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SUBORDINATION PROPERTIES FOR A CERTAIN CLASS OF ANALYTIC FUNCTIONS WITH COMPLEX ORDER

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In this paper, we derive several subordination results for a certain class of analytic functions defined by the generalized Al-Oboudi differential operator. Relevant connections of some of the results obtained with those in earlier works are also provided.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

Further, by $\mathcal{S}$ we will denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

Also let $\mathcal{S}^*(b)$ and $\mathcal{K}(b)$ denote, respectively, the subclasses of $\mathcal{A}$ consisting of functions that are starlike of complex order $b\,(b \in \mathbb{C}\setminus\{0\})$ and convex of complex order $b\,(b \in \mathbb{C}\setminus\{0\})$ in $\mathbb{U}$.

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In particular, the classes $\mathcal{S}^* := \mathcal{S}^*(1)$ and $\mathcal{K} := \mathcal{K}(1)$ are the familiar classes of starlike and convex functions in $\mathbb{U}$, respectively.

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \qquad \text{and} \qquad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Rightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

The following definition of fractional derivative by Owa [10] (also by Srivastava and Owa [15]) will be required in our investigation.

The fractional derivative of order γ is defined, for a function f, by

$$D_z^{\gamma} f(z) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\gamma}} dt \quad (0 \leq \gamma < 1), \tag{2}$$

where the function f is analytic in a simply connected region of the complex z-plane containing the origin, and the multiplicity of $(z-t)^{-\gamma}$ is removed by requiring $\log(z-t)$ to be real when $z-t > 0$.

It readily follows from (2) that

$$D_z^{\gamma} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\gamma)} z^{k-\gamma} \quad (0 \leq \gamma < 1, k \in \mathbb{N} = \{1, 2, \ldots\}).$$

Using the operator $D_z^{\gamma} f$, Owa and Srivastava [11] introduced the operator $\Omega^{\gamma} : \mathcal{A} \to \mathcal{A}$, which is known as an extension of fractional derivative and fractional integral, as follows:

$$\begin{aligned} \Omega^{\gamma} f(z) &= \Gamma(2-\gamma) z^{\gamma} D_z^{\gamma} f(z) \\ &= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)} a_k z^k \qquad \gamma \neq 2, 3, 4, \ldots \end{aligned} \tag{3}$$

Note that

$$\Omega^0 f(z) = f(z).$$

In [2], Al-Oboudi and Al-Amoudi defined the linear multiplier fractional differential operator $D_\lambda^{n,\gamma}$ (which is known as the generalized Al-Oboudi differential operator) as follows:

$$\begin{aligned} D^0 f(z) &= f(z), \\ D_\lambda^{1,\gamma} f(z) &= (1-\lambda)\,\Omega^\gamma f(z) + \lambda z\,(\Omega^\gamma f(z))' \\ &= D_\lambda^\gamma\,(f(z)), \quad \lambda \geq 0,\ 0 \leq \gamma < 1, \qquad (4) \\ D_\lambda^{2,\gamma} f(z) &= D_\lambda^\gamma\left(D_\lambda^{1,\gamma} f(z)\right), \\ &\vdots \\ D_\lambda^{n,\gamma} f(z) &= D_\lambda^\gamma\left(D_\lambda^{n-1,\gamma} f(z)\right), \quad n \in \mathbb{N}. \qquad (5) \end{aligned}$$

If f is given by (1), then by (3), (4) and (5), we see that

$$D_\lambda^{n,\gamma} f(z) = z + \sum_{k=2}^{\infty} \Psi_{k,n}(\gamma,\lambda)\, a_k z^k, \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{6}$$

where

$$\Psi_{k,n}(\gamma,\lambda) = \left[\frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)}\,(1+\lambda\,(k-1))\right]^n. \tag{7}$$

Remark 1.1. (i) When $\gamma = 0$, we get Al-Oboudi differential operator [1].
(ii) When $\gamma = 0$ and $\lambda = 1$, we get Sălăgean differential operator [12].
(iii) When $n = 1$ and $\lambda = 0$, we get Owa-Srivastava fractional differential operator [11].

