$

for all $t \in D(\phi,\varphi)$. Hence, the mapping H is a large contraction in the supremum norm. Indeed, fix $\varepsilon \in (0,1)$ and assume that ϕ and φ are two functions in $\mathbb{M}_L$ satisfying

$$\|\phi-\varphi\| = \sup_{t\in D(\phi,\varphi)} |\phi(t)-\varphi(t)| \geq \varepsilon.$$

If $|\phi(t)-\varphi(t)| \leq \varepsilon/2$ for some $t \in D(\phi,\varphi)$, then from (23) and (24), we get

$$|(H\phi)(t)-(H\varphi)(t)| \leq |\phi(t)-\varphi(t)| \leq \frac{1}{2}\|\phi-\varphi\|. \tag{25}$$

Since h is continuous and strictly increasing, the function $h\left(u+\frac{\varepsilon}{2}\right)-h(u)$ attains its minimum on the closed and bounded interval $[-L,L]$. Thus, if $\frac{\varepsilon}{2} < |\phi(t)-\varphi(t)|$ for some $t \in D(\phi,\varphi)$, then from (H2) and (H3) we conclude that

$$1 \geq \frac{h(\phi(t))-h(\varphi(t))}{\phi(t)-\varphi(t)} > \lambda,$$

where,

$$\lambda := \frac{1}{2L}\min\left\{h\left(u+\frac{\varepsilon}{2}\right)-h(u),\ u\in[-L,L]\right\} > 0.$$

Therefore, from (22), we have

$$|(H\phi)(t)-(H\varphi)(t)| \leq (1-\lambda)\|\phi-\varphi\|. \tag{26}$$

Consequently, it follows from (25) and (26) that

$$|(H\phi)(t)-(H\varphi)(t)| \leq \eta\|\phi-\varphi\|,$$

where

$$\eta = \max\left\{\frac{1}{2}, 1-\lambda\right\} < 1.$$

The proof is complete. □

Example 3.2 ([1]). If $\mathbb{M}_{5^{-1/4}} = \left\{\varphi \in \mathbb{B} : \|\varphi\| \leq 5^{-1/4}\right\}$ and $h(u) = u^5$, then the mapping H defined by (11) is a large contraction on the set $\mathbb{M}_{5^{-1/4}}$.

We shall prove that the mapping $\mathbb{C}$ has a fixed point which solves (1), whenever its derivative exists.

Lemma 3.3. *Suppose that conditions (2)–(7), (14) and (19) hold. Then* $\mathcal{A}: \mathbb{M}_L \to \mathbb{M}_L$ *is compact.*

Proof. Let $\mathcal{A}$ be defined by (15). Obviously, $\mathcal{A}\varphi$ is continuous and it is easy to show that $(\mathcal{A}\varphi)(t+T) = (\mathcal{A}\varphi)(t)$. Observe that in view of (5) and (6), we have

$$\begin{aligned}
|g(t,x)| &\le |g(t,x) - g(t,0) + g(t,0)| \\
&\le |g(t,x) - g(t,0)| + |g(t,0)| \\
&\le k_1 \|x\| + \rho_1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|f(t,x,y)| &\le |f(t,x,y) - f(t,0,0) + f(t,0,0)| \\
&\le |f(t,x,y) - f(t,0,0)| + |f(t,0,0)| \\
&\le k_2 \|x\| + k_3 \|y\| + \rho_2.
\end{aligned}$$

So, for any $\varphi \in \mathbb{M}_L$, we have

$$\begin{aligned}
&|(\mathcal{A}\varphi)(t)| \le \int_t^{t+T} \{|g(s,\varphi(s-\tau(s)))| [|E(t,s)| + |a(s)| |G(t,s)|] \\
&+ |G(t,s)| |f(s,\varphi(s),\varphi(s-\tau(s)))|\} ds \\
&\le \int_t^{t+T} (k_1 L + \rho_1)(\beta + \mu\alpha) + \alpha((k_2 + k_3) L + \rho_2) ds \\
&\le T[(k_1 L + \rho_1)(\beta + \mu\alpha) + \alpha((k_2 + k_3) L + \rho_2)] \le \frac{L}{J} < L.
\end{aligned}$$

That is $\mathcal{A}\varphi \in \mathbb{M}_L$.

To see that $\mathcal{A}$ is continuous, we let $\varphi, \psi \in \mathbb{M}_L$. Given $\varepsilon > 0$, take $\eta = \varepsilon / N$ with $N = T[k_1(\beta + \mu\alpha) + \alpha(k_2 + k_3)]$ where k_1, k_2 and k_3 are given by (5) and (6). Now, for $\|\varphi - \psi\| < \eta$, we obtain

$$\begin{aligned}
\|\mathcal{A}\varphi - \mathcal{A}\psi\| &\le \int_t^{t+T} [k_1(\beta + \mu\alpha)\|\varphi - \psi\| + \alpha(k_2 + k_3)\|\varphi - \psi\|] ds \\
&\le N\|\varphi - \psi\| < \varepsilon.
\end{aligned}$$

This proves that $\mathcal{A}$ is continuous.

