MINIMAX SOLUTIONS FOR A PROBLEM WITH SIGN CHANGING NONLINEARITY AND LACK OF STRICT CONVEXITY

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A result of existence of a nonnegative and a nontrivial solution is proved via critical point theorems for non smooth functionals. The equation considered presents a convex part and a nonlinearity which changes sign.

1. Introduction and main results

Let us consider the problem

\[
\begin{cases}
-\text{div}(\Psi'(\nabla u)) = \lambda u + b(x)|u|^{p-2}u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(\(\mathcal{P}\))

where \(\lambda\) is a real parameter, \(\Omega\) is a bounded open subset of \(\mathbb{R}^N, N \geq 2\), \(b(x) \in \overline{C}(\Omega)\) changes sign in \(\Omega\). Finally \(2 < p < 2^* = \frac{2N}{N-2}\), and we will assume that \(\Psi : \mathbb{R}^N \to \mathbb{R}\) is a convex function of class \(C^1\) satisfying the following conditions:

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\( (\Psi_1) \quad \lim_{\xi \to 0} \frac{\Psi(\xi)}{|\xi|^2} = \frac{1}{2}; \)

\( (\Psi_2) \quad \exists \mu > 0 : \mu |\xi|^2 \leq \Psi(\xi) \leq \frac{1}{\mu} |\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^N; \)

\( (\Psi_3) \quad \lim_{|\xi| \to \infty} \frac{\Psi'(\xi) \cdot \xi - 2\Psi(\xi)}{|\xi|^2} = 0; \)

Moreover the function \( b(x) \) has to be strictly positive in a non zero measure set, and the zero set must be ”thin”, in other words \( b(x) \) must satisfy the following conditions:

\( (b_1) \quad \Omega^+ := \{x \in \Omega : b(x) > 0\} \) is a nonempty open set,

\( (b_2) \quad \Omega^0 := \{x \in \Omega : b(x) = 0\} \) has zero measure.

Conditions \( (b_1) \) and \( (b_2) \) imply that \( b^+(x) = b(x) + b^-(x) \neq 0 \) and that, since \( b \) is continuous, the set \( \Omega^0 \) is closed in \( \Omega \).

Let us also denote by \((\lambda_k)\) the eigenvalues of \(-\Delta\) with homogeneous Dirichlet boundary condition.

In the model case \( \Psi(\xi) = \frac{1}{2} |\xi|^2 \), there is a wide literature on problem \((P)\).

To cite only some of the existing results, in [2] the authors found positive solutions to \((P)\) in case that \( \lambda_1 < \lambda < \Lambda^* \), with \( \Lambda^* \) suitably near to \( \lambda_1 \). In the following many other papers ([1], [2], [3], [5], [6]) were devoted to prove existence of (possibly infinitely many) solutions for \( \lambda \in [\lambda_1, \Lambda^*] \) or also for every \( \lambda \), in case the nonlinearity satisfies some oddness assumption. A result concerning all \( \lambda \) different from the eigenvalues of the Laplacian under some quite general assumptions can be found in [11], while in [8] the authors proved a result of existence of a nontrivial solution (possibly changing sign) for every \( \lambda \).

On the other hand, only a small literature is available when dealing with equations with a non strictly convex principal part. In this framework, in [7] the author applies non smooth variational methods in presence of subcritical, positive, nonlinearities; while using similar techniques a nonlinearity with critical growth was considered in [9].

The aim of this paper is to extend to the setting of non strictly convex functionals some of the results contained in [2] (existence of a positive solution for \( \lambda < \lambda_1 \)) and [8] (existence of a nontrivial solution for any \( \lambda \).)

Problem \((P)\) can be treated by variational techniques. Indeed, weak solutions \( u \) of \((P)\) can be found as critical points of the \( C^1 \) functional \( J : H^1_0(\Omega) \to \mathbb{R} \) defined as

\[
J(u) = \int_\Omega \Psi(\nabla u) \, dx - \frac{\lambda}{2} \int_\Omega u^2 \, dx - \frac{1}{p} \int_\Omega b(x) |u|^p \, dx. \tag{1}
\]
The key point here is that, although $\Psi$ shares some properties with this typical case, there is no assumption of strict convexity with respect to $\xi$.

