LE MATEMATICHE
Vol. LXIX (2014) – Fasc. II, pp. 159–170
doi: 10.4418/2014.69.2.14

MINIMAX SOLUTIONS FOR A PROBLEM WITH SIGN CHANGING NONLINEARITY AND LACK OF STRICT CONVEXITY

PAOLA MAGRONE

A result of existence of a nonnegative and a nontrivial solution is proved via critical point theorems for non smooth functionals. The equation considered presents a convex part and a nonlinearity which changes sign.

1. Introduction and main results

Let us consider the problem

$$\begin{cases} -\mathrm{div}(\Psi'(\nabla u)) = \lambda u + b(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\mathcal{P})$$

where λ is a real parameter, Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $b(x) \in \overline{C}(\Omega)$ changes sign in Ω. Finally $2 < p < 2^* = \frac{2N}{N-2}$, and we will assume that $\Psi : \mathbb{R}^N \to \mathbb{R}$ is a convex function of class C^1 satisfying the following conditions:

Entrato in redazione: 25 settembre 2013

AMS 2010 Subject Classification: 35J65, 58E05.
Keywords: Non strict convexity, sign changing, linking theorem.

(Ψ_1) $\lim_{\xi\to 0}\frac{\Psi(\xi)}{|\xi|^2}=\frac{1}{2}$;

(Ψ_2) $\exists\,\mu>0:\ \mu|\xi|^2\le\Psi(\xi)\le\frac{1}{\mu}|\xi|^2$ for every $\xi\in\mathbb{R}^N$;

(Ψ_3) $\lim_{|\xi|\to\infty}\frac{\Psi'(\xi)\cdot\xi-2\Psi(\xi)}{|\xi|^2}=0$;

Moreover the function $b(x)$ has to be strictly positive in a non zero measure set, and the zero set must be ”thin”, in other words $b(x)$ must satisfy the following conditions:

(b_1) $\Omega^+ := \{x\in\Omega\ :\ b(x)>0\}$ is a nonempty open set,

(b_2) $\Omega^0 := \{x\in\Omega\ :\ b(x)=0\}$ has zero measure.

Conditions (b_1) and (b_2) imply that $b^+(x)=b(x)+b^-(x)\not\equiv 0$ and that, since b is continuous, the set Ω^0 is closed in Ω.

Let us also denote by (λ_k) the eigenvalues of $-\Delta$ with homogeneous Dirichlet boundary condition.
In the model case $\Psi(\xi)=\frac{1}{2}|\xi|^2$, there is a wide literature on problem $(\mathcal{P})$.
To cite only some of the existing results, in [2] the authors found positive solutions to $(\mathcal{P})$ in case that $\lambda_1<\lambda<\Lambda^*$, with Λ^* suitably near to λ_1. In the following many other papers ([1], [2], [3], [5], [6]) were devoted to prove existence of (possibly infinitely many) solutions for $\lambda\in[\lambda_1,\Lambda^*]$ or also for every λ, in case the nonlinearity satisfies some oddness assumption. A result concerning all λ different from the eigenvalues of the Laplacian under some quite general assumptions can be found in [11], while in [8] the authors proved a result of existence of a nontrivial solution (possibly changing sign) for every λ.

On the other hand, only a small literature is available when dealing with equations with a non strictly convex principal part. In this framework, in [7] the author applies non smooth variational methods in presence of subcritical, positive, nonlinearities; while using similar techniques a nonlinearity with critical growth was considered in [9].

The aim of this paper is to extend to the setting of non strictly convex functionals some of the results contained in [2] (existence of a positive solution for $\lambda<\lambda_1$) and [8] (existence of a nontrivial solution for any λ.)

Problem $(\mathcal{P})$ can be treated by variational techniques. Indeed, weak solutions u of $(\mathcal{P})$ can be found as critical points of the C^1 functional $J: H^1_0(\Omega)\to\mathbb{R}$ defined as

$$J(u)=\int_\Omega\Psi(\nabla u)\,dx-\frac{\lambda}{2}\int_\Omega u^2\,dx-\frac{1}{p}\int_\Omega b(x)|u|^p\,dx. \tag{1}$$

The key point here is that, although Ψ shares some properties with this typical case, there is no assumption of strict convexity with respect to ξ.

