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HERMITE-HADAMARD TYPE INEQUALITIES FOR GA-s-CONVEX FUNCTIONS

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In this paper, the author introduces the concepts of the GA-s-convex functions in the first sense and second sense and establishes some integral inequalities of Hermite-Hadamard type related to the GA-s-convex functions. Some applications to special means of real numbers are also given.

1. Introduction

In this section, we firstly list several definitions and some known results.

The following concept was introduced by Orlicz in [11]:

Definition 1.1. Let $0 < s \leq 1$. A function $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ where $\mathbb{R}_0 = [0,\infty)$, is said to be s-convex in the first sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in I$ and $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$. We denote this class of real functions by K_s^1.

In [4], Hudzik and Maligranda considered the following class of functions:

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Definition 1.2. A function $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ where $\mathbb{R}_0 = [0,\infty)$, is said to be s-convex in the second sense if

$$f(\alpha x+\beta y) \leq \alpha^s f(x)+\beta^s f(y)$$

for all $x,y \in I$ and $\alpha,\beta \geq 0$ with $\alpha+\beta = 1$ and s fixed in $(0,1]$. These authors denoted the set of such functions by K_s^2.

It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0,\infty)$.

In [2], Dragomir and Fitzpatrick proved a variant of Hermite-Hadamard inequality which holds for the s-convex functions.

Theorem 1.3. *Suppose that $f : \mathbb{R}_0 \to \mathbb{R}_0$ is an s-convex function in the second sense, where $s \in (0,1]$ and let $a,b \in [0,\infty)$, $a < b$. If $f \in L[a,b]$, then the following inequalities hold*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1}$$

the constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1).

The above inequalities are sharp. For recent results and generalizations concerning s-convex functions see [1, 2, 5, 6, 8].

Definition 1.4 ([9, 10])**.** A function $f : I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ is said to be a GA-convex function on I if

$$f(x^t y^{1-t}) \leq tf(x)+(1-t)f(y)$$

holds for all $x,y \in I$ and $t \in [0,1]$, where $x^t y^{1-t}$ and $tf(x)+(1-t)f(y)$ are respectively called the weighted geometric mean of two positive numbers x and y and the weighted arithmetic mean of $f(x)$ and $f(y)$.

For $b > a > 0$, let $G(a,b) = \sqrt{ab}$, $L(a,b) = (b-a)/(\ln b - \ln a)$, $I(a,b) = (1/e)\left(b^b/a^a\right)^{1/(b-a)}$, $A(a,b) = \frac{a+b}{2}$ and $L_p(a,b) = \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$, $p \in \mathbb{R}\backslash\{-1,0\}$, be the geometric, logarithmic, identric, arithmetic and p-logarithmic means of a and b, respectively. Then

$$\min\{a,b\} < G(a,b) < L(a,b) < I(a,b) < A(a,b) < \max\{a,b\}.$$

In [14], Zhang et al. established some Hermite-Hadamard type integral inequalities for GA-convex functions and applied these inequalities to construct several inequalities for special means and they used the following lemma to prove their results:

Lemma 1.5. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on I°, and $a, b \in I^\circ$, with $a < b$. If $f' \in L[a,b]$, then*

$$bf(b) - af(a) - \int_a^b f(x)dx = (\ln b - \ln a) \int_0^1 b^{2t} a^{2(1-t)} f'\left(b^t a^{1-t}\right) dt.$$

Also, the main inequalities in [14] are pointed out as follows:

Theorem 1.6. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on I°, and $a, b \in I$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is GA-convex on $[a,b]$ for $q \geq 1$, then*

$$\left| bf(b) - af(a) - \int_a^b f(x)dx \right| \leq \frac{[(b-a)A(a,b)]^{1-1/q}}{2^{1/q}}$$
$$\times \left\{ \left[L(a^2,b^2) - a^2\right] \left|f'(a)\right|^q + \left[b^2 - L(a^2,b^2)\right] \left|f'(b)\right|^q \right\}^{1/q}.$$

Theorem 1.7. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on I°, and $a, b \in I$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is GA-convex on $[a,b]$ for $q > 1$, then*

$$\left| bf(b) - af(a) - \int_a^b f(x)dx \right| \leq (\ln b - \ln a)$$
$$\times \left[L(a^{2q/(q-1)}, b^{2q/(q-1)}) - a^{2q/(q-1)}\right]^{1-1/q} \left[A\left(\left|f'(a)\right|^q, \left|f'(b)\right|^q\right)\right]^{1/q}.$$

Theorem 1.8. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on I°, and $a, b \in I$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is GA-convex on $[a,b]$ for $q > 1$ and $2q > p > 0$, then*

$$\left| bf(b) - af(a) - \int_a^b f(x)dx \right|$$
$$\leq \frac{(\ln b - \ln a)^{1-1/q}}{p^{1/q}} \times \left[L(a^{(2q-p)/(q-1)}, b^{(2q-p)/(q-1)})\right]^{1-1/q}$$
$$\times \left\{ [L(a^p,b^p) - a^p] \left|f'(a)\right|^q + [b^p - L(a^p,b^p)] \left|f'(b)\right|^q \right\}^{1/q}.$$