For instance, one could consider

$$\Psi(\xi) = \psi(\xi_1) + \frac{1}{2} \sum_{j=2}^{N} \xi_j^2,$$

(2)

where

$$\psi(t) = \begin{cases} 
\frac{1}{2} t^2 & \text{if } |t| < 1, \\
|t| - \frac{1}{2} & \text{if } 1 \leq |t| \leq 2, \\
\frac{1}{2} |t|^2 - |t| + \frac{3}{2} & \text{if } |t| > 2.
\end{cases}$$

If we look at the principal part of $J$ as the energy stored in the deformation $u$, this means that the material has a plastic behavior when $1 \leq |D_1 u| \leq 2$. We refer the reader to [13, Chapter 6] for a discussion of several models of plasticity.

As shown in [7, 9], it may happen that Palais Smale sequences, even if bounded in $H^1_0(\Omega)$-norm, do not admit any subsequence which converges strongly in this norm. And there is no way to prevent the interaction between the area where $\Psi$ loses strict convexity and the values of $\nabla u$. A possibile strategy is to look for compactness in a weaker norm ($L^2^*$).

Let us introduce the following notations: let $k \geq 1$ be such that $\lambda_k \leq \lambda < \lambda_{k+1}$ and let $e_1, \ldots, e_k$ be eigenfunctions of $-\Delta$ associated to $\lambda_1, \ldots, \lambda_k$, respectively. Finally, let $E_- = \text{span}\{e_1, \ldots, e_k\}$ and $E_+ = E_-^\perp$. The main results of this paper are the following:

**Theorem 1.1.** Let $N \geq 2$ and let $\Psi : \mathbb{R}^N \to \mathbb{R}$ be a convex function of class $C^1$ satisfying $(\Psi_1), (\Psi_2), (\Psi_3)$. Moreover let the function $b(x)$ verify $(b_1), (b_2)$. Then, for every $\lambda \in ]0, \lambda_1[,$ problem $(\mathcal{P})$ admits a nontrivial and nonnegative weak solution $u \in H^1_0(\Omega)$.

**Theorem 1.2.** Let $N \geq 2$ and let $\Psi : \mathbb{R}^N \to \mathbb{R}$ be a convex function of class $C^1$ satisfying $(\Psi_1), (\Psi_2), (\Psi_3)$ and let $\lambda \geq \lambda_1$. Moreover let the function $b(x)$ verify $(b_1), (b_2)$ and the following assumptions:

$$\int_{\Omega} b(x)|v|^p \geq 0 \quad \forall \ v \in E_-.$$  (3)

$$\exists e \in E_+ \setminus \{0\} : \int_{\Omega} b(x)|v|^p \, dx \geq C \int_{\Omega} |v|^p \, dx \quad \forall \ v \in E_- \oplus \text{span}\{e\}. \quad (4)$$

Then problem $(\mathcal{P})$ admits a nontrivial weak solution $u \in H^1_0(\Omega)$. 
Remark 1.3. Arguing as in section 2 of [9] we can deduce the following properties for $\Psi$, up to modifying the constant $\mu$:

\[
\begin{align*}
\Psi'(\xi) \cdot \xi &\geq \mu |\xi|^2 \quad \forall \xi \in \mathbb{R}^N, \\
|\Psi'(\xi)| &\geq \mu |\xi| \quad \forall \xi \in \mathbb{R}^N, \\
|\Psi'(\xi)| &\leq \frac{1}{\mu} |\xi| \quad \forall \xi \in \mathbb{R}^N.
\end{align*}
\]

Furthermore $(\Psi_3)$ yields that $\forall \sigma > 0$, $\exists M_\sigma \in \mathbb{R}$:

\[
\Psi'(\xi) \xi - 2\Psi(\xi) \leq \sigma |\xi|^2 + M_\sigma
\]