For instance, one could consider

$$\Psi(\xi) = \psi(\xi_1) + \frac{1}{2}\sum_{j=2}^{N} \xi_j^2\,, \tag{2}$$

where

$$\psi(t) = \begin{cases} \frac{1}{2}t^2 & \text{if } |t| < 1, \\ |t| - \frac{1}{2} & \text{if } 1 \le |t| \le 2, \\ \frac{1}{2}|t|^2 - |t| + \frac{3}{2} & \text{if } |t| > 2. \end{cases}$$

If we look at the principal part of J as the energy stored in the deformation u, this means that the material has a plastic behavior when $1 \le |D_1 u| \le 2$. We refer the reader to [13, Chapter 6] for a discussion of several models of plasticity.

As shown in [7, 9], it may happen that Palais Smale sequences, even if bounded in $H_0^1(\Omega)$-norm, do not admit any subsequence which converges strongly in this norm. And there is no way to prevent the interaction between the area where Ψ loses strict convexity and the values of ∇u. A possibile strategy is to look for compactness in a weaker norm (L^{2^*}).
Let us introduce the following notations: let $k \ge 1$ be such that $\lambda_k \le \lambda < \lambda_{k+1}$ and let $e_1, \dots, e_k$ be eigenfunctions of $-\Delta$ associated to $\lambda_1, \dots, \lambda_k$, respectively. Finally, let $E_- = \text{span}\{e_1, ..., e_k\}$ and $E_+ = E_-^{\perp}$. The main results of this paper are the following:

Theorem 1.1. *Let $N \ge 2$ and let $\Psi : \mathbb{R}^N \to \mathbb{R}$ be a convex function of class C^1 satisfying $(\Psi_1), (\Psi_2), (\Psi_3)$. Moreover let the function $b(x)$ verify $(b_1), (b_2)$. Then, for every $\lambda \in]0, \lambda_1[$, problem $(\mathcal{P})$ admits a nontrivial and nonnegative weak solution $u \in H_0^1(\Omega)$.*

Theorem 1.2. *Let $N \ge 2$ and let $\Psi : \mathbb{R}^N \to \mathbb{R}$ be a convex function of class C^1 satisfying $(\Psi_1), (\Psi_2), (\Psi_3)$ and let $\lambda \ge \lambda_1$. Moreover let the function $b(x)$ verify $(b_1), (b_2)$ and the following assumptions:*

$$\int_\Omega b(x)|v|^p \ge 0 \qquad \forall\, v \in E_-. \tag{3}$$

$$\exists\, e \in E_-^{\perp} \setminus \{0\} \ :\ \int_\Omega b(x)|v|^p\,dx \ge C \int_\Omega |v|^p\,dx \qquad \forall\, v \in E_- \oplus span\{e\}. \tag{4}$$

Then problem $(\mathcal{P})$ admits a nontrivial weak solution $u \in H_0^1(\Omega)$.

Remark 1.3. Arguing as in section 2 of [9] we can deduce the following properties for Ψ, up to modifying the constant μ :

$$\Psi'(\xi)\cdot\xi \geq \mu|\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^N, \tag{5}$$

$$|\Psi'(\xi)| \geq \mu|\xi| \qquad \forall\, \xi \in \mathbb{R}^N \tag{6}$$

$$|\Psi'(\xi)| \leq \frac{1}{\mu}|\xi| \qquad \forall\, \xi \in \mathbb{R}^N \tag{7}$$

Furthermore (Ψ_3) yields that $\forall\, \sigma > 0$, $\exists M_\sigma \in \mathbb{R}$:

$$\Psi'(\xi)\xi - 2\Psi(\xi) \leq \sigma|\xi|^2 + M_\sigma \tag{8}$$

2. The variational framework

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary and let $\lambda \in \mathbb{R}$. Let us define the following functional $J : H_0^1(\Omega) \to \mathbb{R}$

$$J(u) = \int_\Omega \Psi(\nabla u)\,dx - \frac{\lambda}{2}\int_\Omega u^2\,dx - \frac{1}{p}\int_\Omega b(x)|u|^p\,dx.$$

