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CLASSIFICATION OF GENERAL ABSOLUTE GEOMETRIES WITH LAMBERT-SACCHERI QUADRANGLES

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Without claiming any kind of continuity we show that an absolute geometry has either a singular, a hyperbolic or an elliptic congruence, i.e. for any quadruple (a, b, c, d) of distinct points contained in a plane with $\overline{d, a} \perp \overline{a, b} \perp \overline{b, c} \perp \overline{c, d}$ ((a, b, c, d) is then called *Lambert-Saccheri quadrangle*) if $a' := (a \perp \overline{c, d}) \cap \overline{c, d}$ then either $a' = d$ (*singular case*), or $a' \in]c, d[$ (*hyperbolic case*) or $d \in]c, a'[$ (*elliptic case*).

Introduction.

In their attempt, to deduce the parallel postulate from the other axioms of Euclid, G. Saccheri and H. J. Lambert considered quadrangles (a, b, c, d) with three orthogonal angles α, β, γ and for the forth angle δ they made the three hypotheses: $\delta = R$, $\delta < R$ and $\delta > R$ (where R stands for the right angle). They succeeded to show that Euclid's parallel postulate is equivalent with the hypothesis $\delta = R$, and that the hypothesis $\delta > R$ leads to a contradiction. In order to disprove the hypothesis $\delta < R$ they proved many theorems of hyperbolic geometry but failed to obtain the

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desired contradiction. Of course Euclid's axiomatic and the argumentation of Saccheri and Lambert base on many visual assumptions and as we know since more than 100 years, the mentioned theorems are only valid under the claim of continuity conditions like the Archimedian axiom. M. Dehn [1] gave in 1900 examples of geometries where the hypothesis $\delta = R$ is valid but not the parallel postulate and examples with $\delta > R$.

In this paper we consider absolute planes $(E, \mathcal{G}, \alpha, \equiv)$ in the sense of [4] p. 96. E denotes the set of points, $\mathcal{G}$ of lines, α stands for the order structure and $\equiv$ for the congruence. We claim that the axioms I1, I3, E1, A1, A2, V1, V2,.., V7 of [4] are valid. The order structure which is in [4] a function α from the set of triples $(\mathcal{G}, E, E)' := \{(G, a, b) \in \mathcal{G} \times E \times E \mid a, b \notin G\}$ in the cyclic group $(\{1, -1\}, \cdot)$ of order 2 $((G|a, b) := \alpha(G, a, b) = 1$ resp. $(G|a, b) = -1$ expresses that the points a, b are on the same resp. on different "sides" of the line $G)$ can also be replaced by a betweenness function (cf. e.g. [2] p. 389). If a, b, c are collinear points with $a \neq b, c$ then $(a|b, c) = -1$ or $= 1$ respectively, means that a is *between* b and c or c is a point on the *halfline* $\overrightarrow{a, b}$ respectively. By $]a, b[$ we denote the *open segment*, i.e. the set of all points between a and b and $[a, b] :=]a, b[\cup \{a, b\}$ is the *closed segment*. If $[a, b]$ and $[c, d]$ are two segments then by the axioms V2,V3 (*Streckenabtragen*) there is exactly one point $u \in \overrightarrow{a, b}$ such that $(a, u) \equiv (c, d)$. We set:

$|a, b| = |c, d|$ if $u = b$, $|a, b| < |c, d|$ if $b \in]a, u[$ hence $(b|a, u) = -1$ and $|a, b| > |c, d|$ if $u \in]a, b[$.

A quadruple (a, b, c, d) of distinct complanar but non collinear points will be called LS(*Lambert Saccheri*)-quadrangle if

$$\overline{d, a} \perp \overline{a, b} \perp \overline{b, c} \perp \overline{c, d}.$$

In this case we set
$(a, b, c, d)' := a' := (a \perp \overline{c, d}) \cap \overline{c, d}$.
The set LS of all LS-quadrangles (a, b, c, d) falls into the classes
$LS_r := \{(a, b, c, d) \in LS \mid (a, b, c, d)' = d\}$ of *rectangles*,
$LS_h := \{(a, b, c, d) \in LS \mid (a, b, c, d)' \in]c, d[\}$ of *hyperbolic* and
$LS_e := \{(a, b, c, d) \in LS \mid d \in]c, (a, b, c, d)'[\}$ of *elliptic* quadrangles which are characterized by $|a, b| = |c, d|$, $|a, b| < |c, d|$ and $|a, b| > |c, d|$ resp.