2. The variational framework

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary and let $\lambda \in \mathbb{R}$. Let us define the following functional $J : H^1_0(\Omega) \to \mathbb{R}$

\[
J(u) = \int_\Omega \Psi(\nabla u) \, dx - \frac{\lambda}{2} \int_\Omega u^2 \, dx - \frac{1}{p} \int_\Omega b(x)|u|^p \, dx.
\]

By $(\Psi_1)$, $(\Psi_2)$ the functional $J$ is of class $C^1$ on $H^1_0(\Omega)$. We wish to apply variational methods to functional $J$, but, as already mentioned, it is well known that the Palais Smale (PS) condition for a functional which is not strictly convex is not satisfied on $H^1_0(\Omega)$. So it is convenient to extend the functional $J$ to $L^{2^*}$ with value $+\infty$ outside $H^1_0(\Omega)$.

In other words we define the convex, lower semicontinuous functional (still denoted $J$)

\[
J : L^{2^*}(\Omega) \longrightarrow [-\infty, +\infty]
\]

\[
J(u) = \begin{cases} 
\int_\Omega \Psi(\nabla u) \, dx - \frac{\lambda}{2} \int_\Omega u^2 \, dx - \frac{1}{p} \int_\Omega b(x)|u|^p \, dx & \text{if } u \in H^1_0(\Omega), \\
+\infty & \text{if } u \in L^{2^*}(\Omega) \setminus H^1_0(\Omega)
\end{cases}
\]

This setting will allow us to recover PS condition.

This functional can be written as $J = J_0 + J_1$, where

\[
J_0 = \int_\Omega \Psi(\nabla u) \, dx,
\]

is proper, convex and l.s.c., while

\[
J_1 = -\frac{\lambda}{2} \int_\Omega u^2 \, dx - \frac{1}{p} \int_\Omega b(x)|u|^p \, dx,
\]
is of class $C^1$. We will use the following definitions ([12], [7]) of critical point and PS sequence for functionals of the type $J = J_0 + J_1$:

**Definition 2.1.** Let $X$ be a real Banach space, $u \in X$ is a critical point for $J$ if $J(u) \in \mathbb{R}$ and $-J'_1(u) \in \partial J_0$, where $\partial J_0$ is the subdifferential of $J_0$ at $u$.

**Definition 2.2.** Let $X$ be a real Banach space and let $c \in \mathbb{R}$. We say that $u_k$ is a Palais Smale sequence at level $c$ ((PS)$_c$ sequence for short) for $J$ if $J(u_k) \to 0$ and there exists $\alpha_k \in \partial J_0$ with $(\alpha_k + J'_1(u_k)) \to 0$ in $X^*$.

The following proposition (see [7]) assures that the critical points of the extendend functional already defined gives the solutions of our problem.

**Proposition 2.3.** Let $u \in L^2(\Omega, \mathbb{R}^N)$. Then $u$ is a critical point of $J$ if and only if $u \in H^1_0(\Omega)$ and $u$ is a weak solution of $(P)$.

**Proof.** Let $v \in L^2$. Then $v \in \partial J_0$, if and only if $u \in H^1_0(\Omega)$ and

$$-\text{div}(\Psi'(\nabla u)) = v$$

that is a reformulation of definition 2.1. \hfill \Box

Moreover we will apply the compactness result contained in [7], which we recall.

Let us define the functional $\mathcal{E} : W^{1,2}_0(\Omega, \mathbb{R}^N) \to \mathbb{R}$ as

$$\mathcal{E}(u) = \int_\Omega \Psi(\nabla u) \, dx$$

**Theorem 2.4.** Assume that $\Omega$ is bounded. If $\{u_n\}$ is weakly convergent to $u$ in $W^{1,2}_0(\Omega, \mathbb{R}^N)$ with $\mathcal{E}(\{u_n\}) \to \mathcal{E}(\{u\})$, then $u$ is strongly convergent to $u$ in $L^2(\Omega)$.