By (Ψ_1), (Ψ_2) the functional J is of class C^1 on $H_0^1(\Omega)$. We wish to apply variational methods to functional J, but, as already mentioned, it is well known that the Palais Smale (PS) condition for a functional which is not strictly convex is not satisfied on $H_0^1(\Omega)$. So it is convenient to extend the functional J to L^{2^*} with value $+\infty$ outside $H_0^1(\Omega)$.
In other words we define the convex, lower semicontinuous functional (still denoted J)

$$J : L^{2^*}(\Omega) \longrightarrow]-\infty, +\infty]$$

$$J(u) = \begin{cases} \displaystyle\int_\Omega \Psi(\nabla u)\,dx - \frac{\lambda}{2}\int_\Omega u^2\,dx - \frac{1}{p}\int_\Omega b(x)|u|^p\,dx \text{ if } u \in H_0^1(\Omega), \\ +\infty \qquad \text{if } u \in L^{2^*}(\Omega)\setminus H_0^1(\Omega) \end{cases} \tag{9}$$

This setting will allow us to recover PS condition.
This functional can be written as $J = J_0 + J_1$, where

$$J_0 = \int_\Omega \Psi(\nabla u)\,dx,$$

is proper, convex and l.s.c., while

$$J_1 = -\frac{\lambda}{2}\int_\Omega u^2\,dx - \frac{1}{p}\int_\Omega b(x)|u|^p\,dx,$$

is of class C^1. We will use the following definitions ([12], [7]) of critical point and PS sequence for functionals of the type $J = J_0 + J_1$:

Definition 2.1. Let X be a real Banach space, $u \in X$ is a critical point for J if $J(u) \in \mathbb{R}$ and $-J_1'(u) \in \partial J_0$, where ∂J_0 is the subdifferential of J_0 at u.

Definition 2.2. Let X be a real Banach space and let $c \in \mathbb{R}$. We say that u_k is a Palais Smale sequence at level c ($(PS)_c$ sequence for short) for J if $J(u_k) \to 0$ and there exists $\alpha_k \in \partial J_0$ with $(\alpha_k + J_1'(u_k)) \to 0$ in X^*.

The following proposition (see [7]) assures that the critical points of the extendend functional already defined gives the solutions of our problem.

Proposition 2.3. *Let $u \in L^{2^*}(\Omega, \mathbb{R}^N)$. Then u is a critical point of J if and only if $u \in H_0^1(\Omega)$ and u is a weak solution of $(\mathcal{P})$.*

Proof. Let $v \in L^{2^*}$. Then $v \in \partial J_0$, if and only if $u \in H_0^1(\Omega)$ and

$$-div(\Psi'(\nabla u)) = v$$

that is a reformulation of definition 2.1. □

Moreover we will apply the compactness result contained in [7], which we recall.

Let us define the functional $\mathcal{E} : W_0^{1,2}(\Omega, \mathbb{R}^N) \to \mathbb{R}$ as

$$\mathcal{E}(u) = \int_\Omega \Psi(\nabla u)\, dx$$

Theorem 2.4. *Assume that Ω is bounded. If $\{u_h\}$ is weakly convergent to u in $W_0^{1,2}(\Omega, \mathbb{R}^N)$ with $\mathcal{E}(\{u_h\}) \to \mathcal{E}(\{u\})$, then u is strongly convergent to u in $L^{2^*}(\Omega)$.*

3. Proof of main results

Since $\Psi'(0) = 0$, of course 0 is a solution of $(\mathcal{P})$. Therefore we are interested in *nontrivial* solutions. In order to find nonnegative solutions of $(\mathcal{P})$,we consider the modified functional $\overline{J} : L^{2^*}(\Omega) \to]-\infty,\ +\infty]$ defined as

$$\overline{J}(u) = \begin{cases} \displaystyle\int_\Omega \Psi(\nabla u)\, dx - \frac{\lambda}{2}\int_\Omega (u^+)^2\, dx - \frac{1}{p}\int_\Omega b(x)(u^+)^p\, dx & \text{if } u \in H_0^1(\Omega), \\ +\infty & \text{if } u \in L^{2^*}(\Omega) \setminus H_0^1(\Omega) \end{cases}$$

Proposition 3.1. *Let $\Psi : \mathbb{R}^N \to \mathbb{R}$ be a convex function of class C^1 satisfying (Ψ_2) with $\mu > 0$, and (6). Then each critical point $u \in L^{2^*}$ of $\overline{J}$ is a nonnegative solution of $(\mathcal{P})$.*