In [13], Zhang et al. established the following Hermite-Hadamard type inequality for GA-convex (concave) functions:

Theorem 1.9. *If $b > a > 0$ and $f : [a,b] \to \mathbb{R}$ is a differentiable GA-convex (concave) function then*

$$f(I(a,b)) \leq (\geq) \frac{1}{b-a} \int_a^b f(x)dx \leq (\geq) \frac{b - L(a,b)}{b-a} f(b) + \frac{L(a,b) - a}{b-a} f(a).$$

In [7], the author proved the following identity and established some new Hermite-Hadamard-like type inequalities for the geometrically convex functions.

Lemma 1.10. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on I°, and $a,b \in I$,with $a<b$. If $f' \in L[a,b]$, then*

$$f\left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx =$$

$$\frac{(\ln b - \ln a)}{4}\left[a\int_0^1 t\left(\frac{b}{a}\right)^{\frac{t}{2}} f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right)dt - b\int_0^1 t\left(\frac{a}{b}\right)^{\frac{t}{2}} f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right)dt\right]$$

and

$$\begin{aligned}&\frac{f(a)+f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx \\ &= \frac{(\ln b - \ln a)}{2}\left[a\int_0^1 t\left(\frac{b}{a}\right)^{t} f'\left(a^{1-t}b^t\right)dt - b\int_0^1 t\left(\frac{a}{b}\right)^{t} f'\left(b^{1-t}a^t\right)dt\right]\end{aligned}$$

In this paper, we will give concepts GA-s-convex functions in the first and second sense and establish some new integral inequalities of Hermite-Hadamard like type for these classes of functions by using Lemma 1.10.

2. Definitions of GA-*s*-convex functions in the first and second sense

Now it is time to introduce two concepts, GA-s-convex functions in the first and second sense.

Definition 2.1. Let $0 < s \leq 1$. A function $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ is said to be a GA-s-convex (concave) function in the first sense on I if

$$f(x^t y^{1-t}) \leq (\geq)\ t^s f(x) + (1-t^s) f(y)$$

holds for all $x,y \in I$ and $t \in [0,1]$.

Definition 2.2 ([12, Definition 2.1])**.** Let $0 < s \leq 1$. A function $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ is said to be a GA-s-convex (concave) function in the second sense on I if

$$f(x^t y^{1-t}) \leq (\geq)\ t^s f(x) + (1-t)^s f(y)$$

holds for all $x,y \in I$ and $t \in [0,1]$.

It is clear that when $s=1$, GA-s-convex functions in the first and second sense become GA-convex functions.

3. Inequalities for GA-s-convex functions in the first and second sense

Let $u,n>0,\ m,r\geq 0$ and $q\geq 1$. Throughout this section we will take

$$C(u,m,n,r,q)=\int_0^1 t^m(1-t^n)^r u^{qt}\,dt.$$

Now we are in a position to establish some inequalities of Hermite–Hadamard type for GA-s-convex functions in the first and second sense

Theorem 3.1. *Let $0<s\leq 1$. Suppose that $f:I\subseteq\mathbb{R}_+\to\mathbb{R}$ is GA-s-convex function in the first sense and $a,b\in I$ with $a<b$. If $f\in L[a,b]$, then one has the inequalities:*

$$f\left(\sqrt{ab}\right)\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\leq\frac{f(a)+sf(b)}{s+1}\tag{2}$$

Proof. As f is GA-s-convex function in the first sense, we have, for all $x,y\in I$

$$f(\sqrt{xy})\leq\frac{1}{2^s}f(x)+\left(1-\frac{1}{2^s}\right)f(y).\tag{3}$$

Now, let $x=a^{1-t}b^t$ and $y=a^tb^{1-t}$ with $t\in[0,1]$. Then we get by (3) that:

$$f\left(\sqrt{ab}\right)\leq\frac{1}{2^s}f(a^{1-t}b^t)+\left(1-\frac{1}{2^s}\right)f(a^tb^{1-t})$$

for all $t\in[0,1]$. Integrating this inequality on $[0,1]$, we deduce the first part of (2).

Secondly, we observe that for all $t\in[0,1]$

$$f(a^tb^{1-t})\leq t^sf(a)+(1-t^s)f(b).$$

Integrating this inequality on $[0,1]$, we get

$$\int_0^1 f(a^tb^{1-t})dt\leq\frac{f(a)+sf(b)}{s+1}.$$

As the change of variable $x=a^tb^{1-t}$ gives us that

$$\int_0^1 f(a^tb^{1-t})dt=\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx,$$

the second inequality in (2) is proved. □

Remark 3.2. The constant $k = 1/(s+1)$ for $s \in (0,1]$ is the best possible in the second inequality in (2). Indeed, as the mapping $f : [a,b] \to \mathbb{R}$ given $f(x) = s+1$, $0 < a < b$, is GA-s-convex in the first sense and

$$\frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx = s+1 = \frac{f(a) + sf(b)}{s+1}$$

