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MULTIPLE SOLUTIONS FOR A CLASS OF DEGENERATE NONLOCAL PROBLEMS INVOLVING SUBLINEAR NONLINEARITIES

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In this article, we use the three critical points theorem by G. Bonanno [3] in order to investigate the multiplicity of solutions for nonlocal degenerate problems of the form

$$\begin{cases} -M\Big(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\Big)\,\mathrm{div}\,\Big(|x|^{-ap}|\nabla u|^{p-2}\nabla u\Big) \\ \qquad\qquad = \lambda|x|^{-p(a+1)+c}f(u) \quad \text{in } \Omega, \\ u=0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a smooth bounded domain, $0 \in \Omega$, $0 \leq a < \frac{N-p}{p}$, $1 < p < N$, $c > 0$, and $M : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function that may be degenerate at zero, $f : \mathbb{R} \to \mathbb{R}$ is continuous and sublinear at infinity.

1. Introduction

In this paper, we are concerned with a class of nonlocal degenerate problems of the form

$$\begin{cases} -M\big(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\big)\,\mathrm{div}\,\Big(|x|^{-ap}|\nabla u|^{p-2}\nabla u\Big) \\ \qquad\qquad = \lambda|x|^{-p(a+1)+c}f(u) \quad \text{in } \Omega, \\ u=0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

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where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a smooth bounded domain, $0 \in \Omega$, $0 \leq a < \frac{N-p}{p}$, $1 < p < N$, $c > 0$, and $M : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function, $f : \mathbb{R} \to \mathbb{R}$ is continuous and $(p-1)$-sublinear at infinity. It should be noticed that if $a = 0$ and $c = p$ then problem (1) becomes

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^p dx\right) \Delta_p u = \lambda f(u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{2}$$

Since the first equation in (1) contains an integral over Ω, it is no longer a pointwise identity; therefore it is often called a nonlocal problem. This problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density, see [6]. Problem (1) is related to the stationary version of the Kirchhoff equation which is presented by Kirchhoff in 1883, see [15] for details.

In recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to [2, 10–12, 16, 17, 19, 20], in which the authors have used different methods to get the existence of solutions for (2). One of the important hypotheses in these papers is that the Kirchhoff function M is non-degenerate, i.e.,

$$M(t) \geq m_0 \ \text{ for all } \ t \in \mathbb{R}^+. \tag{3}$$

Motivated by the ideas introduced in [7, 13, 22], the goal of this paper is to study the existence and multiplicity of solutions for problem (1) without condition (3). The approach is based on three critical points theorem by G. Bonanno [3]. Our situation in this paper is different from [8], in which we considered problem (1) with superlinear terms satisfying the (A-R) type condition. Our approach in this work is also different from [9].

We start by recalling some useful results in [4, 5, 22]. We have known that for all $u \in C_0^\infty(\mathbb{R}^N)$, there exists a constant $C_{a,b} > 0$ such that

$$\left(\int_{\mathbb{R}^N} |x|^{-bq} |u|^q \, dx\right)^{\frac{p}{q}} \leq C_{a,b} \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p \, dx, \tag{4}$$

where $-\infty < a < \frac{N-p}{p}$, $a \leq b \leq a+1$, $q = p^*(a,b) = \frac{Np}{N-dp}$, $d = 1 + a - b$.

Let $W_0^{1,p}(\Omega, |x|^{-ap})$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{a,p} = \left(\int_\Omega |x|^{-ap} |\nabla u|^p \, dx\right)^{\frac{1}{p}}.$$

Then $W_0^{1,p}(\Omega, |x|^{-ap})$ is a reflexive Banach space. From the boundedness of Ω and the standard approximation argument, it is easy to see that (4) holds for any