### 3. Proof of main results

Since $\Psi'(0) = 0$, of course 0 is a solution of $(P)$. Therefore we are interested in nontrivial solutions. In order to find nonnegative solutions of $(P)$, we consider the modified functional $\tilde{J} : L^2(\Omega) \to ]-\infty, +\infty]$ defined as

$$\tilde{J}(u) = \begin{cases} 
\int_\Omega \Psi(\nabla u) \, dx - \frac{\lambda}{2} \int_\Omega (u^+)^2 \, dx - \frac{1}{p} \int_\Omega b(x)(u^+)^p \, dx & \text{if } u \in H^1_0(\Omega), \\
+\infty & \text{if } u \in L^2(\Omega) \backslash H^1_0(\Omega)
\end{cases}$$
Proposition 3.1. Let $\Psi : \mathbb{R}^N \rightarrow \mathbb{R}$ be a convex function of class $C^1$ satisfying $(\Psi_2)$ with $\mu > 0$, and (6). Then each critical point $u \in L^{2^*}$ of $J$ is a nonnegative solution of $(P)$.

Proof. Since by Proposition 2.3 we already know that the critical points of $J$ are solutions of our problem, it is only left to prove that the modified functional will give nonnegative solutions. By $(\Psi_2)$ one has

$$\mu \int_{\Omega} |\nabla u^-|^2 \, dx \leq \int_{\Omega} \Psi'(\nabla u) \cdot (-\nabla u) \, dx$$

$$= \lambda \int_{\Omega} u^+ (-u^-) \, dx + \int_{\Omega} (u^+)^{p-1} (-u^-) \, dx = 0$$

whence the assertion. □

Remark 3.2. From now on, to simplify notations, we will keep on using the functional $J$ instead of $\tilde{J}$, since it is understood what has been proved in Proposition 3.1.

Proof of Theorem 1.1

We aim to apply to $J$ a nonsmooth version of Mountain Pass Theorem [12]. First of all, let us observe that, by $(\Psi_1)$, we have

$$\frac{\int_{\Omega} \Psi(\nabla u) \, dx}{\int_{\Omega} |\nabla u|^2 \, dx} \rightarrow \frac{1}{2} \quad \text{as } u \rightarrow 0 \text{ in } L^{2^*}.$$ 

Then, as in the case $\Psi(\xi) = \frac{1}{2}|\xi|^2$ treated in [2, 8], we deduce that there exist $\rho > 0$ and $\alpha > 0$ such that $J(u) \geq \alpha$ whenever $\|u\| = \rho$. On the other hand, there exists $e \in L^{2^*}$ with $e \geq 0$ a.e. in $\Omega$ such that

$$\lim_{t \rightarrow +\infty} J(te) = -\infty,$$ 

again, this is proved in [2] in the case $\Psi(\xi) = \frac{1}{2}|\xi|^2$, but by $(\Psi_2)$ the assertion is true also in our case.

By the Mountain Pass theorem, there exist a sequence $(u_k)$ in $L^{2^*}$ and a sequence $(w_k)$ in $L^{(2^*)'}(\Omega)$ strongly convergent to 0 such that (see definition 2.2)

$$\int_{\Omega} \Psi'(\nabla u_k)(\nabla v - \nabla u_k) \, dx \geq \lambda \int_{\Omega} u_k (v - u_k) \, dx + \int_{\Omega} b(x)|u_k|^{p-1} (v - u_k) \, dx$$

$$+ \int_{\Omega} w_k (v - u_k) \, dx \quad \forall \, v \in L^{(2^*)'} \quad (10)$$
Taking $v = 0$ and $v = 2u_k$ as tests in the previous inequality yield
\[
\int_{\Omega} \Psi'(\nabla u_k) \nabla u_k \, dx = \lambda \int_{\Omega} (u_k)^2 \, dx + \int_{\Omega} b(x)|u_k|^p \, dx + \int_{\Omega} w_k u_k \, dx \quad \forall \, v \in L^{(2^*)'}.
\]

(11)