Proof. Since by Proposition 2.3 we already know that the critical points of J are solutions of our problem, it is only left to prove that the modified functional will give nonnegative solutions. By (Ψ_2) one has

$$\mu \int_\Omega |\nabla u^-|^2\,dx\,dx \le \int_\Omega \Psi'(\nabla u)\cdot(-\nabla u^-)\,dx\,dx$$

$$= \lambda \int_\Omega u^+(-u^-)\,dx\,dx + \int_\Omega (u^+)^{p-1}(-u^-)\,dx\,dx = 0$$

whence the assertion. □

Remark 3.2. From now on, to simplify notations, we will keep on using the functional J instead of $\overline{J}$, since it is understood what has been proved in Proposition 3.1.

Proof of Theorem 1.1

We aim to apply to J a nonsmooth version of Mountain Pass Theorem [12]. First of all, let us observe that, by (Ψ_1), we have

$$\frac{\int_\Omega \Psi(\nabla u)\,dx}{\int_\Omega |\nabla u|^2\,dx} \to \frac{1}{2} \qquad \text{as } u \to 0 \text{ in } L^{2^*}.$$

Then, as in the case $\Psi(\xi) = \frac{1}{2}|\xi|^2$ treated in [2, 8], we deduce that there exist $\rho > 0$ and $\alpha > 0$ such that $J(u) \ge \alpha$ whenever $\|u\| = \rho$. On the other hand, there exists $e \in L^{2^*}$ with $e \ge 0$ a.e. in Ω such that

$$\lim_{t\to+\infty} J(te) = -\infty,$$

again, this is proved in [2] in the case $\Psi(\xi) = \frac{1}{2}|\xi|^2$, but by (Ψ_2) the assertion is true also in our case.

By the Mountain Pass theorem, there exist a sequence (u_k) in L^{2^*} and a sequence (w_k) in $L^{(2^*)'}(\Omega)$ strongly convergent to 0 such that (see definition 2.2)

$$\int_\Omega \Psi'(\nabla u_k)(\nabla v - \nabla u_k)\,dx \ge \lambda \int_\Omega u_k(v-u_k)\,dx + \int_\Omega b(x)|u_k|^{p-1}(v-u_k)\,dx$$

$$+ \int_\Omega w_k(v-u_k)\,dx \quad \forall\, v \in L^{(2^*)'} \qquad (10)$$

Taking $v = 0$ and $v = 2u_k$ as tests in the previous inequality yield

$$\int_\Omega \Psi'(\nabla u_k)\nabla u_k\,dx = \lambda \int_\Omega (u_k)^2\,dx + \int_\Omega b(x)|u_k|^p\,dx + \int_\Omega w_k u_k\,dx \quad \forall\, v \in L^{(2^*)'}. \tag{11}$$

Furthermore also the following relation holds:

$$\lim_{k\to\infty}\left(\int_\Omega \Psi(\nabla u_k)\,dx - \frac{\lambda}{2}\int_\Omega (u_k)^2\,dx - \frac{1}{p}\int_\Omega b(x)(u_k)^p\,dx\right) = c > \alpha. \tag{12}$$

Let us write the expression $pJ(u_k) - J'(u_k)u_k$:

$$\begin{aligned} p\int_\Omega \Psi(\nabla u_k)\,dx - \frac{p}{2}\lambda\int_\Omega (u_k)^2\,dx - \int_\Omega b(x)(u_k)^p\,dx - \int_\Omega \Psi'(\nabla u_k)\cdot\nabla u_k\,dx \\ + \lambda\int_\Omega (u_k)^2\,dx + \int_\Omega b(x)(u_k)^p\,dx \\ = \int_\Omega (p-2)\Psi(\nabla u_k)\,dx + \int_\Omega \left[2\Psi(\nabla u_k) - \Psi'(\nabla u_k)\cdot\nabla u_k\right]\,dx \\ -\lambda\left(\frac{p}{2}-1\right)\int_\Omega (u_k)^2\,dx = (p-2)c - \int_\Omega w_k u_k\,dx + C \end{aligned} \tag{13}$$