$u \in W_0^{1,p}(\Omega, |x|^{-ap})$ in the sense that

$$\left(\int_{\mathbb{R}^N} |x|^{-\alpha} |u|^r \, dx \right)^{\frac{p}{r}} \leq C_{a,b} \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p \, dx, \tag{5}$$

for $1 \leq r \leq p^* = \frac{Np}{N-p}$, $\alpha \leq (1+a)r + N\left(1 - \frac{r}{p}\right)$, that is, the embedding

$$W_0^{1,p}(\Omega, |x|^{-ap}) \hookrightarrow L^r(\Omega, |x|^{-\alpha})$$

is continuous, where $L^r(\Omega, |x|^{-\alpha})$ is the weighted $L^r(\Omega)$ space with the norm

$$|u|_{r,\alpha} := |u|_{L^r(\Omega, |x|^{-\alpha})} = \left(\int_{\Omega} |x|^{-\alpha} |u|^r \, dx \right)^{\frac{1}{r}}.$$

In fact, we have the following compact embedding result which is an extension of the classical Rellich-Kondrachov compactness theorem (see [22]).

Lemma 1.1 (Compact embedding theorem)**.** *Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded domain with C^1 boundary and that $0 \in \Omega$, and $1 < p < N$, $-\infty < a < \frac{N-p}{p}$, $1 \leq r < \frac{Np}{N-p}$ and $\alpha < (1+a)r + N\left(1 - \frac{r}{p}\right)$. Then the embedding $W_0^{1,p}(\Omega, |x|^{-ap}) \hookrightarrow L^r(\Omega, |x|^{-\alpha})$ is compact.*

From Lemma 1.1, B. Xuan proved in [22] that the first eigenvalue λ_1 of the singular quasilinear equation

$$\begin{cases} -\operatorname{div}\left(|x|^{-ap} |\nabla u|^{p-2} \nabla u\right) = \lambda |x|^{-p(a+1)+c} |u|^{p-2} u \text{ in } \Omega, \\ u = 0 \text{ in } \partial\Omega, \end{cases}$$

is isolated, unique (up to a multiplicative constant), that is, the first eigenvalue is simple and it is given by

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega, |x|^{-ap}) \setminus \{0\}} \frac{\int_{\Omega} |x|^{-ap} |\nabla u|^p \, dx}{\int_{\Omega} |x|^{-p(a+1)+c} |u|^p \, dx} > 0. \tag{6}$$

This is a natural extension from the previous results on the case $a = 0$ and $c = p$ relying esstentially on the Caffarelli-Kohn-Nirenberg inequalities.

In our proof, we use the following result, which is obtained by G. Bonanno [3].

Proposition 1.2 (See [3, Theorem 2.1])**.** *Let $(X, \|.\|)$ be a separable and reflexive real Banach space, $\mathcal{A}, \mathcal{F} : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that there exists $x_0 \in X$ such that $\mathcal{A}(x_0) = \mathcal{F}(x_0) = 0$, $\mathcal{A}(x) \geq 0$ for all $x \in X$ and there exist $x_1 \in X$, $\rho > 0$ such that* (*i*) $\rho < \mathcal{A}(x_1)$,

(*ii*) $\sup_{\{\mathcal{A}(x)<\rho\}} \mathcal{F}(x) < \rho \frac{\mathcal{F}(x_1)}{\mathcal{A}(x_1)}$.

Further, put

$$\overline{a} = \frac{\xi\rho}{\rho\frac{\mathcal{F}(x_1)}{\mathcal{A}(x_1)} - \sup_{\{\mathcal{A}(x)<\rho\}} \mathcal{F}(x)}, \text{ with } \xi > 1,$$

and assume that the functional $\mathcal{A} - \lambda\mathcal{F}$ is sequentially weakly lower semicontinuous, satisfies the Palais-Smale condition and

(*iii*) $\lim_{\|x\|\to\infty}[\mathcal{A}(x) - \lambda\mathcal{F}(x)] = +\infty$ *for every* $\lambda \in [0, \overline{a}]$.

Then, there exist an open interval $\Lambda \subset [0, \overline{a}]$ and a positive real number δ such that each $\lambda \in \Lambda$, the equation

$$D\mathcal{A}(u) - \lambda D\mathcal{F}(u) = 0$$

has at least three solutions in X whose $\|.\|$-norms are less than δ.