Furthermore also the following relation holds:
\[
\lim_{k \to \infty} \left( \int_{\Omega} \Psi(\nabla u_k) \, dx - \frac{\lambda}{2} \int_{\Omega} (u_k)^2 \, dx - \frac{1}{p} \int_{\Omega} b(x)(u_k)^p \, dx \right) = c > \alpha.
\]

(12)

Let us write the expression $pJ(u_k) - J'(u_k)u_k$:
\[
\begin{align*}
p \int_{\Omega} \Psi(\nabla u_k) \, dx - \frac{p}{2} \lambda \int_{\Omega} (u_k)^2 \, dx & - \int_{\Omega} b(x)(u_k)^p \, dx - \int_{\Omega} \Psi'(\nabla u_k) \cdot \nabla u_k \, dx \\
& + \lambda \int_{\Omega} (u_k)^2 \, dx + \int_{\Omega} b(x)(u_k)^p \, dx \\
& = \int_{\Omega} (p - 2) \Psi(\nabla u_k) \, dx + \int_{\Omega} \left[ 2 \Psi(\nabla u_k) - \Psi'(\nabla u_k) \cdot \nabla u_k \right] \, dx \\
& - \lambda \left( \frac{p}{2} - 1 \right) \int_{\Omega} (u_k)^2 \, dx = (p - 2)c - \int_{\Omega} w_k u_k \, dx + C
\end{align*}
\]

(13)

By (8) and $(\Psi_2)$ one gets
\[
\mu(p - 2 - \sigma) \int_{\Omega} |\nabla u_k|^2 \, dx - \lambda \left( \frac{p}{2} - 1 \right) \lambda \int_{\Omega} (u_k)^2 \, dx \leq pc - \int_{\Omega} w_k u_k + C
\]

(14)

so
\[
\mu(p - 2 - \sigma) \int_{\Omega} |\nabla u_k|^2 \, dx \leq \lambda \left( \frac{p}{2} - 1 \right) \int_{\Omega} (u_k)^2 \, dx + C
\]

(15)

where the quantity $(p - 2 - \sigma)$ is strictly positive since $\sigma$ is arbitrarily small.

Our aim is to prove the boundedness of the $H^1_0$ norm of the Palais Smale sequences, so arguing by contradiction, let us assume that
\[
||u_k|| \to \infty \quad \text{as} \quad k \to +\infty.
\]

Dividing (12) by $||u_k||^p$ yields
\[
\liminf \left\{ \frac{p \int_{\Omega} \Psi(\nabla u_k)}{||u_k||^p} \, dx - \frac{\lambda p}{2} \frac{\int_{\Omega} (u_k)^2 \, dx}{||u_k||^p} - \frac{1}{p} \int_{\Omega} b(x) \left( \frac{u_k}{||u_k||} \right)^p \, dx \right\} = 0.
\]

Since $p > 2$ and $(\Psi_2)$ holds, the first two terms go to zero. So
\[
\limsup \left( \int_{\Omega} b(x) \left( \frac{u_k}{||u_k||} \right)^p \, dx \right) = 0.
\]

(16)
Since $b$ is bounded, (16) yields that
\[
\left( \frac{u_k}{||u_k||} \right) \rightarrow u_0
\]
strongly in $L^p$ and weakly in $H^1_0(\Omega)$. Arguing by contradiction let us suppose that $u_0 \equiv 0$. Dividing (15) by $||u_k||^2$ yields
\[
\mu (p - 2 - 2\sigma) \leq \lambda \left( \frac{p}{2} - 1 \right) \frac{1}{||u_k||^2} \int_\Omega (u_k)^2 \, dx + \frac{C}{||u_k||^2}
\]
the right hand side goes to zero, which leads to a contradiction since $p - 2 - 2\sigma > 0$ and $\mu > 0$, so $u_0$ must not be identically zero.

