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SEPARATING SEQUENCES OF 0-DIMENSIONAL SCHEMES

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In two previous papers, we defined, for every projective, 0-dimensional, reduced scheme **X**, a set of numerical sequences, which turns out to be a refinement of the Hilbert function of **X**. Here, we extend that definition to the case of a scheme **X** not necessarily reduced; the aim is reached by replacing a point by its corresponding "separating ideal" in its coordinate ring. The numerical sequences are obtained by taking the degrees of the elements appearing in suitable sequences of separating ideals. These latter sequences are themselves a good tool in the search for subschemes of **X** not in general position.

0. Introduction.

In two previous papers ([4] , [5]) we introduced, for any 0-dimensional, projective, reduced scheme **X**, a set of numerical sequences, strictly linked to the Hilbert function of **X**; it was mainly used for a study of all the schemes having the same Hilbert function, in terms of their subschemes not in general position. The numerical sequences allowed for the schemes with a given Hilbert function H are obtained from a very special sequence

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M_Γ ($\Gamma = \Delta H$), closed to the $h - type$ vectors introduced in [7], even if its origin and use are quite different. The hypothesis "**X** reduced" seemed to be essential in the definition of those sequences and in their geometric interpretation. Moreover, the problem of computing the sequences, starting from the coordinate ring of **X**, was not faced.

In this paper, we get rid of the condition "reduced" and produce a technique for computing all the numerical sequences of **X** starting from its projective coordinate ring A^* or from an affine one A. We reach the aim by introducing the definition of "separating ideal", that, in the reduced case, is an algebraic correspondent of "point". If **X** is not reduced, the correspondence between separating ideals of A and points of **X** fails; for instance, if **X** is supported at one point, the set of the separating ideals of A is a projective space, of dimension strictly less then deg **X**.

The concept of "separator", from which the definition of separating ideal is inspired, is not new; the idea of isolating a point of a finite set by means of a hypersurface, passing through the others but avoiding it, is classical; for recent use of this idea and investigation on the degree of a separator, see [2], [9], [10], [12], [14], [16].

The notion of separating ideal allows the definition of a set of sequences, called A-separating sequences, whose elements are separating ideals in the coordinate ring A of **X**, or in some suitable quotient of it. Every element of an A-separating sequence is provided with a well defined degree, so that it is natural to associate to each A-separating sequence the numerical sequence of its degrees; the numerical sequences arising in this way turn out to be the ones already considered in [4] and [5], when **X** is reduced. However, an A-separating sequence contains much more information than the one hidden in its corresponding numerical sequence. In fact, if **X** is reduced, the set $\mathcal{S}_A$ of all A-separating sequences of **X** allows us to point out all the subschemes of **X** not in general position and their mutual intersections.

Even if the scheme is not reduced, the set $\mathcal{S}_A$ and the set $(\delta\mathcal{S})_A$ of its corresponding numerical sequences are still well defined and the link of $(\delta\mathcal{S})_A$ with the Hilbert function survives (see Th.3.10). In this situation, $\mathcal{S}_A$ and $(\delta\mathcal{S})_A$ seem to suggest information on the set of flat families of 0-dimensional schemes containing **X**, but here our investigation is limited to elementary examples.

Some space is devoted to point out the link between $\mathcal{S}_A$ and $\{\mathcal{S}_{A_{\mathcal{M}_i}}\}$, where $\mathcal{M}_i$ spans the set of all maximal ideals of A. Using the canonical isomorphism of an artinian ring with the direct sum of its localizations at the maximal ideals, it is possible to construct $\mathcal{S}_A$ starting from $\{\mathcal{S}_{A_{\mathcal{M}_i}}\}$. Unfortunately, it is impossible to obtain $(\delta\mathcal{S})_A$ from the set $\{(\delta\mathcal{S})_{A_{\mathcal{M}_i}}\}$.

The definition of separator and of separating ideal seems to be more handy in an affine ring, while its link with the Hilbert function can be seen only by using the projective coordinate ring of **X**; that is the reason why the affine coordinate ring of **X** is privileged, but the definitions and results are also translated into the projective language.

1. 0-dimensional schemes and their affine and projective rings.