Let $\mathcal{G}_{\gamma,\lambda}^n(\delta,b,A,B)$ denote the class of functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{1}{b}\left((1-\delta)\,\frac{D_\lambda^{n,\gamma} f(z)}{z} + \delta\left(D_\lambda^{n,\gamma} f(z)\right)' - 1\right) \prec \frac{1+Az}{1+Bz} \tag{8}$$

or satisfying

$$\left|\frac{(1-\delta)\,\frac{D_\lambda^{n,\gamma} f(z)}{z} + \delta\left(D_\lambda^{n,\gamma} f(z)\right)' - 1}{(A-B)\,b - B\left[(1-\delta)\,\frac{D_\lambda^{n,\gamma} f(z)}{z} + \delta\left(D_\lambda^{n,\gamma} f(z)\right)' - 1\right]}\right| < 1, \tag{9}$$

where $z \in \mathbb{U}$, $b \in \mathbb{C}\setminus\{0\}$, $\delta \geq 0$, $-1 \leq B < A \leq 1$ and $D_\lambda^{n,\gamma}$ is the generalized Al-Oboudi differential operator.

In [13], by using the Sălăgean differential operator D^n, Sivasubramanian et al. defined the class

$$\mathcal{G}_{0,1}^{n}(\delta,b,A,B)=\mathcal{G}_{n}(\delta,b,A,B)$$
$$=\left\{f\in\mathcal{A}:1+\frac{1}{b}\left((1-\delta)\frac{D^{n}f(z)}{z}+\delta\left(D^{n}f(z)\right)'-1\right)\prec\frac{1+Az}{1+Bz}\right\}$$

which generalizes the class

$$\mathcal{G}_{0,1}^{n}(\delta,b,1,-1)=\mathcal{G}_{n}(\delta,b)$$
$$=\left\{f\in\mathcal{A}:\Re\left\{1+\frac{1}{b}\left((1-\delta)\frac{D^{n}f(z)}{z}+\delta\left(D^{n}f(z)\right)'-1\right)\right\}>0\right\}$$

introduced by Aouf [3].

We note that, for $z\in\mathbb{U}$,

(i) $\mathcal{G}_{\gamma,\lambda}^{n}(\delta,b,1,-1)=\mathcal{G}_{\gamma,\lambda}^{n}(\delta,b)$
$=\left\{f\in\mathcal{A}:\Re\left\{1+\frac{1}{b}\left((1-\delta)\frac{D_{\lambda}^{n,\gamma}f(z)}{z}+\delta\left(D_{\lambda}^{n,\gamma}f(z)\right)'-1\right)\right\}>0\right\}$,

(ii) $\mathcal{G}_{\gamma,\lambda}^{n}(0,b,1,-1)=\mathcal{G}_{\gamma,\lambda}^{n}(b)=\left\{f\in\mathcal{A}:\Re\left\{1+\frac{1}{b}\left(\frac{D_{\lambda}^{n,\gamma}f(z)}{z}-1\right)\right\}>0\right\}$,

(iii) $\mathcal{G}_{\gamma,\lambda}^{n}(1,b,1,-1)=\mathcal{R}_{\gamma,\lambda}^{n}(b)$
$=\left\{f\in\mathcal{A}:\Re\left\{1+\frac{1}{b}\left(\left(D_{\lambda}^{n,\gamma}f(z)\right)'-1\right)\right\}>0\right\}$,

(iv) $\mathcal{G}_{\gamma,\lambda}^{0}(0,b,1,-1)=\mathcal{G}(b)=\left\{f\in\mathcal{A}:\Re\left\{1+\frac{1}{b}\left(\frac{f(z)}{z}-1\right)\right\}>0\right\}$,

(v) $\mathcal{G}_{\gamma,\lambda}^{0}(1,b,1,-1)=\mathcal{R}(b)=\left\{f\in\mathcal{A}:\Re\left\{1+\frac{1}{b}\left(f'(z)-1\right)\right\}>0\right\}$,

(vi) $\mathcal{G}_{\gamma,\lambda}^{0}(0,1-\alpha,1,-1)=\mathcal{G}_{\alpha}=\left\{f\in\mathcal{A}:\Re\left\{\frac{f(z)}{z}\right\}>\alpha,\ 0\leq\alpha<1\right\}$,

(vii) $\mathcal{G}_{\gamma,\lambda}^{0}(1,1-\alpha,1,-1)=\mathcal{R}_{\alpha}=\left\{f\in\mathcal{A}:\Re\left\{f'(z)\right\}>\alpha,\ 0\leq\alpha<1\right\}$,

(viii) $\mathcal{G}_{\gamma,\lambda}^{0}(\delta,b,1,-1)=\mathcal{G}(\delta,b)$
$=\left\{f\in\mathcal{A}:\Re\left\{1+\frac{1}{b}\left((1-\delta)\frac{f(z)}{z}+\delta f'(z)-1\right)\right\}>0\right\}$.