Similarly to Theorem 3.1, we will give the following theorem for GA-s-convex function in the second sense:

Theorem 3.3. *Suppose that $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ is GA-s-convex function in the second sense and $a,b \in I$ with $a < b$. If $f \in L[a,b]$, then one has the inequalities:*

$$2^{s-1} f\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \leq \frac{f(a) + f(b)}{s+1} \tag{4}$$

Proof. As f is GA-s-convex function in the second sense, we have, for all $x, y \in I$

$$f\left(\sqrt{xy}\right) \leq \frac{f(x) + f(y)}{2^s}. \tag{5}$$

Now, let $x = a^{1-t} b^t$ and $y = a^t b^{1-t}$ with $t \in [0,1]$. Then we get by (5) that:

$$f\left(\sqrt{ab}\right) \leq \frac{f(a^{1-t} b^t) + f(a^t b^{1-t})}{2^s}$$

for all $t \in [0,1]$. Integrating this inequality on $[0,1]$, we deduce the first part of (4).

Secondly, we observe that for all $t \in [0,1]$

$$f(a^t b^{1-t}) \leq t^s f(a) + (1-t)^s f(b).$$

Integrating this inequality on $[0,1]$, we get

$$\int_0^1 f(a^t b^{1-t}) dt \leq \frac{f(a) + f(b)}{s+1}.$$

As the change of variable $x = a^t b^{1-t}$ gives us that

$$\int_0^1 f(a^t b^{1-t}) dt = \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx,$$

the second inequality in (4) is proved. □

Theorem 3.4. *Let $f: I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on I°, and $a,b \in I^\circ$ with $a<b$ and $f' \in L[a,b]$.*

a) If $|f'|^q$ is GA-s-convex function in the second sense on $[a,b]$ for $q \geq 1$ and $s \in (0,1]$, then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx\right| \leq \ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}}$$
$$\times\left[a\left\{C\left(\frac{b}{a},1,1,s,q\right)\left|f'(a)\right|^q + C\left(\frac{b}{a},s+1,1,0,q\right)\left|f'(b)\right|^q\right\}^{\frac{1}{q}}\right. \tag{6}$$
$$\left.+b\left\{C\left(\frac{a}{b},1,1,s,q\right)\left|f'(b)\right|^q + C\left(\frac{a}{b},s+1,1,0,q\right)\left|f'(a)\right|^q\right\}^{\frac{1}{q}}\right],$$

$$\left|f\left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx\right| \leq \ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{3-\frac{1}{q}}$$
$$\times\left[a\left\{C\left(\frac{b}{a},1,1,s,\frac{q}{2}\right)\left|f'(a)\right|^q + C\left(\frac{b}{a},s+1,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right. \tag{7}$$
$$\left.+b\left\{C\left(\frac{a}{b},1,1,s,\frac{q}{2}\right)\left|f'(b)\right|^q + C\left(\frac{a}{b},s+1,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right].$$

b) If $|f'|^q$ is GA-s-convex function in the first sense on $[a,b]$ for $q \geq 1$ and $s \in (0,1]$, then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx\right| \leq \ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}}$$
$$\times\left[a\left\{C\left(\frac{b}{a},1,s,1,q\right)\left|f'(a)\right|^q + C\left(\frac{b}{a},s+1,1,0,q\right)\left|f'(b)\right|^q\right\}^{\frac{1}{q}}\right. \tag{8}$$
$$\left.+b\left\{C\left(\frac{a}{b},1,s,1,q\right)\left|f'(b)\right|^q + C\left(\frac{a}{b},s+1,1,0,q\right)\left|f'(a)\right|^q\right\}^{\frac{1}{q}}\right],$$

$$\left|f\left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx\right| \leq \ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{3-\frac{1}{q}}$$
$$\times\left[a\left\{C\left(\frac{b}{a},1,s,1,\frac{q}{2}\right)\left|f'(a)\right|^q + C\left(\frac{b}{a},s+1,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right. \tag{9}$$
$$\left.+b\left\{C\left(\frac{a}{b},1,s,1,\frac{q}{2}\right)\left|f'(b)\right|^q + C\left(\frac{a}{b},s+1,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right].$$