By (8) and (Ψ_2) one gets

$$\mu(p-2-\sigma)\int_\Omega |\nabla u_k|^2\,dx - \lambda\left(\frac{p}{2}-1\right)\lambda\int_\Omega (u_k)^2\,dx \le pc - \int_\Omega w_k u_k + C \tag{14}$$

so

$$\mu(p-2-\sigma)\int_\Omega |\nabla u_k|^2\,dx \le \lambda\left(\frac{p}{2}-1\right)\int_\Omega (u_k)^2\,dx + C \tag{15}$$

where the quantity $(p-2-\sigma)$ is strictly positive since σ is arbitrarily small. Our aim is to prove the boundedness of the H_0^1 norm of the Palais Smale sequences, so arguing by contradiction, let us assume that

$$||u_k|| \to \infty \quad \text{as } k \to +\infty.$$

Dividing (12) by $||u_k||^p$ yields

$$\liminf\left\{\frac{p\int_\Omega \Psi(\nabla u_k)}{||u_k||^p}\,dx - \frac{\lambda p}{2}\frac{\int_\Omega (u_k)^2\,dx}{||u_k||^p}\,dx - \frac{1}{p}\int_\Omega b(x)\left(\frac{u_k}{||u_k||}\right)^p\,dx\right\} = 0.$$

Since $p > 2$ and (Ψ_2) holds, the first two terms go to zero. So

$$\limsup\left(\int_\Omega b(x)\left(\frac{u_k}{||u_k||}\right)^p\,dx\right) = 0. \tag{16}$$

Since b is bounded, (16) yields that

$$\left(\frac{u_k}{||u_k||}\right) \to u_0$$

strongly in L^p and weakly in $H_0^1(\Omega)$. Arguing by contradiction let us suppose that $u_0 \equiv 0$. Dividing (15) by $||u_k||^2$ yields

$$\mu(p-2-2\sigma) \leq \lambda\left(\frac{p}{2}-1\right)\frac{1}{||u_k||^2}\int_\Omega (u_k)^2\,dx + \frac{C}{||u_k||^2} \tag{17}$$

the right hand side goes to zero, which leads to a contradiction since $p-2-2\sigma > 0$ and $\mu > 0$, so u_0 must not be identically zero.

Now let $\phi \in C_0^\infty(\Omega^+)$ be a compact support function, $\phi \geq 0$ and $\phi \not\equiv 0$. Let us use the function $t\phi v$, $v \in H_0^1(\Omega)$ as a test in (10):

$$\begin{aligned}\forall\, v \in H_0^1(\Omega): \quad & \int_{\Omega^+} \Psi'(\nabla u_k)(t\phi\nabla v + tv\nabla\phi - \nabla u_k) \\ & \geq \lambda\int_{\Omega^+} u_k(tv\phi - u_k) + \int_{\Omega^+} b(x)(u_k)^{p-1}(tv\phi - u_k) + \int_{\Omega^+} w_k(tv\phi - u_k).\end{aligned}$$

Then let us divide the previous inequality by t and then let t go to $+\infty$:

$$\begin{aligned}&\int_{\Omega^+} \Psi'(\nabla u_k)(\phi\nabla v) + \Psi'(\nabla u_k)v\nabla\phi \\ &\geq \lambda\int_{\Omega^+} u_k v\phi + \int_{\Omega^+} b^+(x)(u_k)^{p-1}v\phi + \int_\Omega w_k v\phi \qquad \forall\, v \in H_0^1(\Omega)\end{aligned} \tag{18}$$

On the other hand, if $t \to -\infty$, one gets the opposite inequality, so we can deduce that the equality holds in the last expression, that is

$$\begin{aligned}&\int_{\Omega^+} \Psi'(\nabla u_k)(\phi\nabla v) + \Psi'(\nabla u_k)v\nabla\phi \\ &\quad = \lambda\int_{\Omega^+} u_k v\phi + \int_{\Omega^+} b^+(x)(u_k)^{p-1}v\phi + \int_\Omega w_k v\phi \quad \forall\, v \in H_0^1(\Omega).\end{aligned} \tag{19}$$

Now let us choose $v = u_k$ and divide both handsides of (19) by $||u_k||^p$. It is easily seen that the terms containing λ and w_k go to 0 as $k \to +\infty$. Then

$$\int_{\Omega^+} \frac{\Psi'(\nabla u_k)\nabla u_k\phi}{||u_k||^p}$$

goes to 0 since $p > 2$ and (7) holds.