It is well known that $LS_r \neq \emptyset$ implies $LS = LS_r$. Then the absolute plane $(E, \mathcal{G}, \alpha, \equiv)$ is called *singular* (cf. e.g. [3] p. 162) and the congruence *Euclidean*. The Euclidean planes are a subclass of the singular planes.

In this paper we give a geometric proof that, for the *ordinary* case, characterized by $LS_r = \emptyset$, we have also the statements:

$$\text{“}LS_h \neq \emptyset \Rightarrow LS_h = LS\text{”}, \quad \text{“}LS_e \neq \emptyset \Rightarrow LS_e = LS\text{”}.$$

If $LS = LS_h$, or $LS = LS_e$, we say that the congruence "$\equiv$" is *hyperbolic* or *elliptic* respectively. The hyperbolic planes have hyperbolic congruence. The elliptic planes fall out of our axiomatic frame. But their congruence is elliptic.

Since in a spatial absolute geometry all planes are absolute planes and these planes are isomorphic, this classification is also here valid.

We have the main results:

Theorem 1. *Let $(P, \mathcal{G}, \alpha, \equiv)$ be a general absolute geometry (in the sense of [2]) then either $LS = LS_r$ or $LS = LS_h$ or $LS = LS_e$, i.e. the congruence "$\equiv$" is either Euclidean or hyperbolic or elliptic.*

If $\widetilde{P} := \{\tilde{p} \mid p \in P\}$ denotes the set of all point-reflections then:

(1) $LS = LS_r \Longleftrightarrow \widetilde{P}^3 = \widetilde{P}$.

(2) If $(P, \mathcal{G}, \alpha, \equiv)$ is an Euclidean geometry (i.e. the Parallel Axiom is valid) then $LS = LS_r$.

(3) If $(P, \mathcal{G}, \alpha, \equiv)$ is a hyperbolic geometry (i.e. $(P, \mathcal{G}, \alpha)$ is an ordered space with hyperbolic incidence structure in the sense of [2] Sec.2) then $LS = LS_h$.

(4) If $LS_r = \emptyset$ then: $\forall a, b, c \in P : \{a, b, c\}$ are collinear $\Longleftrightarrow \tilde{a} \circ \tilde{b} \circ \tilde{c}$ is an involution.

Theorem 2. *There are absolute geometries $(P, \mathcal{G}, \alpha, \equiv)$ such that:*

(1) $LS = LS_r$ and $(P, \mathcal{G}, \alpha, \equiv)$ is not an Euclidean geometry (cf.[1]).

(2) $LS = LS_h$ and $(P, \mathcal{G}, \alpha, \equiv)$ is not a hyperbolic geometry.

(3) $LS = LS_e$ (cf. [1]).

1. Notations and known Results.

In the absolute plane $(E, \mathcal{G}, \alpha, \equiv)$, given by the axioms I1, I3, A1, A2, V1, V2,...V7 of [4], let $a, b, c, \ldots \in E$ and $A, B, C, \ldots \in \mathcal{G}$ denote points and lines respectively. If $a \neq b$ let $\overline{a, b} \in \mathcal{G}$ be the uniquely determined line joining a and b and let $\tilde{a}(b) \in \overline{a, b} \setminus \{b\}$ be the unique point such that $(\tilde{a}(b), a) \equiv (a, b)$ and let $\tilde{a}(a) := a$. The map