Let us fix some notation and recall some well known facts about homogenization and dehomogenization in polynomial rings (see [6] ,chap. 2, n. 6; [17], chap. VII, § 5).

K is an algebraically closed field.

$S = K[Y_0, ..., Y_n]$, with its ordinary graduation, is the coordinate ring of $\mathbf{P}^n$.

J is a homogeneous, saturated ideal of S, defining a projective scheme **X**, with projective coordinate ring $S/J = K[y_0, ..., y_n]$.

Without any loss of generality, we suppose Y_0 a regular form with respect to S/J and dehomogenize S with respect to it.

Following a usual notation (see [6]), for any $F(Y_0, ..., Y_n) \in S$, we set $F_*(X_1, ..., X_n) = F(1, X_1, ..., X_n)$. So, in the affine ring $R = K[X_1, ..., X_n]$, the image of J in the morphism just defined is $J_* = I$, which turns out to be the affine ideal of **X** in R; moreover $R/I = K[x_1, ..., x_n]$ is an affine coordinate ring of **X**. The hypothesis of regularity, required for Y_0, guaranties that we don't loose any irreducible component of **X** when we pass from S/J to R/I. Conversely, we can homogenize R, passing from it to S by setting, for any $f(X_1, ..., X_n) \in R$ of degree d, $f^*(Y_0, ..., Y_n) = Y_0^d f(Y_1/Y_0, ..., Y_n/Y_0) \in S$.

Recall 1.0. The following relations are well known (see [6], chap.2,Prop.5):

i) $(FG)_* = F_*G_*, \quad (fg)^* = f^*g^*$.

$ii)$ If r is the highest power of Y_0 which divides F, then $Y_0^r(F_*)^* = F$.

$iii) (f^*)_* = f$.

$iv)$ $(F + G)_* = F_* + G_*$.

$v)$ $Y_0^t(f + g)^* = Y_0^r f^* + Y_0^s g^*$, where $r = deg\ g, s = deg\ f$, $t = r + s - deg(f + g)$ or, equivalently,

$(f+g)^* = f^* + Y_0^{s-r} g^*$, if $s > r$,
$Y_0^u (f+g)^* = f^* + g^*$, $u = r - deg(f+g)$, if $r = s$.

We recall that homogenization is an injective map of R in S, whose image consists of the forms not in $(Y_0)R$. Such a map induces an injective map of ideals, sending $\mathcal{A} \subset R$ to $\mathcal{A}^* \subset R^*$, where $\mathcal{A}^*$ is the ideal generated by the image of $\mathcal{A}$. Moreover, we have the following well known relations:

$vi)(\mathcal{A}^*)_* = \mathcal{A}$, for any ideal $\mathcal{A} \subset R$;

$vii)(\mathcal{C}_*)^* = \mathcal{C}$, iff Y_0 is regular with respect to $S/\mathcal{C}$.

Moreover, it is immediate to prove that:

$viii) p+\mathcal{A} = q+\mathcal{A} \Leftrightarrow p^*+\mathcal{A}^* = (q^*+\mathcal{A}^*)Y_0^h$, where $h = degp - degq$.

From now on we will denote $A^* = S/J$ the homogeneous coordinate ring of $\mathbf{X}$ and $A = R/I$, $I = J_*$, its affine coordinate ring in the chart $Y_0 \neq 0$. As we will always add the hypothesis that Y_0 is regular for A^*, we have also the equality: $J = I^*$.

Now we need to define a map $\psi : A \longrightarrow A^*$, generalizing the homogenization $* : R \longrightarrow S$ and a map $D : A \longrightarrow \mathbf{N}$, that we will call “degree”, generalizing the usual degree in R. Let us remark in advance that the ring A, with the degree D, is not going to be a graduate ring.

Definition 1.1.

$i)$ For each $a = p+I \in A$, we denote by $\widetilde{p}$ any polynomial of minimal degree in $p+I$ and we set $Da = deg\ \widetilde{p}$.

$ii)$ $\psi : A \longrightarrow A^*$ is the correspondence defined by $\psi(p+I) = \widetilde{p}^* + I^*$.