Proof. $a)$ (1) Since $|f'|^q$ is GA-s-convex function in the second sense on $[a,b]$, from lemma 1.10 and power mean inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right| \\
&\leq \frac{\ln\left(\frac{b}{a}\right)}{2}\left[a\int_0^1 t\left(\frac{b}{a}\right)^t\left|f'\left(a^{1-t}b^t\right)\right|dt+b\int_0^1 t\left(\frac{a}{b}\right)^t\left|f'\left(b^{1-t}a^t\right)\right|dt\right] \\
&\leq \frac{a\ln\left(\frac{b}{a}\right)}{2}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{b}{a}\right)^{qt}\left|f'\left(a^{1-t}b^t\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{b\ln\left(\frac{b}{a}\right)}{2}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{a}{b}\right)^{qt}\left|f'\left(b^{1-t}a^t\right)\right|^q dt\right)^{\frac{1}{q}} \qquad (10)\\
&\leq \frac{a\ln\left(\frac{b}{a}\right)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{b}{a}\right)^{qt}\left((1-t)^s\left|f'(a)\right|^q+t^s\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&+\frac{b\ln\left(\frac{b}{a}\right)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{a}{b}\right)^{qt}\left((1-t)^s\left|f'(b)\right|^q+t^s\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&\leq a\ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}}\left\{C\left(\frac{b}{a},1,1,s,q\right)\left|f'(a)\right|^q+C\left(\frac{b}{a},s+1,1,0,q\right)\left|f'(b)\right|^q\right\}^{\frac{1}{q}} \\
&+b\ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}}\left\{C\left(\frac{a}{b},1,1,s,q\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},s+1,1,0,q\right)\left|f'(a)\right|^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

(2) Since $|f'|^q$ is GA-s-convex function in the second sense on $[a,b]$, from lemma 3.4 and power mean inequality, we have

$$\begin{aligned}
&\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right| \\
&\leq \frac{\ln\frac{b}{a}}{4}\left[a\int_0^1 t\left(\frac{b}{a}\right)^{\frac{t}{2}}\left|f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right)\right|dt+b\int_0^1 t\left(\frac{a}{b}\right)^{\frac{t}{2}}\left|f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\right|dt\right] \\
&\leq \frac{a\ln\frac{b}{a}}{4}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{b}{a}\right)^{\frac{qt}{2}}\left|f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}}+
\end{aligned}$$

$$+\frac{b\ln\frac{b}{a}}{4}\left(\int_0^1 t\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{a}{b}\right)^{\frac{qt}{2}}\left|f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{a\ln\frac{b}{a}}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{b}{a}\right)^{\frac{qt}{2}}\left((1-t)^s\left|f'(a)\right|^q+t^s\left|f'\left(\sqrt{ab}\right)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$+\frac{b\ln\frac{b}{a}}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{a}{b}\right)^{\frac{qt}{2}}\left((1-t)^s\left|f'(b)\right|^q+t^s\left|f'\left(\sqrt{ab}\right)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$\leq \ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{3-\frac{1}{q}}\left[a\left\{C\left(\frac{b}{a},1,1,s,\frac{q}{2}\right)\left|f'(a)\right|^q\right.\right. \tag{11}$$

$$\left.+C\left(\frac{b}{a},s+1,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}$$

$$\left.+b\left\{C\left(\frac{a}{b},1,1,s,\frac{q}{2}\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},s+1,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right],$$

b) (1) Since $|f'|^q$ is GA-s-convex function in the first sense on $[a,b]$, from the inequality (10), we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right|$$

$$\leq \frac{a\ln\left(\frac{b}{a}\right)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{b}{a}\right)^{qt}\left((1-t^s)\left|f'(a)\right|^q+t^s\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$+\frac{b\ln\left(\frac{b}{a}\right)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\frac{a}{b}\right)^{qt}\left((1-t^s)\left|f'(b)\right|^q+t^s\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$\leq a\ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}}\left\{C\left(\frac{b}{a},1,s,1,q\right)\left|f'(a)\right|^q+\right.$$

$$\left.+C\left(\frac{b}{a},s+1,1,0,q\right)\left|f'(b)\right|^q\right\}^{\frac{1}{q}}$$

$$+b\ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}}\left\{C\left(\frac{a}{b},1,s,1,q\right)\left|f'(b)\right|^q\right.$$

$$\left.+C\left(\frac{a}{b},s+1,1,0,q\right)\left|f'(a)\right|^q\right\}^{\frac{1}{q}}.$$

(2) Since $|f'|^q$ is GA-s-convex function in the first sense on $[a,b]$, the inequality (9) is easily obtained by using the inequality (11). $\square$

If taking $s=1$ in Theorem 3.4, we can derive the following inequalities for GA-convex.

Corollary 3.5. *Let $f: I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on I°, and $a,b \in I^\circ$ with $a<b$ and $f' \in L[a,b]$. If $|f'|^q$ is GA-convex on $[a,b]$ for $q \geq 1$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right| \leq \ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}} \\
&\times\left[a\left\{C\left(\frac{b}{a},1,1,1,q\right)\left|f'(a)\right|^q+C\left(\frac{b}{a},2,1,0,q\right)\left|f'(b)\right|^q\right\}^{\frac{1}{q}}\right. \qquad (12)\\
&\left.+b\left\{C\left(\frac{a}{b},1,1,1,q\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},2,1,0,q\right)\left|f'(a)\right|^q\right\}^{\frac{1}{q}}\right],
\end{aligned}$$

$$\begin{aligned}
&\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right| \leq \ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{3-\frac{1}{q}} \\
&\times\left[a\left\{C\left(\frac{b}{a},1,1,1,\frac{q}{2}\right)\left|f'(a)\right|^q+C\left(\frac{b}{a},2,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right. \qquad (13)\\
&\left.+b\left\{C\left(\frac{a}{b},1,1,1,\frac{q}{2}\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},2,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right],
\end{aligned}$$