$$\tilde{a} : E \to E; x \mapsto \tilde{a}(x)$$

is called *point reflection*. We set
$A \perp B :\Longleftrightarrow |A \cap B| = 1$ and if $c \in A \cap B$, $a \in A \setminus B$, $b \in B \setminus A$ then $(a, b) \equiv (a, \tilde{c}(b)) \equiv (b, \tilde{c}(a))$
and call then A and B *orthogonal*.
$(a \perp B) \in \mathcal{G}$ denotes the unique line with $a \in (a \perp B)$and $(a \perp B) \perp B$ (cf.[4] (16.9)).
If $\{a'\} := (a \perp B) \cap B$ is the *foot of a on B* let $\widetilde{B}(a) := \widetilde{a'}(a)$ then

$$\widetilde{B} : E \to E; x \mapsto \widetilde{B}(x)$$

is called *line reflection* (cf. [4] p. 105). Let $\widetilde{\mathcal{G}} := \{\widetilde{G} \mid G \in \mathcal{G}\}$ be the set of all line reflections.

(1.1) If $\mathcal{M}$ denotes the group of all motions of the absolute plane $(E, \mathcal{G}, \alpha, \equiv)$ then:
(1) $\mathcal{M} = \widetilde{\mathcal{G}} \circ \widetilde{\mathcal{G}} \dot{\cup} \widetilde{\mathcal{G}} \circ \widetilde{\mathcal{G}} \circ \widetilde{\mathcal{G}}$ (cf.[4](17.6) and (17.9)).
(2) $\mathcal{M}^+ := \widetilde{\mathcal{G}} \circ \widetilde{\mathcal{G}}$ is the subgroup of *proper* motions (cf.[4](17.18)).
(3) $\forall a, b, c, d$ with $a \neq b, c \neq d$ $\exists_1 \sigma \in \mathcal{M}^+ : \sigma(\overrightarrow{a, b}) = \overrightarrow{c, d}$ (cf. [4](17.15)).
(4) $\forall a, b, c, x, y, z$ with $b \neq a, c,$ $\overline{b, a} \perp \overline{b, c}$ and $y \neq x, z,$ $\overline{y, x} \perp \overline{y, z}$ $\exists_1 \sigma \in \mathcal{M} : \sigma(\overrightarrow{y, x}) = \overrightarrow{b, a}$ and $\sigma(\overrightarrow{y, z}) = \overrightarrow{b, c}$ (follows from [4](17.15) and Axiom V2).

(1.2) If $A, B, C \perp G$ and $\{a\} := A \cap G$ then:
(1) $\tilde{a} = \widetilde{A} \circ \widetilde{G}$.
(2) There is a line D with $\widetilde{A} \circ \widetilde{B} \circ \widetilde{C} = \widetilde{D}$ and $D \perp G$. (cf. [4](17.7) and (17.13.2)).

From (1.2) and [4](17.10) follows:

(1.3) If a, b, c are collinear then $Fix(\tilde{a} \circ \tilde{b} \circ \tilde{c}) \neq \emptyset$ and if $d \in Fix(\tilde{a} \circ \tilde{b} \circ \tilde{c})$ then $\tilde{a} \circ \tilde{b} \circ \tilde{c} = \tilde{d} = \tilde{c} \circ \tilde{b} \circ \tilde{a}$ and a, b, c, d are collinear.

By [4](18.3):

(1.4) $\widetilde{\tilde{a}(b)} = \tilde{a} \circ \tilde{b} \circ \tilde{a}$, $\widetilde{\tilde{a}(B)} = \tilde{a} \circ \widetilde{B} \circ \tilde{a}$, $\widetilde{\widetilde{A}(b)} = \widetilde{A} \circ \tilde{b} \circ \widetilde{A}$, $\widetilde{\widetilde{A}(B)} = \widetilde{A} \circ \widetilde{B} \circ \widetilde{A}$.