On the other hand, by (7), since $p > 2$ and ϕ is of class C^∞ in Ω^+ bounded,

$$\frac{1}{||u_k||^p}\int_{\Omega^+}\Psi'(\nabla u_k)u_k\nabla\phi \le C\frac{||u_k||}{||u_k||^{p-1}}\frac{||u_k||_{L^2}}{||u_k||}$$

The term $\frac{||u_k||_{L^2}}{||u_k||}$ is bounded, while $\frac{||u_k||}{||u_k||^{p-1}}$ converges to 0.
By (19) We can conclude that

$$\int_{\Omega^+}\frac{1}{||u_k||^p}b^+(x)(u_k)^p\phi \ \mapsto 0 \text{ as } k\to\infty.$$

Applying Fatou's Lemma yields

$$\liminf\int_{\Omega^+}\frac{1}{||u_k||^p}b^+(x)(u_k)^p\phi \le 0$$

and since the integrand is nonnegative, this means that $\frac{u_k^p}{||u_k||^p}$ must tend to 0, *a.e.* in Ω^+ as $k\to\infty$. Arguing in the same way $\frac{u_k^p}{||u_k||^p}\to 0$ a.e. as $k\to\infty$, in Ω^-. This yields that $\frac{u_k^p}{||u_k||^p}\to 0$ a.e. in Ω since the remaining part is negligible. This is in contradiction with the fact that it converges to a nonzero function u_0.
Then u_k must have bounded norm in $H_0^1(\Omega)$ and admits a subsequence weakly converging in L^{2^*}.

According to (10) and taking $v=u$ as a test function yields

$$\int_\Omega \Psi'(\nabla u_k)(\nabla u-\nabla u_k)\,dx \\ \ge \lambda\int_\Omega u_k(u-u_k)\,dx+\int_\Omega b(x)(u_k)^{p-1}(u-u_k)\,dx+o(1)$$

so as $k\to\infty$ the right hand-side terms go to zero, and we obtain

$$\liminf\int_\Omega \Psi'(\nabla u_k)(\nabla u-\nabla u_k)\,dx \ge 0. \tag{20}$$

On the other hand, by convexity

$$\int_\Omega \Psi(\nabla u)\,dx \ge \int_\Omega \Psi(\nabla u_k)\,dx+\int_\Omega \Psi'(\nabla u_k)(\nabla u-\nabla u_k)\,dx \tag{21}$$

So by (20) and (21)

$$\limsup\int_\Omega \Psi(\nabla u_k)\,dx \le \limsup\left(\int_\Omega \Psi(\nabla u)\,dx-\int_\Omega \Psi'(\nabla u_k)(\nabla u-\nabla u_k)\,dx\right) \\ \le \int_\Omega \Psi(\nabla u)\,dx-\liminf\int_\Omega \Psi'(\nabla u_k)(\nabla u-\nabla u_k)\,dx \le \int_\Omega \Psi(\nabla u)\,dx$$

By lower semicontinuity and convexity

$$\liminf \int_\Omega \Psi(\nabla u_k)\, dx \geq \int_\Omega \Psi(\nabla u)\, dx \tag{22}$$

We can conclude that

$$\int_\Omega \Psi(\nabla u_k)\, dx \to \int_\Omega \Psi(\nabla u)\, dx.$$

By Theorem 2.4 u_k admits a subsequence strongly converging in L^{2^*}, which concludes the proof of PS condition and of Theorem 1.1.

Proof of Theorem 1.2

We are now concerned with the existence of (possibly sign-changing) nontrivial solutions u of $(\mathcal{P})$. Let (λ_k) denote the sequence of the eigenvalues of $-\Delta$ with homogeneous Dirichlet condition, repeated according to multiplicity.

Since the case $0 < \lambda < \lambda_1$ is already contained in Theorem 1.1, we may assume that $\lambda \geq \lambda_1$. Let $k \geq 1$ be such that $\lambda_k \leq \lambda < \lambda_{k+1}$, $e_1, \dots, e_k$ are eigenfunctions of $-\Delta$, as defined in the introduction. Finally, let $E_- = \text{span}\{e_1, ..., e_k\}$ and $E_+ = E_-^\perp$.