$iii)$ As usual, we will call ”form” any homogeneous element of a graded ring.

Lemma 1.2. *The correspondence $\psi : A \longrightarrow A^*$ of Definition 1.1 is an injective map. Moreover:*

$i)$ *$Im\ \psi$ is the subset of all the forms of A^* not in $(y_0)A^*$.*

$ii)$ *$\psi(p+I)\psi(q+I) = \psi(p\,q+I){y_0}^h$*
$\psi(p+I)+\psi(q+I)y_0^{d-d'} = \psi(p+q+I){y_0}^h$, where $d = D(p+I) \geq d' = D(q+I)$.
In both relations, h is the maximal power of y_0 dividing the first member of the equality.

$iii)$ *For every $a \in A$, $D(a)$ turns out to be the degree of $\psi(a)$ as an element of A^*.*

iv) ψ induces a $1-1$ correspondence Ψ between the ideals of A and the homogeneous ideals $\mathcal{C}$ of A^*, such that y_0 is regular for $A^*/\mathcal{C}$, defined as follows: $\Psi(\mathcal{Q})$ is the ideal generated in A^* by $\psi(\mathcal{Q})$. The following properties hold for Ψ:

iv_1) If Q is the lifting of $\mathcal{Q}$ in R, then $\Psi(\mathcal{Q}) = Q^*/I^*$.

iv_2) Ψ preserves height and primary decompositions.

iv_3) All the elements of the image of Ψ are saturated.

iv_4) $\Psi(Ann\mathcal{Q}) = Ann\Psi(\mathcal{Q})$

Proof. Let us verify that ψ is a map. If $deg(\widetilde{p}+i) = deg(\widetilde{p})$, then $deg\ i \leq deg\widetilde{p}$, so that $(\widetilde{p}+i)^* = \widetilde{p}^* + i^*Y_0{}^{(deg\ \widetilde{p}-deg\ i)}$ and, as a consequence, $(\widetilde{p}+i)^* + I^* = \widetilde{p}^* + I^*$.

The map ψ is injective. In fact, if $\widetilde{p}^*+I^* = \widetilde{q}^*+I^*$, then $\widetilde{p}^*-\widetilde{q}^* \in I^*$; moreover, $\widetilde{p}-\widetilde{q} = (\widetilde{p}^*)_*-(\widetilde{q}^*)_* = (\widetilde{p}^*-\widetilde{q}^*)_*$. As a consequence, $\widetilde{p}-\widetilde{q} \in I$.

i) $Im\ \psi$ is a set of forms, not divisible by y_0, as $\widetilde{p}^*$ is not in $(Y_0)R^*$ and $\widetilde{p}$ is of minimal degree in its coset. In fact, if $\widetilde{p}^* = Y_0^h q+j$, $j \in I^*$, $h > 0$, $deg\ \widetilde{p}^* = deg\ j = deg\ q+h$, then $\widetilde{p} = q_*+j_*$,where $deg\ q_* < deg\ \widetilde{p}$, impossible, for the choice of $\widetilde{p}$. Viceversa, every form $f+I^*$, not in $(y_0)A^*$, comes from f_*+I. In fact, let us suppose the existence of $i \in I$ such that $deg\ (f_*+i) = deg\ (f_*)-h$, $h > 0$; then $(f_*+i)^* = ((f_*)^*+i^*)Y_0{}^{-h}$, or, equivalently, $f+i^* = (f_*+i)^*Y_0{}^h \in (Y_0)R^*$, a contradiction.

ii) Let us suppose $D(p+I) = deg\ p$ and $D(q+I) = deg\ q$, or, equivalently, p and q of minimal degree in their cosets. With this choice, we have:

$$\psi(p+I) = p^*+I^*, \qquad \psi(q+I) = q^*+I^*.$$

As a consequence:

$$\psi(p+I)\psi(q+I) = p^*q^*+I^* = (pq)^*+I^* = \psi(pq+I)y_0^h,$$

thanks to $viii$) in Recall 1.0. Analogously:

$$\psi(p+q+I) = ((p+q)^*+I^*)y_0^{-k} = ((p^*+q^*y_0^{d-d'})+I^*)y_0^{-h}$$

$$\psi(p+q+I)y_0^h = p^*+I^*+(q^*+I^*)y_0^{d-d'} = \psi(p+I)+(\psi(q+I))y_0^{d-d'}.$$