Let $a \neq b$, if there is a point m or a line M with $\widetilde{m}(a) = b$ or $\widetilde{M}(a) = b$ respectively, then m is called the *midpoint* or M the *midline* of a and b respectively. The following point sets determined by a and b with $a \neq b$

are called:
$]a,b[\;:=\; \{x \in \overline{a,b} \mid (x|a,b) = -1\}$ *open segment* and $[a,b] :=]a,b[\cup \{a,b\}$ *closed segment*,
$\overrightarrow{a,b} := \{x \in \overline{a,b} \mid (a|b,x) = 1\}$ and $\overset{-}{\overrightarrow{a,b}} := \{x \in \overline{a,b} \mid (a|b,x) = -1\}$ *halflines*.
A subset $S \subseteq E$ is called *convex* if for all $\{a,b\} \in \binom{S}{2}$ it holds $]a,b[\subseteq S$.

Let $\mathfrak{C}$ be the set of all convex subsets of E.

From [4]p.87-89 follows:

(1.5) If $a \neq b$ then:
(1) $\overline{a,b} = \overrightarrow{a,b} \,\dot{\cup}\, \{a\} \,\dot{\cup}\, \overset{-}{\overrightarrow{a,b}}$, $]a,b[= \overrightarrow{a,b} \cap \overrightarrow{b,a}$.
(2) $]a,b[$, $[a,b]$, $\overrightarrow{a,b} \in \mathfrak{C}$.
(3) If $\{a,b,c\} \in \binom{E}{3}$ are collinear then: $(b|a,c) = -1 \iff \overrightarrow{b,a} \cap \overrightarrow{b,c} = \emptyset$.

By [4](16.11), (16.12) and (18.3) resp. (16.10.2) holds:

(1.6) Two distinct points a, b have exactly one midpoint m and one midline M and $(m|a,b) = (M|a,b) = -1$, i.e. $m \in]a,b[$, $M = (m \perp \overline{a,b})$, $\widetilde{m} = \widetilde{\overline{a,b}} \circ \widetilde{M}$, $\widetilde{m} \circ \tilde{a} = \tilde{b} \circ \widetilde{m}$ or $\tilde{b} = \widetilde{m} \circ \tilde{a} \circ \widetilde{m}$.

(1.7) If $A, B \perp C$ and $A \neq B$ then $A \cap B = \emptyset$.

(1.8) Let $a, b \notin C$ and $a \neq b$ then:
(1) $]a,b[\cap C \neq \emptyset \iff (C|a,b)) = -1$ (cf. [4],p.83,V1).
(2) If there is a $D \in \mathcal{G}$ with $\overline{a,b}, C \perp D$ then $(C|a,b) = 1$ (by (1) and [4](16.10.2)).
(3) If $\overline{a,b} \cap C \neq \emptyset$ and $\{c\} := \overline{a,b} \cap C$ then $(C|a,b) = (c|a,b)$ ([4](13.9)).
(4) If a, b, c, or respectively a', b', c', are distinct and collinear and if there are A, B, C distinct with $a, a' \in A$, $b, b' \in B$, $c, c' \in C$ and $A \cap B = A \cap C = \emptyset$ (in particular if there is a D with $A, B, C \perp D$ - in this case we write
$(a,b,c) \perp_D (a',b',c')$ -) then $(a|b,c) = (a'|b',c')$ ([4](13.10)).

(1.9) Let a, b, c be three distinct and collinear points, let m_1, m_2, m_3 be the midpoints of a, b, of b, c, of a, c respectively, then:

(1) $\exists_1\ d \in \overline{a, b}$ such that $\tilde{c} \circ \tilde{a} \circ \tilde{b} = \tilde{d}$ (cf. (1.3)).
(2) If $\tilde{c} \circ \tilde{a} \circ \tilde{b} = \tilde{d}$ then $(b|a, c) = (c|d, b)$.
(3) $(b|a, c) = -1 \Rightarrow (m_3|m_1, m_2) = -1$.