Consider the functional J defined in (9) We aim to apply the version of the Linking Theorem for convex functional presented by Szulkin in [12]. Since

$$\frac{\int_\Omega \Psi(\nabla u)\, dx}{\int_\Omega |\nabla u|^2\, dx} \to \frac{1}{2} \qquad \text{as } u \to 0 \text{ in } H_0^1(\Omega),$$

as in the case $\Psi(\xi) = \frac{1}{2}|\xi|^2$ treated in [8], we deduce that there exist $\rho > 0$ and $\alpha > 0$ such that $J(u) \geq \alpha$ whenever $u \in E_+$ with $\|u\| = \rho$. On the other hand, there exists $e \in H_0^1(\Omega) \setminus E_-$ such that

$$\lim_{\substack{\|u\|\to\infty \\ u \in \mathbb{R}e \oplus E_-}} J(u) = -\infty$$

Again, this is proved in [8] when $\Psi(\xi) = \frac{1}{2}|\xi|^2$, but by (Ψ_2) the assertion is true also in our case. Finally, it is clear that $J(u) \leq 0$ for every $u \in E_-$.

By the Linking type theorem in [12] (Theorem 3.4), there exist a PS sequence (u_k) in $H_0^1(\Omega)$ and we can continue, up to minor changes, as in the proof of Theorem 1.1 to prove that there exists a subsequence of (u_k) strongly converging in L^{2^*}. This concludes the proof of Theorem 1.2, since the nontriviality of the solution comes directly from the characterization of the critical level of the solution.

Acknowledgment

The author thanks Prof. Marco Degiovanni for very helpful conversations.

REFERENCES

[1] S. Alama - M. del Pino, *Solutions of elliptic equations with indefinite nonlinearities via Morse theory and linking*, Ann. I.H.P. Analyse non linéaire 13 (1996), 95–115.

[2] S. Alama - G. Tarantello, *On semilinear elliptic equations with indefinite nonlinearities*, Calc. Var. 1 (1993), 439–475.

[3] S. Alama - G. Tarantello, *Elliptic problems with nonlinearities indefinite in sign*, J. Funct. An. 141 (1996), 159–215.

[4] A. Ambrosetti - P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. 14 (1973), 349–381.

[5] M. Badiale, *Infinitely many solutions for some indefinite nonlinear elliptic problems*, Comm. on Appl. Non. Anal. 3 (1996), 61–76.

[6] M. Badiale - E. Nabana, *A remark on multiplicity solutions for semilinear elliptic problems with indefinite nonlinearities*, C. R. Acad. Sc. Paris 323 serie 1 (1996), 151–156.

[7] M. Degiovanni, *Variational methods for functionals with lack of strict convexity*, in "Nonlinear Equations: Methods, Models and Applications" (Bergamo, 2001), D. Lupo, C. Pagani and B. Ruf, eds., 127–139, Progress in Nonlinear Differential Equations and their Applications, Vol. 54, Birkhäuser, Boston, Inc., Boston, Ma, 2003.

[8] M. Grossi - P. Magrone - M. Matzeu, *Linking type solutions for elliptic equations with indefinite nonlinearities up to the critical growth*, Discrete Contin. Dynam. Systems 7 (4), (2001), 703–718.

[9] P. Magrone, *An existence result for a problem with critical growth and lack of strict convexity*, NoDEA Nonlinear Differential Equations Appl. 15 (6) (2008), 717–728.

[10] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, "CBMS Regional Conference Series in Mathematics", Vol. 65, American Mathematical Society, Providence, R.I., 1986.

[11] M. Ramos - S. Terracini - C. Troestler, *Superlinear indefinite elliptic problems and Pohozaev type identities*, J. Funct. Anal. 159 (1998), 596–628.

[12] A. Szulkin, *Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems*, Ann. Inst. H. Poincar Anal. Non Linéaire 3 (2) (1986), 77–109.

[13] H.-C. Wu, *Continuum mechanics and plasticity*, CRC Series: Modern Mechanics and Mathematics, Chapman & Hall/CRC, Boca Raton, FL., 2005.

PAOLA MAGRONE
Dipartimento di Architettura
Università degli Studi Roma Tre
e-mail: magrone@mat.uniroma3.it