where

$$\begin{aligned}
C\left(\frac{b}{a},1,1,1,q\right) &= \frac{L(a,b)}{qa^q(b-a)}\left[2b^q-3L(a^q,b^q)\right],\\
C\left(\frac{b}{a},2,1,0,q\right) &= \frac{L(a,b)}{qa^q(b-a)}\left[2L(a^q,b^q)-b^q\right],\\
C\left(\frac{a}{b},1,1,1,q\right) &= \frac{L(a,b)}{qb^q(b-a)}\left[3L(a^q,b^q)-2a^q\right],\\
C\left(\frac{a}{b},2,1,0,q\right) &= \frac{L(a,b)}{qb^q(b-a)}\left[a^q-2L(a^q,b^q)\right].
\end{aligned}$$

If taking $q=1$ in Theorem 3.4, we can derive the following corollary.

Corollary 3.6. *Let $f: I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on I°, and $a,b \in I^\circ$ with $a<b$ and $f' \in L[a,b]$.*

a) If $|f'|$ is GA-s-convex function in the second sense on $[a,b]$, $s\in(0,1]$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right|$$
$$\leq\frac{\ln\left(\frac{b}{a}\right)}{2}\left[\left(aC\left(\frac{b}{a},1,1,s,1\right)+bC\left(\frac{a}{b},s+1,1,0,1\right)\right)\left|f'(a)\right|\right.$$
$$\left.+\left(bC\left(\frac{a}{b},1,1,s,1\right)+aC\left(\frac{b}{a},s+1,1,0,1\right)\right)\left|f'(b)\right|\right]$$

$$\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right|$$
$$\leq\frac{\ln\left(\frac{b}{a}\right)}{4}\left[aC\left(\frac{b}{a},1,1,s,\frac{1}{2}\right)\left|f'(a)\right|+bC\left(\frac{a}{b},1,1,s,\frac{1}{2}\right)\left|f'(b)\right|\right.$$
$$\left.+\left(aC\left(\frac{b}{a},s+1,1,0,\frac{1}{2}\right)+bC\left(\frac{a}{b},s+1,1,0,\frac{1}{2}\right)\right)\left|f'\left(\sqrt{ab}\right)\right|\right].$$

b) If $|f'|^q$ is GA-s-convex function in the first sense on $[a,b]$, $s\in(0,1]$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right|$$
$$\leq\frac{\ln\left(\frac{b}{a}\right)}{2}\left[\left(aC\left(\frac{b}{a},1,s,1,1\right)+bC\left(\frac{a}{b},s+1,1,0,1\right)\right)\left|f'(a)\right|\right.$$
$$\left.+\left(aC\left(\frac{b}{a},s+1,1,0,1\right)+bC\left(\frac{a}{b},1,s,1,1\right)\right)\left|f'(b)\right|\right],$$

$$\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right|$$
$$\leq\frac{\ln\left(\frac{b}{a}\right)}{4}\left[aC\left(\frac{b}{a},1,s,1,\frac{1}{2}\right)\left|f'(a)\right|+bC\left(\frac{a}{b},1,s,1,\frac{1}{2}\right)\left|f'(b)\right|\right.$$
$$\left.+\left(aC\left(\frac{b}{a},s+1,1,0,\frac{1}{2}\right)+bC\left(\frac{a}{b},s+1,1,0,\frac{1}{2}\right)\right)\left|f'\left(\sqrt{ab}\right)\right|\right].$$

Theorem 3.7. *Let $f:I\subseteq\mathbb{R}_+\to\mathbb{R}$ be differentiable on I°, and $a,b\in I^\circ$ with $a<b$ and $f'\in L[a,b]$.*

a) If $|f'|^q$ *is GA-s-convex function in the second sense on* $[a,b]$ *for* $q>1$ *and* $s\in(0,1]$, *then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right|\leq\frac{\ln\left(\frac{b}{a}\right)}{2}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}$$
$$\times\left[a\left\{C\left(\frac{b}{a},0,1,s,q\right)\left|f'(a)\right|^q+C\left(\frac{b}{a},s,1,0,q\right)\left|f'(b)\right|^q\right\}^{\frac{1}{q}}\right.\tag{14}$$
$$\left.+b\left\{C\left(\frac{a}{b},0,1,s,q\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},s,1,0,q\right)\left|f'(a)\right|^q\right\}^{\frac{1}{q}}\right],$$

$$\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right|\leq\frac{\ln\left(\frac{b}{a}\right)}{4}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}$$
$$\times\left[a\left\{C\left(\frac{b}{a},0,1,s,\frac{q}{2}\right)\left|f'(a)\right|^q+C\left(\frac{b}{a},s,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right.\tag{15}$$
$$\left.+b\left\{C\left(\frac{a}{b},0,1,s,\frac{q}{2}\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},s,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right].$$