The class $\mathcal{R}(b)$ was studied by Halim [8], the class $\mathcal{G}_{\alpha}$ was studied by Chen [5, 6] and the class $\mathcal{R}_{\alpha}$ was studied by Ezrohi [7].

Definition 1.2 (Hadamard product or Convolution)**.** Given two functions f and g in the class $\mathcal{A}$, where f is given by (1) and g is given by

$$g(z)=z+\sum_{k=2}^{\infty}b_{k}z^{k},$$

the Hadamard product (or convolution) $f*g$ is defined by

$$(f*g)(z)=z+\sum_{k=2}^{\infty}a_{k}b_{k}z^{k}=(g*f)(z)\quad(z\in\mathbb{U}).$$

Definition 1.3 (Subordinating Factor Sequence)**.** A sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever f of the form (1) is analytic, univalent and convex in $\mathbb{U}$, we have the subordination given by

$$\sum_{k=1}^{\infty} a_k b_k z^k \prec f(z) \quad (z \in \mathbb{U};\ a_1 = 1).$$

Lemma 1.4 ([16])**.** *The sequence $\{b_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\Re\left\{1 + 2\sum_{k=1}^{\infty} b_k z^k\right\} > 0 \quad (z \in \mathbb{U}).$$

2. Main Result

Now, we prove the following theorem which gives a sufficient condition for functions belonging to the class $\mathcal{G}_{\gamma,\lambda}^{n}(\delta,b,A,B)$.

Theorem 2.1. *Let the function f which is defined by (1) satisfy the following condition:*

$$\sum_{k=2}^{\infty} (1+|B|)(1+\delta(k-1))\Psi_{k,n}(\gamma,\lambda)|a_k| \le (A-B)|b|, \tag{10}$$

then $f \in \mathcal{G}_{\gamma,\lambda}^{n}(\delta,b,A,B)$.

Proof. Suppose that the inequality (10) holds. Then we have for $z \in \mathbb{U}$,

$$\begin{aligned}
&\left|(1-\delta)\frac{D_{\lambda}^{n,\gamma} f(z)}{z} + \delta\left(D_{\lambda}^{n,\gamma} f(z)\right)' - 1\right| \\
&- \left|(A-B)b - B\left[(1-\delta)\frac{D_{\lambda}^{n,\gamma} f(z)}{z} + \delta\left(D_{\lambda}^{n,\gamma} f(z)\right)' - 1\right]\right| \\
&= \left|\sum_{k=2}^{\infty}(1+\delta(k-1))\Psi_{k,n}(\gamma,\lambda)a_k z^{k-1}\right| \\
&- \left|(A-B)b - B\sum_{k=2}^{\infty}(1+\delta(k-1))\Psi_{k,n}(\gamma,\lambda)a_k z^{k-1}\right| \\
&\le \sum_{k=2}^{\infty}(1+\delta(k-1))\Psi_{k,n}(\gamma,\lambda)|a_k||z|^{k-1} \\
&- \left[(A-B)|b| - |B|\sum_{k=2}^{\infty}(1+\delta(k-1))\Psi_{k,n}(\gamma,\lambda)|a_k||z|^{k-1}\right] <
\end{aligned}$$

$$< \sum_{k=2}^{\infty} (1+|B|)(1+\delta(k-1))\Psi_{k,n}(\gamma,\lambda)|a_k| - (A-B)|b| \leq 0,$$

which shows that f belongs to the class $\mathcal{G}^{n}_{\gamma,\lambda}(\delta,b,A,B)$. □

In view of Theorem 2.1, we now introduce the subclass $\mathcal{G}^{n*}_{\gamma,\lambda}(\delta,b,A,B)$ which consists of functions $f \in \mathcal{A}$ whose Taylor-Maclaurin coefficients satisfy the inequality (10). We note that

$$\mathcal{G}^{n*}_{\gamma,\lambda}(\delta,b,A,B) \subset \mathcal{G}^{n}_{\gamma,\lambda}(\delta,b,A,B).$$

In this work, we prove several subordination relationships involving the function class $\mathcal{G}^{n*}_{\gamma,\lambda}(\delta,b,A,B)$ employing the technique used earlier by Attiya [4] and Srivastava and Attiya [14].