Proof. "(1)" follows from [4](17.7) and (17.13.2).
"(2)" Since $\widetilde{m_2} \circ \tilde{c} = \tilde{b} \circ \widetilde{m_2}$ (cf.(1.4)),the points a, b, c, m_2 are collinear and we have: $\widetilde{\widetilde{m_2}(d)} = \widetilde{m_2} \circ \tilde{d} \circ \widetilde{m_2} = \widetilde{m_2} \circ \tilde{c} \circ \tilde{a} \circ \tilde{b} \circ \widetilde{m_2} = \tilde{b} \circ (\widetilde{m_2} \circ \tilde{a} \circ \tilde{b}) \circ \widetilde{m_2} = \tilde{b} \circ (\tilde{b} \circ \tilde{a} \circ \widetilde{m_2}) \circ \widetilde{m_2} = \tilde{a}$, i.e. $\widetilde{m_2}(d) = a$. Therefore by (1.1)
$(b|a, c) = (\widetilde{m_2}(b)|\widetilde{m_2}(a), \widetilde{m_2}(c)) = (c|d, b)$.
"(3)" By (1.6) and (1.5) $m_1 \in]a, b[\subseteq \overrightarrow{b, a}$,i.e. $(b|m_1, a) = 1$ and $m_2 \in]b, c[\subseteq \overrightarrow{b, c}$, i.e. $(b|m_2, c) = 1$. Now $(b|a, c) = -1$ implies $(b|m_1, m_2) = (b|m_1, a) \cdot (b|a, c) \cdot (b|c, m_2) = 1 \cdot (-1) \cdot 1 = -1$ thus $(m_1|b, m_2) = (m_2|b, m_1) = 1$. By $\widetilde{m_2} \circ \widetilde{m_3} \circ \widetilde{m_1}(b) = \widetilde{m_2} \circ \widetilde{m_3}(a) = \widetilde{m_2}(c) = b$ we have, according to (1.3), $\widetilde{m_1} \circ \widetilde{m_3} \circ \widetilde{m_2} = \widetilde{m_2} \circ \widetilde{m_3} \circ \widetilde{m_1} = \tilde{b}$ and so by (2), $(m_2|m_3, m_1) = (m_1|b, m_2) = 1$ and $(m_1|m_3, m_2) = (m_2|b, m_1) = 1$. But this implies by [4] Axiom A1, $(m_3|m_1, m_2) = -1$. □

2. Lambert-Saccheri Quadrangles.

A quadruple $(a, b, c, d) \in \binom{E}{4}$ is called *Lambert-Saccheri quadrangle* if $\overline{d, a} \perp \overline{a, b} \perp \overline{b, c} \perp \overline{c, d}$. Let LS be the set of all Lambert-Saccheri quadrangles and let $(a, b, c, d) \in LS$, then also $(c, b, a, d) \in LS$ and if $(a, b, c, d)' := a' := (a \perp \overline{c, d}) \cap \overline{c, d}$ then also $(a', c, b, a) \in LS$.
We have the three types
$LS_r := \{(a, b, c, d) \in LS \mid (a, b, c, d)' = d\}$,
$LS_h := \{(a, b, c, d) \in LS \mid (a, b, c, d)' \in]c, d[\}$ and
$LS_e := \{(a, b, c, d) \in LS \mid d \in]c, (a, b, c, d)'[\}$
of LS-quadrangles: the *rectangles*, the *hyperbolic* and the *elliptic* ones.

Since $LS_r \neq \emptyset$ implies $LS = LS_r$ - the absolute plane $(E, \mathcal{G}, \alpha, \equiv)$ is then called *singular* (cf. [4] (21.3) and [3] p.162) - we consider from now on only *ordinary* absolute planes characterized by $LS_r = \emptyset$.

(2.1) For $(a, b, c, d) \in LS$ holds:
$(a, b, c, d) \in LS_h \Longleftrightarrow |a, b| < |c, d|$.
$(a, b, c, d) \in LS_e \Longleftrightarrow |c, d| < |a, b|$.