b) If $|f'|^q$ *is GA-s-convex function in the first sense on* $[a,b]$ *for* $q>1$ *and* $s\in(0,1]$, *then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right|\leq\frac{\ln\left(\frac{b}{a}\right)}{2}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}$$
$$\times\left[a\left\{C\left(\frac{b}{a},0,s,1,q\right)\left|f'(a)\right|^q+C\left(\frac{b}{a},s,1,0,q\right))\left|f'(b)\right|^q\right\}^{\frac{1}{q}}\right.\tag{16}$$
$$\left.+b\left\{C\left(\frac{a}{b},0,s,1,q\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},s,1,0,q\right)\left|f'(a)\right|^q\right\}^{\frac{1}{q}}\right]$$

$$\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right|\leq\frac{\ln\left(\frac{b}{a}\right)}{4}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}$$
$$\times\left[a\left\{C\left(\frac{b}{a},0,s,1,\frac{q}{2}\right)\left|f'(a)\right|^q+C\left(\frac{b}{a},s,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right.\tag{17}$$
$$\left.+b\left\{C\left(\frac{a}{b},0,s,1,\frac{q}{2}\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},s,1,0,q\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right].$$

Proof. a) (1) Since $|f'|^q$ is GA-s-convex function in the second sense on $[a,b]$, from lemma 1.10 and Hölder inequality, we have

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right| \\
&\leq \frac{\ln\left(\frac{b}{a}\right)}{2}\left[a\int_0^1 t\left(\frac{b}{a}\right)^t \left|f'\left(a^{1-t}b^t\right)\right|dt+b\int_0^1 t\left(\frac{a}{b}\right)^t \left|f'\left(b^{1-t}a^t\right)\right|dt\right] \\
&\leq \frac{a\ln\left(\frac{b}{a}\right)}{2}\left(\int_0^1 t^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{b}{a}\right)^{qt}\left|f'\left(a^{1-t}b^t\right)\right|^q dt\right)^{\frac{1}{q}} \qquad (18)\\
&+\frac{b}{2}\ln\left(\frac{b}{a}\right)\left(\int_0^1 t^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{a}{b}\right)^{qt}\left|f'\left(b^{1-t}a^t\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{a\ln\left(\frac{b}{a}\right)}{2}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{b}{a}\right)^{qt}\left((1-t)^s\left|f'(a)\right|^q+t^s\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&+\frac{b}{2}\ln\left(\frac{b}{a}\right)\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{a}{b}\right)^{qt}\left((1-t)^s\left|f'(b)\right|^q+t^s\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&\leq \frac{\ln\left(\frac{b}{a}\right)}{2}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left[a\left\{C\left(\frac{b}{a},0,1,s,q\right)\left|f'(a)\right|^q\right.\right. \\
&\left.+C\left(\frac{b}{a},s,1,0,q\right)\left|f'(b)\right|^q\right\}^{\frac{1}{q}} \\
&\left.+b\left\{C\left(\frac{a}{b},0,1,s,q\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},s,1,0,q\right)\left|f'(a)\right|^q\right\}^{\frac{1}{q}}\right].
\end{aligned}
$$

(2) Since $|f'|^q$ is GA-s-convex function in the first sense on $[a,b]$, from lemma 1.10 and Hölder inequality, we have

$$
\begin{aligned}
&\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right| \\
&\leq \frac{\ln\frac{b}{a}}{4}\left[a\int_0^1 t\left(\frac{b}{a}\right)^{\frac{t}{2}}\left|f'\left(a^{1-t}(ab)^{\frac{t}{2}}\right)\right|dt+b\int_0^1 t\left(\frac{a}{b}\right)^{\frac{t}{2}}\left|f'\left(b^{1-t}(ab)^{\frac{t}{2}}\right)\right|dt\right]
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{a\ln\frac{b}{a}}{4}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{b}{a}\right)^{\frac{qt}{2}}\left((1-t)^s\left|f'(a)\right|^q+t^s\left|f'\left(\sqrt{ab}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\\
&+\frac{b\ln\frac{b}{a}}{4}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{a}{b}\right)^{\frac{qt}{2}}\left((1-t)^s\left|f'(b)\right|^q+t^s\left|f'\left(\sqrt{ab}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\\
&\le \frac{\ln\left(\frac{b}{a}\right)}{4}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} \qquad (19)\\
&\times\left[a\left\{C\left(\frac{b}{a},0,1,s,\frac{q}{2}\right))\left|f'(a)\right|^q+C\left(\frac{b}{a},s,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right.\\
&\left.+b\left\{C\left(\frac{a}{b},0,1,s,\frac{q}{2}\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},s,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right].
\end{aligned}
$$

b) (1) Since $|f'|^q$ is GA-s-convex function in the first sense on $[a,b]$, from the inequality (18), we have