2. Main result

In this section, we shall state and prove the main result of this paper. Let us introduce the following hypotheses:

(M_0) $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and satisfies

$$M(t) \geq m_0 t^{\alpha-1} \text{ for all } t \in \mathbb{R}^+,$$

where $m_0 > 0$ and $1 < \alpha < \min\left\{\frac{N}{N-p}, \frac{N-p(a+1)+c}{N-p(a+1)}\right\}$;

(F_1) $\lim_{|t|\to+\infty} \frac{f(t)}{|t|^{\alpha p-1}} = 0$, i.e., f is $(\alpha p - 1)$-sublinear at infinity;

(F_2) $\lim_{|t|\to 0} \frac{f(t)}{|t|^{\alpha p-1}} = 0$;

(F_3) There exists $t_0 \in \mathbb{R}$ such that $F(t_0) = \int_0^{t_0} f(s)\, ds > 0$.

Definition 2.1. We say that $u \in X = W_0^{1,p}(\Omega, |x|^{-ap})$ is a weak solution of problem (1) if

$$M\left(\int_\Omega |x|^{-ap}|\nabla u|^p\, dx\right) \int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u \cdot \nabla v\, dx \\ - \int_\Omega |x|^{-(a+1)p+c} f(u) v\, dx = 0$$

for all $v \in C_0^\infty(\Omega)$.

Theorem 2.2. *Assume that the conditions* (M_0)*,* (F_1)*-*(F_3) *are satisfied. Then there exist an open interval* $\Lambda \subset (0,+\infty)$ *and a constant* μ *such that for every* $\lambda \in \Lambda$*, problem (1) has at least three distinct weak solutions whose norms are less than* μ*.*

For all $\lambda \in \mathbb{R}$, we consider the functional $J_\lambda : X \to \mathbb{R}$ given by

$$\begin{aligned} J_\lambda(u) &= \frac{1}{p}\widehat{M}\left(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\right) - \lambda\int_\Omega |x|^{-p(a+1)+c}F(u)\,dx \\ &= \mathcal{A}(u) - \lambda\mathcal{F}(u), \end{aligned} \tag{7}$$

where

$$\begin{aligned} \mathcal{A}(u) &= \frac{1}{p}\widehat{M}\left(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\right), \\ \mathcal{F}(u) &= \int_\Omega |x|^{-p(a+1)+c}F(u)\,dx, \quad u \in X. \end{aligned} \tag{8}$$

By the condition (F_1), using Lemma 1.1, we can show that the functional J_λ is of $C^1(X,\mathbb{R})$ and thus the weak solutions of (1) are exactly the critical points of J_λ. Moreover, the following result holds.

Lemma 2.3. *For every* $\lambda \in \mathbb{R}$*, the functional* $J_\lambda : X \to \mathbb{R}$ *is sequentially weakly lower semicontinuous.*

Lemma 2.4. *For every* $\lambda \in \mathbb{R}$*, the functional* J_λ *is coercive and satisfies the Palais-Smale condition.*

Proof. First, since $1 < \alpha < \min\left\{\frac{N}{N-p}, \frac{N-p(a+1)+c}{N-p(a+1)}\right\}$, the embedding

$$X \hookrightarrow L^{\alpha p}(\Omega, |x|^{-p(a+1)+c})$$

is compact, see Lemma 1.1. Then there exists $C_1 > 0$ such that

$$C_1\|u\|_{L^{\alpha p}(\Omega,|x|^{-p(a+1)+c})} \le \|u\|_{a,p} \text{ for all } u \in X$$

or

$$C_1^{\alpha p}\int_\Omega |x|^{-p(a+1)+c}|u|^{\alpha p}\,dx \le \left(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\right)^\alpha \text{ for all } u \in X.$$

It follows that the number

$$\lambda_\alpha := \inf_{u\in X\setminus\{0\}} \frac{\left(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\right)^\alpha}{\int_\Omega |x|^{-p(a+1)+c}|u|^{\alpha p}\,dx} > 0. \tag{9}$$

Let us fix $\lambda \in \mathbb{R}$, arbitrary. By (F_1) there exists $\delta = \delta(\lambda)$ such that

$$|f(t)| \leq \frac{m_0\lambda_\alpha}{1+|\lambda|}|t|^{\alpha p-1} \text{ for every } |t| \geq \delta. \tag{10}$$