Theorem 2.2. *Let the function f defined by (1) be in the class $\mathcal{G}^{n*}_{\gamma,\lambda}(\delta,b,A,B)$ and suppose that $g \in \mathcal{K}$. Then*

$$\frac{(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)}{2[(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)+(A-B)|b|]}(f*g)(z) \prec g(z) \quad (z \in \mathbb{U}) \quad (11)$$

and

$$\Re\{f(z)\} > -\frac{(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)+(A-B)|b|}{(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)} \quad (z \in \mathbb{U}). \quad (12)$$

The constant factor

$$\frac{(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)}{2[(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)+(A-B)|b|]}$$

in the subordination result (11) cannot be replaced by a larger one.

Proof. Let $f \in \mathcal{G}^{n*}_{\gamma,\lambda}(\delta,b,A,B)$ and let $g(z) = z + \sum_{k=2}^{\infty} c_k z^k \in \mathcal{K}$. Then we have

$$\begin{aligned}&\frac{(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)}{2[(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)+(A-B)|b|]}(f*g)(z)\\ &= \frac{(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)}{2[(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)+(A-B)|b|]}\left(z+\sum_{k=2}^{\infty} a_k c_k z^k\right). \quad (13)\end{aligned}$$

Thus, by Definition 1.3, the subordination result (11) will hold true if the sequence

$$\left\{\frac{(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)}{2[(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)+(A-B)|b|]}a_k\right\}_{k=1}^{\infty} \quad (14)$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1.4, this is equivalent to the following inequality:

$$\Re\left\{1+\sum_{k=1}^{\infty}\frac{(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)}{[(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)+(A-B)|b|]}a_k z^k\right\}>0 \quad (z\in\mathbb{U}). \tag{15}$$

Since

$$(1+\delta(k-1))\Psi_{k,n}(\gamma,\lambda)$$

is an increasing function of k $(k\geq 2)$, when $|z|=r$ $(0<r<1)$, we have

$$\begin{aligned}
&\Re\left\{1+\sum_{k=1}^{\infty}\frac{(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)}{[(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)+(A-B)|b|]}a_k z^k\right\}\\
&=\Re\left\{1+\frac{(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)}{(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)+(A-B)|b|}z\right.\\
&\left.+\frac{1+|B|}{(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)+(A-B)|b|}\sum_{k=2}^{\infty}(1+\delta)\Psi_{2,n}(\gamma,\lambda)a_k z^k\right\}\\
&\geq 1-\frac{(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)}{(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)+(A-B)|b|}r\\
&-\frac{1+|B|}{(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)+(A-B)|b|}\sum_{k=2}^{\infty}(1+\delta(k-1))\Psi_{k,n}(\gamma,\lambda)|a_k|r^k\\
&>1-\frac{(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)}{(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)+(A-B)|b|}r\\
&-\frac{(A-B)|b|}{(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)+(A-B)|b|}r\\
&=1-r>0,
\end{aligned}$$

where we have also made use of the assertion (10) of Theorem 2.1. Then (15) holds true in $\mathbb{U}$. This proves the inequality (11). The inequality (12) follows from (11) by taking the convex function

$$g(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k.$$

To prove the sharpness of the constant

$$\frac{(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)}{2[(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)+(A-B)|b|]},$$

we consider the function $f_0\in\mathcal{G}_{\gamma,\lambda}^{n*}(\delta,b,A,B)$ given by

$$f_0(z)=z-\frac{(A-B)|b|}{(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)}z^2. \tag{16}$$

Thus from (11), we have

$$\frac{(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)}{2[(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)+(A-B)|b|]}f_0(z) \prec \frac{z}{1-z} \quad (z\in\mathbb{U}). \tag{17}$$

It can easily be verified for the function f_0 given by (16) that

$$\min_{|z|\leq r}\left\{\Re\left(\frac{(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)}{2[(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)+(A-B)|b|]}f_0(z)\right)\right\} = -\frac{1}{2}. \tag{18}$$

This shows that the constant

$$\frac{(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)}{2[(1+|B|)(1+\delta)\Psi_{2,n}(\gamma,\lambda)+(A-B)|b|]}$$

is the best possible, which completes the proof of Theorem 2.2. □

For the choices $\gamma = 0$ and $\lambda = 1$ in Theorem 2.2, we get the following corollary.