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right|\\
&\le \frac{a\ln\left(\frac{b}{a}\right)}{2}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{b}{a}\right)^{qt}\left((1-t^s)\left|f'(a)\right|^q+t^s\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\\
&+\frac{b\ln\left(\frac{b}{a}\right)}{2}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{a}{b}\right)^{qt}\left((1-t^s)\left|f'(b)\right|^q+t^s\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}}\\
&\le \frac{\ln\left(\frac{b}{a}\right)}{2}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left[a\left\{C\left(\frac{b}{a},0,s,1,q\right)\left|f'(a)\right|^q\right.\right.\\
&\left.+C\left(\frac{b}{a},s,1,0,q\right)\left|f'(b)\right|^q\right\}^{\frac{1}{q}}\\
&\left.+b\left\{C\left(\frac{a}{b},0,s,1,q\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},s,1,0,q\right)\left|f'(a)\right|^q\right\}^{\frac{1}{q}}\right]
\end{aligned}
$$

(2) Since $|f'|^q$ is GA-s-convex function in the first sense on $[a,b]$, the inequality (9) is easily obtained by using the inequality (19). □

If taking $s=1$ in Theorem 3.7, we can derive the following inequalities for GA-convex.

Corollary 3.8. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on I°, and $a,b \in I^\circ$ with $a<b$ and $f' \in L[a,b]$. If $|f'|^q$ is GA-convex function in the second sense on $[a,b]$ for $q>1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right| \leq \frac{\ln\left(\frac{b}{a}\right)}{2}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}$$
$$\times\left[a\left\{C\left(\frac{b}{a},0,1,1,q\right)\left|f'(a)\right|^q+C\left(\frac{b}{a},1,1,0,q\right)\left|f'(b)\right|^q\right\}^{\frac{1}{q}}\right. \tag{20}$$
$$\left.+b\left\{C\left(\frac{a}{b},0,1,1,q\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},1,1,0,q\right)\left|f'(a)\right|^q\right\}^{\frac{1}{q}}\right],$$

$$\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right| \leq \frac{\ln\left(\frac{b}{a}\right)}{4}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}$$
$$\times\left[a\left\{C\left(\frac{b}{a},0,1,1,\frac{q}{2}\right)\left|f'(a)\right|^q+C\left(\frac{b}{a},1,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right. \tag{21}$$
$$\left.+b\left\{C\left(\frac{a}{b},0,1,1,\frac{q}{2}\right)\left|f'(b)\right|^q+C\left(\frac{a}{b},1,1,0,\frac{q}{2}\right)\left|f'\left(\sqrt{ab}\right)\right|^q\right\}^{\frac{1}{q}}\right],$$

where

$$C\left(\frac{b}{a},0,1,1,q\right) = \frac{L(a,b)}{qa^q(b-a)}\left[L(a^q,b^q)-a^q\right],$$
$$C\left(\frac{b}{a},1,1,0,q\right) = \frac{L(a,b)}{qa^q(b-a)}\left[b^q-L(a^q,b^q)\right],$$

$$C\left(\frac{a}{b},0,1,1,q\right) = \frac{L(a,b)}{qb^q(b-a)}\left[b^q-L(a^q,b^q)\right],$$
$$C\left(\frac{a}{b},1,1,0,q\right) = \frac{L(a,b)}{qb^q(b-a)}\left[L(a^q,b^q)-a^q\right].$$

4. Application to Special Means

Proposition 4.1. *Let $0<a<b$, $n>0$, $q\geq 1$ and $nq\neq 1$. Then*

$$\left|A\left(a^{n+1},b^{n+1}\right)-L_n^n(a,b)L(a,b)\right| \leq \frac{n+1}{q^{1/q}}\left(\frac{b-a}{L(a,b)}\right)^{1-\frac{1}{q}}\left(\frac{1}{2}\right)^{2-\frac{1}{q}}$$
$$\times\left[\left\{a^{nq}\left(2b^q-3L(a^q,b^q)\right)+b^{nq}\left(2L(a^q,b^q)-b^q\right)\right\}^{1/q}\right.$$
$$\left.+\left\{b^{nq}\left(3L(a^q,b^q)-2a^q\right)+a^{nq}\left(a^q-2L(a^q,b^q)\right)\right\}^{1/q}\right],$$

$$\left|G\left(a^{n+1},b^{n+1}\right)-L_n^n(a,b)L(a,b)\right| \leq \frac{n+1}{q^{1/q}}\left(\frac{b-a}{L(a,b)}\right)^{1-\frac{1}{q}}\left(\frac{1}{2}\right)^{3-\frac{2}{q}}$$
$$\times\left[\sqrt{a}\left\{a^{nq}\left(2b^{q/2}-3L(a^{q/2},b^{q/2})\right)+G^{nq}(a,b)\left(2L(a^{q/2},b^{q/2})-b^{q/2}\right)\right\}^{1/q}\right.$$
$$\left.+\sqrt{b}\left\{b^{nq}\left(3L(a^{q/2},b^{q/2})-2a^{q/2}\right)+G^{nq}(a,b)\left(a^{q/2}-2L(a^{q/2},b^{q/2})\right)\right\}^{1/q}\right],$$