Now let $\phi \in C^\infty_0(\Omega^+)$ be a compact support function, $\phi \geq 0$ and $\phi \not\equiv 0$. Let us use the function $t\phi v$, $v \in H^1_0(\Omega)$ as a test in (10):
\[
\forall v \in H^1_0(\Omega) : \int_{\Omega^+} \Psi'(\nabla u_k)(t\phi \nabla v + tv\nabla \phi - \nabla u_k)
\geq \lambda \int_{\Omega^+} u_k(tv\phi - u_k) + \int_{\Omega^+} b(x)(u_k)^{p-1}(tv\phi - u_k) + \int_{\Omega^+} w_k(tv\phi - u_k).
\]

Then let us divide the previous inequality by $t$ and then let $t$ go to $+\infty$:
\[
\int_{\Omega^+} \Psi'(\nabla u_k)(\phi \nabla v) + \Psi'(\nabla u_k)v \nabla \phi
\geq \lambda \int_{\Omega^+} u_kv\phi + \int_{\Omega^+} b^+(x)(u_k)^{p-1}v\phi + \int_{\Omega} w_kv\phi \quad \forall v \in H^1_0(\Omega)
\]

On the other hand, if $t \rightarrow -\infty$, one gets the opposite inequality, so we can deduce that the equality holds in the last expression, that is
\[
\int_{\Omega^+} \Psi'(\nabla u_k)(\phi \nabla v) + \Psi'(\nabla u_k)v \nabla \phi
= \lambda \int_{\Omega^+} u_kv\phi + \int_{\Omega^+} b^+(x)(u_k)^{p-1}v\phi + \int_{\Omega} w_kv\phi \quad \forall v \in H^1_0(\Omega). \quad (19)
\]

Now let us choose $v = u_k$ and divide both hand sides of (19) by $||u_k||^p$. It is easily seen that the terms containing $\lambda$ and $w_k$ go to 0 as $k \rightarrow +\infty$. Then
\[
\int_{\Omega^+} \frac{\Psi'(\nabla u_k) \nabla u_k \phi}{||u_k||^p}
\]
go to 0 since $p > 2$ and (7) holds.

On the other hand, by (7), since $p > 2$ and $\phi$ is of class $C^\infty$ in $\Omega^+$ bounded,
\[
\frac{1}{||u_k||^p} \int_{\Omega^+} \Psi'(\nabla u_k) u_k \nabla \phi \leq C \frac{||u_k||}{||u_k||^{p-1}} \frac{||u_k||}{L^2}
\]

The term \(\frac{||u_k||^2}{||u_k||^p}\) is bounded, while \(\frac{||u_k||}{||u_k||^{p-1}}\) converges to 0.

By (19) We can conclude that
\[
\int_{\Omega^-} \frac{1}{||u_k||^p} b^+(x)(u_k)^p \phi \to 0 \text{ as } k \to \infty.
\]

Applying Fatou’s Lemma yields
\[
\liminf_{k \to \infty} \int_{\Omega^+} \frac{1}{||u_k||^p} b^+(x)(u_k)^p \phi \leq 0
\]

and since the integrand is nonnegative, this means that \(\frac{u_k^p}{||u_k||^p}\) must tend to 0, a.e. in \(\Omega^+\) as \(k \to \infty\). Arguing in the same way \(\frac{u_k^p}{||u_k||^p}\) → 0 a.e. as \(k \to \infty\), in \(\Omega^-\). This yields that \(\frac{u_k^p}{||u_k||^p}\) → 0 a.e. in \(\Omega\) since the remaining part is negligible. This is in contradiction with the fact that it converges to a nonzero function \(u_0\).

Then \(u_k\) must have bounded norm in \(H^1_0(\Omega)\) and admits a subsequence weakly converging in \(L^{2^*}\).