(Here m_0 is from (M_0)). Integrating the above inequality we have

$$|F(t)| \leq \frac{m_0\lambda_\alpha}{\alpha p(1+|\lambda|)}|t|^{\alpha p} + \max_{|t|\leq\delta}|f(t)||t| \text{ for every } t \in \mathbb{R}. \tag{11}$$

Thus, for every $u \in X$ we obtain

$$\begin{aligned}
J_\lambda(u) &\geq \mathcal{A}(u) - |\lambda||\mathcal{F}(u)| \\
&\geq \frac{m_0}{\alpha p}\left(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\right)^\alpha - \frac{m_0\lambda_\alpha|\lambda|}{\alpha p(1+|\lambda|)}\int_\Omega |x|^{-(a+1)p+c}|u|^{\alpha p}\,dx \\
&\quad - |\lambda|\max_{|t|\leq\delta}|f(t)|\int_\Omega |x|^{-(a+1)p+c}|u|\,dx \\
&\geq \frac{m_0}{\alpha p(1+|\lambda|)}\|u\|_{a,p}^{\alpha p} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (12) \\
&\quad - |\lambda|\max_{|t|\leq\delta}|f(t)|\left(\int_\Omega |x|^{-(a+1)p+c}|u|^p\,dx\right)^{\frac{1}{p}}\cdot\left(\int_\Omega |x|^{-(a+1)p+c}\,dx\right)^{\frac{p-1}{p}} \\
&\geq \frac{m_0}{\alpha p(1+|\lambda|)}\|u\|_{a,p}^{\alpha p} - \lambda_1^{\frac{1}{p}}|\lambda|\max_{|t|\leq\delta}|f(t)|\|u\|_{a,p}\cdot\left(\int_\Omega |x|^{-(a+1)p+c}\,dx\right)^{\frac{p-1}{p}},
\end{aligned}$$

where λ_1 is given by (6) and λ_α is given by (9). Since $\alpha > 1$, $p > 1$ and the fact that

$$\int_\Omega |x|^{-(a+1)p+c}\,dx < +\infty$$

we deduce that the functional $J_\lambda(u) \to +\infty$ as $\|u\|_{a,p} \to +\infty$, i.e., J_λ is coercive.

Let $\{u_m\}$ be a sequence such that

$$J_\lambda(u_m) \to \overline{c} < \infty, \quad J'_\lambda(u_m) \to 0 \text{ in } X^* \text{ as } m \to \infty, \tag{13}$$

where X^* is the dual space of X.

Since T_λ is coercive on X, relation (13) implies that the sequence $\{u_m\}$ is bounded in X. Since X is reflexive, there exists $u \in X$ such that, passing to a subsequence, still denoted by $\{u_m\}$, it converges weakly to u in X. Hence, $\{\|u_m - u\|_{a,p}\}$ is bounded. This and (13) imply that $J'_\lambda(u_m)(u_m - u)$ converges to 0 as $m \to \infty$. Using the condition (F_1) combined with Hölder's inequality, we conclude that

$$\begin{aligned}
&\int_\Omega |x|^{-p(a+1)+c}|f(x,u_m)||u_m-u|\,dx\\
&\le C_2\int_\Omega |x|^{-p(a+1)+c}(1+|u_m|^{\alpha p-1})|u_m-u|\,dx\\
&\le C_3\left(\int_\Omega |x|^{-(a+1)p+c}\,dx\right)^{\frac{\alpha p-1}{\alpha p}}\|u_m-u\|_{L^{\alpha p}(\Omega,|x|^{-p(a+1)+c})}\\
&\qquad + C_3\|u_m\|^{\alpha p-1}_{L^{\alpha p}(\Omega,|x|^{-p(a+1)+c})}\|u_m-u\|_{L^{\alpha p}(\Omega,|x|^{-p(a+1)+c})},
\end{aligned}$$

which shows that

$$\lim_{m\to\infty}\mathcal{F}'(u_m)(u_m-u)=0. \tag{14}$$

By (13), (14) and the definition of the functional J, it follows that

$$\lim_{m\to\infty} M\left(\int_\Omega |x|^{-ap}|\nabla u_m|^p\,dx\right)\int_\Omega |x|^{-ap}|\nabla u_m|^{p-2}\nabla u_m\cdot(\nabla u_m-\nabla u)\,dx=0. \tag{15}$$