Proof. Let $D := \overline{a, d}$, $C := \overline{c, d}$, $D' := (a \perp C)$, $\{a'\} := D' \cap C$, M the midline of b and c, $D_1 := \widetilde{M}(D)$ and $a_1 := \widetilde{M}(a)$. Then $\{a_1\} =$

$D_1 \cap C$ $(a_1, c) \equiv (a, b)$ thus $|a_1, c| = |a, b|$ and $D_1, D' \perp C$ and $(M|b, c) = -1$ (by (1.6)), $(M|a, b) = (M|c, d) = 1$ by (1.8.2) implying $(M|a, d) = (M|a, b) \cdot (M|b, c) \cdot (M|c, d) = 1 \cdot (-1) \cdot 1 = -1$.
Therefore the point $\{m\} := M \cap]a, d[$ exists and we have $m \in D, D_1$, $(m|a, d) = -1$ and $(a, m, d) \perp_C (a', a_1, d)$ hence by (1.8.4), $(a_1|a', d) = = (m|a, d) = -1$, i.e. $a_1 \in]a', d[$ and $(a'|a_1, d) = 1$.
Now if $(a, b, c, d) \in LS_h$ then $a' \in]c, d[$ hence $(a'|a_1, c) = (a'|c, d) \cdot (a'|a_1, d) = (-1) \cdot 1 = -1$ implying $(a_1|a', c) = 1$ and so $(a_1|c, d) = (a_1|a', d) \cdot (a_1|a', c) = (-1) \cdot 1 = -1$, i.e. $a_1 \in]c, d[$. Therefore together $|a, b| = |a_1, c| < |c, d|$.
If $(a, b, c, d) \in LS_e$ then $d \in]a', c[$ hence $(d|a', c) = -1$ and $(a_1|a', d) = -1$ implies $(d|a', a_1) = 1$. Consequently $(d|a_1, c) = (d|a', c) \cdot (d|a', a_1) = (-1) \cdot 1 = -1$, i.e. $d \in]a_1, c[$ and so $|c, d| < |a_1, c| = |a, b|$. □

(2.2) Let $(a, b, c_1, d_1), (a, b, c_2, d_2) \in LS$ with $d_1 \in]a, d_2[$.
If $(a, b, c_1, d_1) \in LS_h$ then $(a, b, c_2, d_2) \in LS_h$,
if $(a, b, c_1, d_1) \in LS_e$ then $(a, b, c_2, d_2) \in LS_e$.

Proof. Let $D := \overline{a, d_1}$, $A := \overline{a, b}$, $B := \overline{b, c_1}$ $(= \overline{b, c_2})$, $C_i := \overline{c_i, d_i}$, let M_1, M_2, M_3 be the midlines of b, c_1, of c_1, c_2, of b, c_2 resp.. Then $M_1, M_2, M_3, A, C_i \perp B$, $\widetilde{M_3} \circ \widetilde{M_2} \circ \widetilde{M_1}(b) = \widetilde{M_3} \circ \widetilde{M_2}(c_1) = \widetilde{M_3}(c_2) = b$, $b \in A$ and $D \perp A$ imply by (1.2):

(1) $\widetilde{M_3} \circ \widetilde{M_2} \circ \widetilde{M_1} = \widetilde{A}$, $\widetilde{M_3}(D) = \widetilde{M_2} \circ \widetilde{M_1}(D)$, $\widetilde{M_1}(A) = C_1$, $\widetilde{M_3}(A) = C_2$ and $\widetilde{M_2}(C_1) = C_2$ and $\{m_1'\} := M_1 \cap]b, c_1[= M_1 \cap B$, $\{m_2'\} := M_2 \cap]c_1, c_2[= M_2 \cap B$, $\{m_3'\} := M_3 \cap]b, c_2[= M_3 \cap B$ resp. are the midpoints of b, c_1 of c_1, c_2 of b, c_2 respectively.

By (1.6) $(M_1|b, c_1) = (M_2|c_1, c_2) = (M_3|b, c_2) = -1$ and by (1.8.2) $(M_1|b, a) = (M_1|c_1, d_1) = (M_2|c_1, d_1) = (M_2|c_2, d_2) = (M_3|c_2, d_2) = = (M_3|b, a) = 1$. Consequently $(M_1|a, d_1) = (M_1|a, b) \cdot (M_1|b, c_1) \cdot (M_1|c_1, d_1) = 1 \cdot (-1) \cdot 1 = -1$ and in the same way $(M_2|d_1, d_2) = (M_3|a, d_2) = -1$, i.e. the points $\{m_1\} := M_1 \cap]a, d_1[= M_1 \cap D$, $\{m_2\} := M_2 \cap D$ and $\{m_3\} := M_3 \cap D$ exist (cf.(1.8.1)) and we have $(m_1|a, d_1) = (m_2|d_1, d_2) = (m_3|d_2, a) = -1$.