The assertion about h comes immediately from i).

iii) This assertion is a consequence of Definition 1.1.

iv_1) The set $\{\widetilde{q}^*+I^*, q \in Q\}$ generates $\Psi(\mathcal{Q})$; hence $\widetilde{q}^*+I^* \in Q^*/I^*$ implies $\Psi(\mathcal{Q}) \subseteq Q^*/I^*$. Viceversa, a set of generators for Q^*/I^* is

$\{q^* + I^*, q \in Q\}$, where $q^* + I^* = (\widetilde{q}^* + I^*)y_0^h$, for some $h \geq 0$, so that $q^* + I^* \in \Psi(\mathcal{Q})$, which proves the other inclusion.

Thanks to iv_1), we have: $A^*/\Psi(\mathcal{Q}) \simeq S/Q^*$; so, taking into account that Y_0 is regular for S/Q^*, we conclude that it is also regular for $A^*/\Psi(\mathcal{Q})$. Viceversa, let $\mathcal{C}$ be any homogeneous ideal of A^* and C its lifting in S. If Y_0 is regular for $A^*/\mathcal{C} \simeq S/C$, then, for the regularity of Y_0, $(C_*)^* = C$, so that $\Psi(C_* + I/I) = C/I^* = \mathcal{C}$.

iv_2) Thanks to iv_1), Ψ preserves the height and the primary decomposition of ideals, because the same property holds for the homogenization in a polynomial ring ([17], cap.VII, Th. 17).

iv_3) This is a consequence of the property of regularity of Y_0 with respect to $S/\Psi(\mathcal{Q})$; in fact the irrelevant ideal contains Y_0 and, as a consequence, it is not associated to $\Psi(\mathcal{Q})$.

iv_4) Let $a \in Ann\mathcal{Q}$; then $\psi(a)\psi(q) = 0$ for every $q \in \mathcal{Q}$, as $\psi(a)\psi(q) = \psi(aq)y_0^h = \psi(0)y_0^h = 0$. Viceversa, let us consider a form $\alpha \in Ann\Psi(\mathcal{Q})$; then there exist $a \in A, h \in \mathbf{N}$ such that $\alpha = \psi(a)y_0^h$ and $\psi(a)y_0^h\psi(q) = 0$, for every $q \in \mathcal{Q}$. As a consequence, $\psi(aq)y_0^k = 0$, for some $k \in \mathbf{N}$ and, by the regularity of y_0, $\psi(aq) = 0$; as ψ is injective, this implies $aq = 0$, or, equivalently, $a \in Ann(\mathcal{Q})$. □

Remark-Notation 1.3. Let $a = p + I$; then $\psi(a) = (p^* + I^*){y_0}^{-h}$, where h is the maximal power of y_0 dividing $p^* + I^*$. We will denote: $a^* = \psi(a)$ and call it the homogenization of a (as an element of A, with respect to y_0). Analogously, we will denote $\Psi(\mathcal{C}) = \mathcal{C}^*$.

From now on, we will be interested in 0-dimensional schemes, so that **X** will be always supposed 0-dimensional.

Let us recall the following basic property of an artinian ring (see [1], Th.8.7):

Proposition 1.4. *Let $A = R/I$ be a zero dimensional ring and $M_1, ..., M_t$ the maximal ideals of R associated to I. There is a canonical isomorphism*

$$\phi : A \longrightarrow A_1 \oplus ... \oplus A_t, \quad A_j = R/I_j \simeq A_{\mathcal{M}_j},$$

defined by:

$\phi(\alpha + I) = (\alpha + I_1, ..., \alpha + I_t)$ *, with I_j the contraction to R of the extension of I to R_{M_j}.*

Remark. The ideals $I_1, ..., I_t$ are just the primary ideals appearing in the primary decomposition of I, with $\sqrt{I}_j = M_j$, so that I_j/I, $j = 1, ..., t$, are the ideals appearing in the primary decomposition of (0) in A.