Since $\{u_m\}$ is bounded in X, passing to a subsequence, if necessary, we may assume that

$$\int_\Omega |x|^{-ap}|\nabla u_m|^p\,dx\to t_0\ge 0 \text{ as } m\to\infty.$$

If $t_0=0$ then $\{u_m\}$ converges strongly to $u=0$ in X and the proof is finished. If $t_0>0$ then by (M_0) and the continuity of M, we get

$$M\left(\int_\Omega |x|^{-ap}|\nabla u_m|^p\,dx\right)\to M(t_0)>0 \text{ as } m\to\infty.$$

Thus, for m sufficiently large, we have

$$0<C_4\le M\left(\int_\Omega |x|^{-ap}|\nabla u_m|^p\,dx\right)\le C_5. \tag{16}$$

From (15) and (16) and the condition (M_0), we have

$$\lim_{m\to\infty}\int_\Omega |x|^{-ap}|\nabla u_m|^{p-2}\nabla u_m\cdot(\nabla u_m-\nabla u)\,dx=0. \tag{17}$$

On the other hand, since $\{u_m\}$ converges weakly to u in X, we have

$$\lim_{m\to\infty}\int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u\cdot(\nabla u_m-\nabla u)\,dx=0. \tag{18}$$

By (17) and (18),

$$\lim_{m\to\infty}\int_\Omega |x|^{-ap}\left(|\nabla u_m|^{p-2}\nabla u_m-|\nabla u|^{p-2}\nabla u\right)\cdot(\nabla u_m-\nabla u)\,dx=0.$$

or

$$\lim_{m\to\infty}\int_\Omega \left(|\nabla v_m|^{p-2}\nabla v_m - |\nabla v|^{p-2}\nabla v\right)\cdot(\nabla v_m - \nabla v)\,dx = 0, \tag{19}$$

where $\nabla v_m = |x|^{-a}\nabla u_m$, $\nabla v = |x|^{-a}\nabla u \in L^p(\Omega)$.

We recall that the following inequalities hold

$$\begin{aligned}
&\left\langle |\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi-\eta\right\rangle \geq C_6\Big(|\xi|+|\eta|\Big)^{p-2}|\xi-\eta|^2 \text{ if } 1<p<2,\\
&\left\langle |\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi-\eta\right\rangle \geq C_7|\xi-\eta|^p \text{ if } p\geq 2,
\end{aligned} \tag{20}$$

for all $\xi,\eta\in\mathbb{R}^N$, where $\langle .,.\rangle$ denote the usual product in $\mathbb{R}^N$.

If $1<p<2$, using the Hölder inequality, by (19) and (20), we have

$$\begin{aligned}
0 &\leq \|u_m-u\|_{a,p}^p = \||\nabla v_m - \nabla v|\|_{L^p(\Omega)}^p\\
&\leq \int_\Omega |\nabla v_m - \nabla v|^p\Big(|\nabla v_m|+|\nabla v|\Big)^{\frac{p(p-2)}{2}}\Big(|\nabla v_m|+|\nabla v|\Big)^{\frac{p(2-p)}{2}}dx\\
&\leq \left(\int_\Omega |\nabla v_m-\nabla v|^2(|\nabla v_m|+|\nabla v|)^{p-2}\,dx\right)^{\frac{p}{2}}\left(\int_\Omega(|\nabla v_m|+|\nabla v|)^p\,dx\right)^{\frac{2-p}{2}}\\
&\leq C_8\left(\int_\Omega\left\langle|\nabla v_m|^{p-2}\nabla v_m - |\nabla v|^{p-2}\nabla v, \nabla v_m-\nabla v\right\rangle dx\right)^{\frac{p}{2}}\\
&\qquad\qquad \times\left(\int_\Omega(|\nabla v_m|+|\nabla v|)^p\,dx\right)^{\frac{2-p}{2}}\\
&\leq C_9\left(\int_\Omega\left\langle|\nabla v_m|^{p-2}\nabla v_m - |\nabla v|^{p-2}\nabla v, \nabla v_m-\nabla v\right\rangle dx\right)^{\frac{p}{2}},
\end{aligned}$$