From $d_1 \in]a, d_2[$ follows by (1.8.4) $c_1 \in]b, c_2[$, i.e. $(c_1|b, c_2) = -1$ and so by (1.9.3), $(m_3'|m_1', m_2') = -1$. Since $A, C_i, M_i \perp B$ we obtain again by (1.8.4):

(2) $(m_1|a, d_1) = (m_2|d_1, d_2) = (m_3|a, d_2) = (m_3|m_1, m_2) = -1$.

Now let $D_1 := \widetilde{M}_1(D)$ and $D_2 := \widetilde{M}_2(D_1) = \widetilde{M}_2 \circ \widetilde{M}_1(D)$, i.e. by (1), $\widetilde{M}_3(D) = D_2$. If we set $a_1 := \widetilde{M}_1(a)$, $a_2 := \widetilde{M}_2(a_1)$ and since $D = \overline{a, m_1} = \overline{a, m_3}$, $A \perp D$ and by (1) we have:

(3) $a_i \in C_i$, $(a, b) \equiv (a_1, c_1) \equiv (a_2, c_2)$, $D_1 = \overline{a_1, m_1}$, $D_2 = \overline{a_2, m_3}$, $C_1 \perp D_1$, $C_2 \perp D_2$ and $\widetilde{D_2} = \widetilde{M}_2 \circ \widetilde{D}_1 \circ \widetilde{M}_2 = \widetilde{M_2(D_1)}$.

From (3) and (2) follows:

(4) $(D_1|a, d_1) = (m_1|a, d_1) = -1$, $(D_2|m_1, m_2) = (m_3|m_1, m_2) = -1$.

Since $B, D_1 \perp C_1$ and $B, D_2 \perp C_2$ we have:

(5) $(D_1|c_1, m_2') = (D_2|c_2, m_2') = 1$.

From (2) follows $(d_1|a, m_1) = (d_1|m_2, d_2) = 1$ and since $d_1 \in]a, d_2[$, i.e. $(d_1|a, d_2) = -1$ we have $(d_1|m_1, m_2) = (d_1|m_1, a) \cdot (d_1|a, d_2) \cdot (d_1|d_2, m_2) = 1 \cdot (-1) \cdot 1 = -1$ and so $(m_1|d_1, m_2) = 1$. Together with (3) and (5) follows $(D_1|m_2, m_2') = (D_1|m_2, d_1) \cdot (D_1|d_1, c_1) \cdot (D_1|c_1, m_2') = (m_1|m_2, d_1) \cdot (a_1|d_1, c_1)$ hence:

(6) $(D_1|m_2, m_2') = (a_1|d_1, c_1)$.

Finally $\widetilde{m_3'} \circ \widetilde{m_1'} \circ \widetilde{m_2'}(c_2) = c_2$ implies $\widetilde{m_3'} \circ \widetilde{m_1'} \circ \widetilde{m_2'} = \widetilde{c_2}$ (cf. (1.3)) hence by (1.9.2), $(m_2'|m_1', m_3') = (m_3'|c_2, m_2')$, by (1.8.4), $(m_2|m_1, m_3) = (m_3|d_2, m_2)$ and so by (2) and Axiom A2 $(m_3|d_2, m_2) = (m_2|m_1, m_3) = 1$. Since by (3), $D_2 = \overline{a_2, m_3}$ and $d_2, a_2, c_2 \in C_2$ we have $(D_2|m_2, d_2) = (m_3|m_2, d_2) = 1$, $(D_2|d_2, c_2) = (a_2|d_2, c_2)$ and by (5), $(D_2|c_2, m_2') = 1$. Consequently $(D_2|m_2, m_2') = (D_2|m_2, d_2) \cdot (D_2|d_2, c_2) \cdot (D_2|c_2, m_2')$ implies:

(7) $(D_2|m_2, m_2') = (a_2|d_2, c_2)$.