Corollary 2.3 ([13, Theorem 2.2]). *Let the function f defined by* (1) *be in the class $\mathcal{G}_n^*(\delta,b,A,B)$ and suppose that $g\in\mathcal{K}$. Then*

$$\frac{(1+|B|)(1+\delta)2^n}{2[(1+|B|)(1+\delta)2^n+(A-B)|b|]}(f*g)(z) \prec g(z) \quad (z\in\mathbb{U}) \tag{19}$$

and

$$\Re\{f(z)\} > -\frac{(1+|B|)(1+\delta)2^n+(A-B)|b|}{(1+|B|)(1+\delta)2^n} \quad (z\in\mathbb{U}).$$

The constant factor

$$\frac{(1+|B|)(1+\delta)2^n}{2[(1+|B|)(1+\delta)2^n+(A-B)|b|]}$$

in the subordination result (19) *cannot be replaced by a larger one.*

For the choices of $\gamma = 0$, $\lambda = 1$ and $A = 1$, $B = -1$ in Theorem 2.2, we get the following corollary.

Corollary 2.4 ([3, Theorem 1]). *Let the function f defined by* (1) *be in the class $\mathcal{G}_n^*(\delta,b)$ and suppose that $g\in\mathcal{K}$. Then*

$$\frac{(1+\delta)2^n}{2[(1+\delta)2^n+|b|]}(f*g)(z) \prec g(z), \quad (z\in\mathbb{U}) \tag{20}$$

and

$$\Re\{f(z)\} > -\frac{(1+\delta)2^n+|b|}{(1+\delta)2^n}, \quad (z\in\mathbb{U}).$$

The constant factor

$$\frac{(1+\delta)\,2^n}{2\,[(1+\delta)\,2^n+|b|]}$$

in the subordination result (20) *cannot be replaced by a larger one.*

For the choices of $n=0$, $\gamma=0$, $\lambda=1$ and $A=1$, $B=-1$ in Theorem 2.2, we get the following corollary.

Corollary 2.5. *Let the function f defined by* (1) *be in the class $\mathcal{G}^*(\delta,b)$ and suppose that $g\in\mathcal{K}$. Then*

$$\frac{1+\delta}{2\,(1+\delta+|b|)}\,(f*g)\,(z)\prec g(z), \quad (z\in\mathbb{U}) \tag{21}$$

and

$$\Re\{f(z)\}>-\frac{1+\delta+|b|}{1+\delta}, \quad (z\in\mathbb{U}).$$

The constant factor

$$\frac{1+\delta}{2\,(1+\delta+|b|)}$$

in the subordination result (21) *cannot be replaced by a larger one.*

For the choices of $\delta=0$, $n=0$, $\gamma=0$, $\lambda=1$ and $A=1$, $B=-1$ in Theorem 2.2, we get the following corollary.

Corollary 2.6. *Let the function f defined by* (1) *be in the class $\mathcal{G}^*(b)$ and suppose that $g\in\mathcal{K}$. Then*

$$\frac{1}{2\,(1+|b|)}\,(f*g)\,(z)\prec g(z), \quad (z\in\mathbb{U}) \tag{22}$$

and

$$\Re\{f(z)\}>-\,(1+|b|)\,, \quad (z\in\mathbb{U}).$$

The constant factor

$$\frac{1}{2\,(1+|b|)}$$

in the subordination result (22) *cannot be replaced by a larger one.*

For the choices of $b=1-\alpha$ $(0\le\alpha<1)$, $\delta=0$, $n=0$, $\gamma=0$, $\lambda=1$ and $A=1$, $B=-1$ in Theorem 2.2, we get the following corollary.