Proof. Let

$$f(x)=\frac{x^{n+1}}{n+1},\ x>0.$$

Then $|f'(x)|^q = x^{nq}$ is a GA-convex function on $\mathbb{R}_+$. The assertion follows from the inequalities (12) and (13) in Corollary 3.5 for the function f. □

Corollary 4.2. *Under conditions of Proposition 4.1, when $q=1$ and $n\neq 1$, we have*

$$\left|A\left(a^{n+1},b^{n+1}\right)-L_n^n(a,b)L(a,b)\right|$$
$$\leq \frac{n+1}{2}\left\{a^n\left(a+2b-5L(a,b)\right)+b^n\left(5L(a,b)-2a-b\right)\right\},$$

$$\left|G\left(a^{n+1},b^{n+1}\right)-L_n^n(a,b)L(a,b)\right|$$
$$\leq \frac{n+1}{2}\left\{a^n\left(2G(a,b)-3\sqrt{a}L(\sqrt{a},\sqrt{b})\right)-2G^n(a,b)\left(\sqrt{b}-\sqrt{a}\right)(\sqrt{a},\sqrt{b})\right.$$
$$\left.+b^n\left(3\sqrt{b}L(\sqrt{a},\sqrt{b})-2G(a,b)\right)\right\}$$

Proposition 4.3. *Let $0<a<b$,and $q\geq 1$. Then*

$$|A(a,b)-L(a,b)| \leq \left[\ln\left(\frac{b}{a}\right)\right]^{1-\frac{1}{q}}\left(\frac{1}{2}\right)^{2-\frac{1}{q}}\left(\frac{1}{q}\right)^{\frac{1}{q}}$$
$$\times\left[\left\{b^q-L(a^q,b^q)\right\}^{\frac{1}{q}}+\left\{L(a^q,b^q)-a^q\right\}^{\frac{1}{q}}\right]$$

$$|G(a,b)-L(a,b)| \leq \left[\ln\left(\frac{b}{a}\right)\right]^{1-\frac{1}{q}}\left(\frac{1}{2}\right)^{3-\frac{1}{q}}\left(\frac{2}{q}\right)^{\frac{1}{q}}$$
$$\times\left[\sqrt{a}\left\{b^{q/2}-L(a^{q/2},b^{q/2})\right\}^{\frac{1}{q}}+\sqrt{b}\left\{L(a^{q/2},b^{q/2})-a^{q/2}\right\}^{\frac{1}{q}}\right].$$

Proof. The assertion follows from the inequalities (12) and (13) in Corollary 3.5 for $f(x)=x,\ x>0$. □

Proposition 4.4. *Let* $0 < a < b$, $n > 0$, $q > 1$ *and* $nq \neq 1$. *Then*

$$\left|A\left(a^{n+1},b^{n+1}\right) - L_n^n(a,b)L(a,b)\right| \le \frac{n+1}{2q^{1/q}}\left(\frac{(q-1)(b-a)}{(2q-1)L(a,b)}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left\{a^{nq}\left(L(a^q,b^q)-a^q\right)+b^{nq}\left(b^q-L(a^q,b^q)\right)\right\}^{1/q}\right.$$
$$\left.+\left\{b^{nq}\left(b^q-L(a^q,b^q)\right)+a^{nq}\left(L(a^q,b^q)-a^q\right)\right\}^{1/q}\right],$$

$$\left|G\left(a^{n+1},b^{n+1}\right) - L_n^n(a,b)L(a,b)\right| \le \frac{(n+1)}{2^{2-1/q}q^{1/q}}\left(\frac{(q-1)(b-a)}{(2q-1)L(a,b)}\right)^{1-\frac{1}{q}}$$
$$\times\left[\sqrt{a}\left\{a^{nq}\left(L(a^{q/2},b^{q/2})-a^{q/2}\right)+G^{nq}(a,b)\left(b^{q/2}-L(a^{q/2},b^{q/2})\right)\right\}^{1/q}\right.$$
$$\left.+\sqrt{b}\left\{b^{nq}\left(b^{q/2}-L(a^{q/2},b^{q/2})\right)+G^{nq}(a,b)\left(L(a^{q/2},b^{q/2})-a^{q/2}\right)\right\}^{1/q}\right].$$

Proof. The assertion follows from the inequalities (20) and (21) in Corollary 3.8 for $f(x) = \frac{x^{n+1}}{n+1}$, $x > 0$. □

Proposition 4.5. *Let* $0 < a < b$ *and* $q > 1$. *Then*

$$|A(a,b) - L(a,b)| \le \ln\left(\frac{b}{a}\right)\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} L^{\frac{1}{q}}(a^q,b^q)$$

$$|G(a,b) - L(a,b)| \le \frac{\ln\left(\frac{b}{a}\right)}{2}\left(\frac{q-1}{(2q-1)}\right)^{1-\frac{1}{q}} L^{\frac{1}{q}}(a^{q/2},b^{q/2})A\left(\sqrt{a},\sqrt{b}\right).$$

Proof. The assertion follows from the inequalities (20) and (21) in Corollary 3.8 for $f(x) = x$, $x > 0$. □

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