If $D_1 \cap M_2 = \emptyset$ then by (3), $D_2 \cap M_2 = \emptyset$ and therefore $(D_1|m_2, m_2') = (D_2|m_2, m_2') = 1$, i.e. by (6) and (7), $(a_1|d_1, c_1) = (a_2|d_2, c_2)$. If

$\{t\} := D_1 \cap M_2 \neq \emptyset$ then by (3) $t \in D_2$ and so by [4] Axiom A1, $(D_1|m_2, m_2') = (D_2|m_2, m_2')$. Therefore in any case:

(8) $(a_1|d_1, c_1) = (a_2|d_2, c_2)$.

By (3) $|a, b| = |a_1, c_1| = |a_2, c_2|$ and since $a_i, c_i, d_i \in C_i$ we have: $(a, b, c_1, d_1) \in LS_h \iff |a, b| = |a_1, c_1| < |d_1, c_1| \iff a_1 \in]c_1, d_1[\iff -1 = (a_1|d_1, c_1) \overset{(8)}{=} (a_2|d_2, c_2) \iff a_2 \in]c_2, d_2[\iff |a, b| = |a_2, c_2| < |d_2, c_2| \iff (a, b, c_2, d_2) \in LS_h$ and
$(a, b, c_1, d_1) \in LS_e \iff |a, b| = |a_1, c_1| > |d_1, c_1| \iff d_1 \in]c_1, a_1[\iff -1 = (d_1|a_1, c_1) \overset{(8)}{=} (d_2|a_2, c_2) \iff d_2 \in]c_2, a_2[\iff |a, b| = |a_2, c_2| > |d_2, c_2| \iff (a, b, c_2, d_2) \in LS_e$. □

From (2.2) follows:

(2.3) An ordinary absolute plane has either a *hyperbolic congruence*, i.e. for all $(a, b, c, d) \in LS$ it holds $(a, b, c, d) \in LS_h$ or a *elliptic congruence*, i.e. for all $(a, b, c, d) \in LS$ it holds $(a, b, c, d) \in LS_e$.

Proof. Let $(a, b, c, d), (u, x, y, z) \in LS$. We set $(a, b, c, d) \sim (u, x, y, z)$ if both quadruples are in LS_h or both in LS_e contained. Clearly if τ is a motion then $(a, b, c, d) \sim (\tau(a), \tau(b), \tau(c), \tau(d))$.
By (1.1.4) there is exactly one motion $\sigma \in \mathcal{M}$ with $\sigma(x) = b$, $\sigma(u) \in \overrightarrow{b, a}$ and $\sigma(y) \in \overrightarrow{b, c}$. Therefore we may assume $x = b$, $y \in \overrightarrow{b, c}$ and $u \in \overrightarrow{b, a}$ and we have to consider the three cases:
1. $u = a$ resp. $c = y$ then by (2.2), $(a, b, y, z) \sim (a, b, c, d)$ resp. $(u, b, c, z) \sim (a, b, c, d)$.
2. $u \in]a, b[$. If $U := (u \perp \overline{b, a})$ then $(U|a, b) = (u|a, b) = -1$, $(U|a, d) = (U|b, c) = 1$ hence $(U|c, d) = (U|c, b) \cdot (U|b, a) \cdot (U|a, d) = 1 \cdot (-1) \cdot 1 = -1$ and so $\{s\} := U \cap]c, d[$ exists and $(u, b, c, s) \in LS$. By (2.2), $(u, b, y, z) \sim (u, b, c, s) \sim (a, b, c, d)$.
3. $a \in]u, b[$. Let $D := (a \perp \overline{a, b})$ then in the same way,
$(D|y, z) = (D|u, b) = (a|u, b) = -1$ hence $\{t\} := D \cap]y, z[$ exists and by (2.2) and case 1., $(u, b, y, z) \sim (a, b, y, t) \sim (a, b, c, d)$. □

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