Corollary 2.7. *Let the function f defined by (1) be in the class $\mathcal{G}_{\alpha}^{*}$ and suppose that $g \in \mathcal{K}$. Then*

$$\frac{1}{2(2-\alpha)}(f * g)(z) \prec g(z), \quad (z \in \mathbb{U}) \tag{23}$$

and

$$\Re\{f(z)\} > -(2-\alpha), \quad (z \in \mathbb{U}).$$

The constant factor

$$\frac{1}{2(2-\alpha)}$$

in the subordination result (23) cannot be replaced by a larger one.

For the choices of $b = 1-\alpha$ $(0 \le \alpha < 1)$, $\delta = 1$, $n = 0$, $\gamma = 0$, $\lambda = 1$ and $A = 1$, $B = -1$ in Theorem 2.2, we get the following corollary.

Corollary 2.8. *Let the function f defined by (1) be in the class $\mathcal{R}_{\alpha}^{*}$ and suppose that $g \in \mathcal{K}$. Then*

$$\frac{1}{3-\alpha}(f * g)(z) \prec g(z), \quad (z \in \mathbb{U}) \tag{24}$$

and

$$\Re\{f(z)\} > -\frac{3-\alpha}{2}, \quad (z \in \mathbb{U}).$$

The constant factor

$$\frac{1}{3-\alpha}$$

in the subordination result (24) cannot be replaced by a larger one.

REFERENCES

[1] F. M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Int. J. Math. Math. Sci. 25-28 (2004), 1429–1436.

[2] F. M. Al-Oboudi - K. A. Al-Amoudi, *On classes of analytic functions related to conic domains*, J. Math. Anal. Appl. 339 (1) (2008), 655–667.

[3] M. K. Aouf, *Subordination properties for a certain class of analytic functions defined by the Salagean operator*, Appl. Math. Lett. 22 (10) (2009), 1581–1585.

[4] A. A. Attiya, *On some applications of a subordination theorem*, J. Math. Anal. Appl. 311 (2) (2005), 489–494.

[5] M. -P. Chen, *On functions satisfying* $\Re\left\{\frac{f(z)}{z}\right\} > \alpha$, Tamkang J. Math. 5 (1974), 231–234.

[6] M. -P. Chen, *On the regular functions satisfying* $\Re\left\{\frac{f(z)}{z}\right\} > \alpha$, Bull. Inst. Math. Acad. Sinica 3 (1) (1975), 65–70.

[7] T. G. Ezrohi, *Certain estimates in special classes of univalent functions regular in the circle* $|z| < 1$, Dopovidi Akademiji Nauk Ukrajins Koji RSR (1965), 984–988.

[8] S. A. Halim, *On a class of functions of complex order*, Tamkang J. Math. 30 (2) (1999), 147–153.

[9] S. S. Miller - P. T. Mocanu, *Differential subordinations: Theory and applications*, Monographs and Textbooks in Pure and Applied Mathematics 225, Marcel Dekker Inc., New York, 2000.

[10] S. Owa, *On the distortion theorems. I*, Kyungpook Math. J. 18 (1) (1978), 53–59.

[11] S. Owa - H. M. Srivastava, *Univalent and starlike generalized hypergeometric functions*, Canad. J. Math. 39 (5) (1987), 1057–1077.

[12] G. Ş. Sălăgean, *Subclasses of univalent functions*, Complex Analysis: Fifth Romanian-Finnish seminar, Part 1 (Bucharest, 1981), Lecture Notes in Math. 1013, Springer, Berlin, 1983, 362–372.

[13] S. Sivasubramanian, A. Mohammed - M. Darus, *Certain Subordination Properties for Subclasses of Analytic Functions Involving Complex Order*, Abstr. Appl. Anal. 2011, Article ID 375897, 8 pages.

[14] H. M. Srivastava - A. A. Attiya, *Some subordination results associated with certain subclasses of analytic functions*, J. Inequal. Pure Appl. Math. 5 (4) (2004), Article 82, 6 pp. (electronic).

[15] H. M. Srivastava - S. Owa, (Eds.), *Univalent Functions, Fractional Calculus, and Their Applications*, Ellis Horwood Series: Mathematics and Its Applications, Ellis Horwood, Chichester, UK; JohnWiley & Sons, New York, NY, USA, 1989.

[16] H. S. Wilf, *Subordinating factor sequences for convex maps of the unit circle*, Proc. Amer. Math. Soc. 12 (1961), 689–693.

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