which converges to 0 as $m\to\infty$. If $p\geq 2$, one has

$$\begin{aligned}
0 &\leq \|u_m-u\|_{a,p}^p = \||\nabla v_m - \nabla v|\|_{L^p(\Omega)}^p\\
&\leq C_8\int_\Omega\left\langle|\nabla v_m|^{p-2}\nabla v_m - |\nabla v|^{p-2}\nabla v, \nabla v_m-\nabla u\right\rangle dx,
\end{aligned}$$

which converges to 0 as $m\to\infty$. So we deduce that $\{u_m\}$ converges strongly to u in X and the functional J satisfies the (PS) condition. □

Lemma 2.5. *The following property holds:*

$$\lim_{\rho\to 0^+}\frac{\sup\{\mathcal{F}(u):\ \mathcal{A}(u)<\rho\}}{\rho} = 0.$$

Proof. Due to (F_2), for an arbitrary small $\varepsilon>0$, there exists $\delta>0$ such that

$$|f(s)|\leq \varepsilon\alpha p c_{\alpha p}^{-\alpha p}|s|^{\alpha p-1} \text{ for all } |s|\leq\delta.$$

Combining the above inequality with

$$|f(s)| \le C_2(1+|s|^{\alpha p-1}) \text{ for all } s\in\mathbb{R},$$

we obtain

$$|F(s)| \le \varepsilon c_{\alpha p}^{-\alpha p}|s|^{\alpha p} + C(\delta)|s|^q \text{ for all } s\in\mathbb{R}, \tag{21}$$

where $\alpha p < q < \min\left\{\frac{Np}{N-p}, \frac{p(N-p(a+1)+c)}{N-p(a+1)}\right\}$ is fixed, and $C(\delta)$ does not depend on s. For $\rho>0$, define the sets

$$B_\rho^1 := \{u\in X:\ \mathcal{A}(u)<\rho\},$$

and

$$B_\rho^2 := \left\{u\in X:\ \frac{m_0}{\alpha p}\|u\|_{a,p}^{\alpha p}<\rho\right\}.$$

By the condition (M_0), $B_\rho^1\subset B_\rho^2$.

On the other hand, we have $0\in B_\rho^1$. From this and the fact that $\mathcal{F}(0)=0$, it implies that $0\le \sup_{u\in B_\rho^1}\mathcal{F}(u)$. Now, if $u\in B_\rho^2$ then we have $\|u\|_{a,p}\le \left(\frac{m_0}{\alpha p}\right)^{-\frac{1}{\alpha p}}\rho^{\frac{1}{\alpha p}}$, so it follows from (21) that

$$\begin{aligned} 0\le \frac{\sup_{u\in B_\rho^1}\mathcal{F}(u)}{\rho} &\le \frac{\sup_{u\in B_\rho^2}\mathcal{F}(u)}{\rho}\\ &\le \frac{m_0}{\alpha p}\varepsilon + C(\delta)c_q^{-q}\left(\frac{m_0}{\alpha p}\right)^{-\frac{q}{\alpha p}}\rho^{\frac{q}{\alpha p}-1}. \end{aligned}$$

Since $\varepsilon>0$ is arbitrary and $q>\alpha p$, let $\rho\to 0^+$, we conclude the proof of Lemma 2.5. $\square$

Proof of Theorem 2.2. In order to prove Theorem 2.2, we shall apply Proposition 1.2 by choosing $X=W_0^{1,p}(\Omega,|x|^{-ap})$ as well as $\mathcal{A}$ and $\mathcal{F}$ as in (8). Now, we shall check all assumptions of Proposition 1.2. Indeed, we have $\mathcal{A}(0)=\mathcal{F}(0)=0$ and we have $\mathcal{A}(u)\ge 0$ for any $u\in X$.