To show that the image of $\mathcal{A}$ is contained in a compact set. Let $\varphi_n \in \mathbb{M}_L$, where n is a positive integer. Then, as above, we see that

$$\|\mathcal{A}\varphi_n\| \le L.$$

Next we calculate $\frac{d}{dt}(\mathcal{A}\varphi_n)(t)$ and show that it is uniformly bounded. By making use of (2), (3) and (4) we obtain by taking the derivative in (15) that

$$\begin{aligned}
&\frac{d}{dt}(\mathcal{A}\varphi_n)(t) \\
&= \frac{\exp\left(\int_t^{t+T} b(v)dv\right)-1}{\exp\left(\int_0^T b(v)dv\right)-1} g(t, \varphi_n(t-\tau(t))) \\
&+ \int_t^{t+T} g(s, \varphi_n(s-\tau(s))) \\
&\times \left[-b(t)E(t,s) - a(s)\left(-b(t)G(t,s) + \frac{\exp(\int_t^s a(v)dv)}{\exp\left(\int_0^T a(v)dv\right)-1}\right)\right] ds \\
&+ \int_t^{t+T} \left(-b(t)G(t,s) + \frac{\exp(\int_t^s a(v)dv)}{\exp\left(\int_0^T a(v)dv\right)-1}\right) f(s, \varphi_n(s), \varphi_n(s-\tau(s))) ds.
\end{aligned}$$

Consequently, by invoking (5), (6) and (17), we obtain

$$\begin{aligned}
&\left|\frac{d}{dt}(\mathcal{A}\varphi_n)(t)\right| \\
&\leq \beta(k_1 L + \rho_1) \\
&+ T[(k_1 L + \rho_1)(\lambda\beta + \mu(\lambda\alpha + \beta)) + (\lambda\alpha + \beta)((k_2 + k_3)L + \rho_2)] \\
&\leq D,
\end{aligned}$$

for some positive constant D. Hence the sequence $(\mathcal{A}\varphi_n)$ is uniformly bounded and equicontinuous. The Ascoli-Arzela theorem implies that a subsequence $(\mathcal{A}\varphi_{n_k})$ of $(\mathcal{A}\varphi_n)$ converges uniformly to a continuous T-periodic function. Thus $\mathcal{A}$ is continuous and $\mathcal{A}(\mathbb{M}_L)$ is contained in a compact subset of $\mathbb{M}_L$. □

Lemma 3.4. *For $\mathcal{B}$ defined in (16), suppose that (H1)–(H3), (18) and (20) hold. Then $\mathcal{B} : \mathbb{M}_L \to \mathbb{M}_L$ is a large contraction.*

Proof. Let $\mathcal{B}$ be defined by (16). Obviously, $\mathcal{B}\varphi$ is continuous and it is easy to show that $(\mathcal{B}\varphi)(t+T) = (\mathcal{B}\varphi)(t)$. So, for any $\varphi \in \mathbb{M}_L$, we have

$$\begin{aligned}
|(\mathcal{B}\varphi)(t)| &\leq \int_t^{t+T} |G(t,s)||q(s)||H(\varphi(s))| ds \\
&\leq \alpha\sigma T \max(|H(-L)|, |H(L)|) \leq \frac{(J-1)L}{J} < L,
\end{aligned}$$

by (18) and (20). Then, for any $\varphi \in \mathbb{M}_L$, we have

$$\|\mathcal{B}\varphi\| \leq L.$$

Thus $\mathcal{B}\varphi \in \mathbb{M}_L$. Consequently, we have $\mathcal{B} : \mathbb{M}_L \to \mathbb{M}_L$.

It remains to show that $\mathcal{B}$ is large contraction. From the proof of Proposition 3.1 we have for $\varphi, \psi \in \mathbb{M}_L$, with $\varphi \neq \psi$

$$|(\mathcal{B}\varphi)(t) - (\mathcal{B}\psi)(t)| \leq \alpha\sigma T \|\varphi - \psi\| \leq \|\varphi - \psi\|.$$

Then $\|\mathcal{B}\varphi - \mathcal{B}\psi\| \leq \|\varphi - \psi\|$. Now, let $\varepsilon \in (0,1)$ be given and let $\varphi, \psi \in \mathbb{M}_L$ with $\|\varphi - \psi\| \geq \varepsilon$. From the proof of the Proposition 3.1 we have found $\delta < 1$ such that

$$|(\mathcal{B}\varphi)(t) - (\mathcal{B}\psi)(t)| \leq \alpha\sigma T\delta \|\varphi - \psi\| \leq \delta \|\varphi - \psi\|.$$

Then $\|\mathcal{B}\varphi - \mathcal{B}\psi\| \leq \delta \|\varphi - \psi\|$. Consequently, $\mathcal{B}$ is a large contraction. □

Theorem 3.5. *Let $(P_T, \|.\|)$ be the Banach space of continuous T-periodic real valued functions and $\mathbb{M}_L = \{\varphi \in P_T : \|\varphi\| \leq L\}$, where L is positive constant. Suppose (H1)–(H3), (2)–(7), (14) and (18)–(20) hold. Then equation (1) has a T-periodic solution φ in the subset $\mathbb{M}_L$.*

Proof. By Lemma 3.3, the operator $\mathcal{A} : \mathbb{M}_L \to \mathbb{M}_L$ is compact and continuous. Also, from Lemma 3.4, the operator $\mathcal{B} : \mathbb{M}_L \to \mathbb{M}_L$ is a large contraction. Moreover, if $\varphi, \psi \in \mathbb{M}_L$, we see that

$$\|\mathcal{A}\varphi + \mathcal{B}\psi\| \leq \|\mathcal{A}\varphi\| + \|\mathcal{B}\psi\| \leq \frac{L}{J} + \frac{(J-1)L}{J} \leq L.$$

Thus $\mathcal{A}\varphi + \mathcal{B}\psi \in \mathbb{M}_L$.

Clearly, all the hypotheses of the Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $\varphi \in \mathbb{M}_L$ such that $\varphi = \mathcal{A}\varphi + \mathcal{B}\varphi$. By Lemma 2.4 this fixed point is a solution of (1) and the proof is complete. □

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