According to (10) and taking \(v = u\) as a test function yields
\[
\int_{\Omega} \Psi'(\nabla u_k)(\nabla u - \nabla u_k) \, dx \\
\geq \lambda \int_{\Omega} u_k(u - u_k) \, dx + \int_{\Omega} b(x)(u_k)^{p-1}(u - u_k) \, dx + o(1)
\]

so as \(k \to \infty\) the right hand-side terms go to zero, and we obtain
\[
\liminf_{k \to \infty} \int_{\Omega} \Psi'(\nabla u_k)(\nabla u - \nabla u_k) \, dx \geq 0. \quad (20)
\]

On the other hand, by convexity
\[
\int_{\Omega} \Psi(\nabla u) \, dx \geq \int_{\Omega} \Psi(\nabla u_k) \, dx + \int_{\Omega} \Psi'(\nabla u_k)(\nabla u - \nabla u_k) \, dx \quad (21)
\]

So by (20) and (21)
\[
\limsup_{k \to \infty} \int_{\Omega} \Psi(\nabla u_k) \, dx \leq \limsup_{k \to \infty} \left( \int_{\Omega} \Psi(\nabla u) \, dx - \int_{\Omega} \Psi'(\nabla u_k)(\nabla u - \nabla u_k) \, dx \right) \\
\leq \int_{\Omega} \Psi(\nabla u) \, dx - \liminf_{k \to \infty} \int_{\Omega} \Psi'(\nabla u_k)(\nabla u - \nabla u_k) \, dx \leq \int_{\Omega} \Psi(\nabla u) \, dx
\]
By lower semicontinuity and convexity
\[
\liminf \int_\Omega \Psi(\nabla u_k) \, dx \geq \int_\Omega \Psi(\nabla u) \, dx
\] (22)

We can conclude that
\[
\int_\Omega \Psi(\nabla u_k) \, dx \to \int_\Omega \Psi(\nabla u) \, dx.
\]

By Theorem 2.4 \( u_k \) admits a subsequence strongly converging in \( L^2^* \), which concludes the proof of PS condition and of Theorem 1.1.

**Proof of Theorem 1.2**

We are now concerned with the existence of (possibly sign-changing) nontrivial solutions \( u \) of (P). Let \( (\lambda_k) \) denote the sequence of the eigenvalues of \(-\Delta\) with homogeneous Dirichlet condition, repeated according to multiplicity.

Since the case \( 0 < \lambda < \lambda_1 \) is already contained in Theorem 1.1, we may assume that \( \lambda \geq \lambda_1 \). Let \( k \geq 1 \) be such that \( \lambda_k \leq \lambda < \lambda_{k+1} \), \( e_1, \ldots, e_k \) are eigenfunctions of \(-\Delta\), as defined in the introduction. Finally, let \( E_- = \text{span}\{e_1, \ldots, e_k\} \) and \( E_+ = E_-^\perp \).

Consider the functional \( J \) defined in (9). We aim to apply the version of the Linking Theorem for convex functional presented by Szulkin in [12]. Since
\[
\frac{\int_\Omega \Psi(\nabla u) \, dx}{\int_\Omega |\nabla u|^2 \, dx} \to \frac{1}{2} \quad \text{as} \ u \to 0 \ \text{in} \ H_0^1(\Omega),
\]
as in the case \( \Psi(\xi) = \frac{1}{2} |\xi|^2 \) treated in [8], we deduce that there exist \( \rho > 0 \) and \( \alpha > 0 \) such that \( J(u) \geq \alpha \) whenever \( u \in E_+ \) with \( ||u|| = \rho \). On the other hand, there exists \( e \in H_0^1(\Omega) \setminus E_- \) such that
\[
\lim_{||u|| \to \infty, u \in \text{Re} \oplus E_-} J(u) = -\infty
\]
Again, this is proved in [8] when \( \Psi(\xi) = \frac{1}{2} |\xi|^2 \), but by (\( \Psi_2 \)) the assertion is true also in our case. Finally, it is clear that \( J(u) \leq 0 \) for every \( u \in E_- \).

By the Linking type theorem in [12] (Theorem 3.4), there exist a PS sequence \( (u_k) \) in \( H_0^1(\Omega) \) and we can continue, up to minor changes, as in the proof of Theorem 1.1 to prove that there exists a subsequence of \( (u_k) \) strongly converging in \( L^2^* \). This concludes the proof of Theorem 1.2, since the non triviality of the solution comes directly from the characterization of the critical level of the solution.
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