From (F_3), let $t_0\in\mathbb{R}$ be such that $F(t_0)>0$. Here and in the sequel, let $x_0\in\Omega$ and $R_0>0$ so small such that $|x_0|>R_0$ and $B(x_0,R_0)\subset\Omega$. For $\sigma\in(0,1)$, we define the function u_σ by

$$u_\sigma(x)=\begin{cases} 0, & \text{for } x\in\mathbb{R}^N\backslash B_{R_0}(x_0),\\ t_0, & \text{for } x\in B_{\sigma R_0}(x_0),\\ \frac{t_0}{R_0(1-\sigma)}(R_0-|x|) & \text{for } x\in B_{R_0}(x_0)\backslash B_{\sigma R_0}(x_0), \end{cases}$$

where $B_r(0)$ denotes the N-dimensional open ball with center 0 and radius $r > 0$, and $|.|$ denotes the usual Euclidean norm in $\mathbb{R}^N$. It is clear that $u_\sigma \in H_0^1(\Omega)$. From the definition of u_σ, simple computations show that

$$\|u_\sigma\|_{a,p}^p = \int_\Omega |x|^{-ap}|\nabla u|^p\,dx \geq |t_0|^p(|x_0|+R_0)^{-ap}(1-\sigma)^{-p}(1-\sigma^N)\omega_N R_0^{N-2}$$

and

$$\begin{aligned}\mathcal{F}(u_\sigma) &= \int_{B_{\sigma R_0}(x_0)} F(u_\sigma)\,dx + \int_{B_{R_0}(x_0)\setminus B_{\sigma R_0}(x_0)} F(u_\sigma)\,dx\\ &\geq F(t_0)(|x_0|+R_0)^{-p(a+1)+c}\sigma^N R_0^N\omega_N\\ &\qquad - \max_{|t|\leq R_0}|F(t)|(|x_0|-R_0)^{-p(a+1)+c}(1-\sigma)^N R_0^N\omega_N,\end{aligned}$$

where ω_N is the volume of the N-dimensional unit ball. If we choose $\sigma \in (0,1)$ close enough to 1, says σ_0, then the right-hand side of the last inequality becomes strictly positive. By Lemma 2.5, we can choose $\rho_0 \in (0,1)$ such that

$$\rho_0 < \frac{m_0}{\alpha p}\|u_{\sigma_0}\|^{\alpha p} \leq \mathcal{A}(u_{\sigma_0})$$

and

$$\begin{aligned}&\frac{\sup_{\mathcal{A}(u)<\rho_0}\mathcal{F}(u)}{\rho_0}\\ &< \frac{F(t_0)(|x_0|+R_0)^{-p(a+1)+c}\sigma^N R_0^N\omega_N}{2\mathcal{A}(u_{\sigma_0})}\\ &\qquad - \frac{\max_{|t|\leq R_0}|F(t)|(|x_0|-R_0)^{-p(a+1)+c}(1-\sigma)^N R_0^N\omega_N}{2\mathcal{A}(u_{\sigma_0})}\\ &< \frac{\mathcal{F}(u_{\sigma_0})}{\mathcal{A}(u_{\sigma_0})}.\end{aligned}$$

Now, in Proposition 1.2, we choose $x_0 = 0$, $x_1 = u_{\sigma_0}$, $\xi = 1+\rho_0$ and

$$\overline{a} = \frac{1+\rho_0}{\frac{\mathcal{F}(u_{\delta_0})}{\mathcal{A}(u_{\delta_0})} - \frac{\sup_{\mathcal{A}(u)<\rho\sigma_0}\mathcal{F}(u)}{\rho_0}} > 0.$$

Taking into account the above lemmas, all assumptions of Proposition 1.2 are verified. Then there exist an open interval $\Lambda \subset [0,\overline{a}]$ and a number μ, such that for each $\lambda \in \Lambda$, the equation $D\mathcal{A}(u) - \lambda D\mathcal{F}(u) = 0$ has at least three distinct solutions in X whose X-norms are less than μ. By (F_2), $f(0) = 0$, one of them may be the trivial one, so problem (1) has at least two non-trivial weak solutions with the required properties. □

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