Proposition 1.4 can be translated into the projective situation, but the statement is less impressive. We will apply it in paragraph 4.

Proposition 1.5. *The morphism:*
$\phi^* : A^* \longrightarrow \oplus_{j=1}^t A_j^*$, *where* $A^* = S/I^*$, $A_j^* = S/I_j^*$,
defined by
$\phi^*(P + I^*) = (P + I_1^*, ..., P + I_j^*, ..., P + I_t^*)$,
is injective.
The morphism ϕ^* is not surjective; however, for every $u \in \oplus_{j=1}^t A_j^*$, there is a natural integer n_u such that $y_0^n u \in Im\ \phi^*$ iff $n \geq n_u$. In particular, if $u_j = (0, ..., 1 + I_j, ..., 0)$, $n_{u_j} = m_j$ is the minimal degree of i_j in a relation of the kind: $1 = i_j + V_j$, $\quad i_j \in I_j$, $V_j \in \cap_{t \neq j} I_t$.

Proof. Let us use the notation of Proposition 1.4.

The injectivity of ϕ is equivalent to $\bigcap_{j=1}^{t} I_j = (0)$. This implies $\bigcap_{j=1}^{t} I_j^* = (0)$, which is equivalent to the injectivity of ϕ^*. On the contrary, $\bigcup_{j=1}^{t} I_j^*$ is contained in the irrelevant ideal, so that ϕ^* is not surjective. However, the surjectivity of ϕ is equivalent to say that $1 = i_j + V_j$, $\quad i_j \in I_j$, $\quad V_j \in \cap_{h \neq j} I_h$, for every j.

Let us denote m_j the minimal degree of i_j in a relation of the type just described. In this situation, homogenizing $1 - i_j$, we get:
${Y_0}^{m_j} - i_j^* \in \bigcap_{h \neq j} I_h^*$ and, as a consequence,
$\phi^*(({Y_0}^{m_j} - i_j^*) + I^*) = (0, \ldots, 0, {Y_0}^{m_j} + I_j^*, 0, \ldots, 0) = u_{m_j, j}$.

Moreover, if $f + I^*$ is a form such that $\phi^*(f + I^*) = u_{n,j}$, then there is $a_j \in I_j^*$, such that $deg\ a_j = n$, $f = {Y_0}^n - a_j \in \bigcap_{t \neq j} I_t^*$. By dehomogenization we get: $1 - (a_j)_* \in \bigcap_{h \neq j} I_h$, $deg(a_j)_* \leq n$; as a consequence, $n \geq m_j$.

Finally, it is enough to show that for every $p_j = (0, \ldots, 0, P + I_j^*, 0, \ldots, 0)$, there is an integer $m_{p_j} \leq m_j$ such that ${y_0}^n p_j \in Im\ \phi^*$ iff $n \geq m_{p_j}$. In fact we have: $\phi^*(P({Y_0}^{m_j} - i_j^*) + I^*) = \phi^*(P + I^*)\phi^*(({Y_0}^{m_j} -$

$i_j^*) + I^*) = (0, \dots, 0, P{Y_0}^{m_j} + I_j^*, 0, \dots, 0) = p_j {y_0}^{m_j}$. □

Working directly in the projective situation, we can restate Proposition 1.5 as follows.

Proposition 1.6. *Let $B = K[Y_0, \dots, Y_n]/J = K[y_0, \dots, y_n]$ be an arithmetically Cohen-Macaulay graded ring of dimension 1, L any regular linear form of B and $(0) = \mathcal{Q}_1 \cap \dots \cap \mathcal{Q}_t$ a primary decomposition. The morphism $\chi : B \longrightarrow \oplus_{j=1}^t B_j$, $B_j = B/\mathcal{Q}_j$, defined by $\chi(b) = (b + \mathcal{Q}_1, ..., b + \mathcal{Q}_j, ..., b + \mathcal{Q}_t)$, satisfies the following properties:*

$i)$ χ is injective

$ii)$ For each j, there exists an integer m_j such that:

$$(0, \dots, 0, L^n + \mathcal{Q}_j, 0, \dots, 0) \in Im\chi \text{ iff } n \geq m_j.$$

2. Separators and separating ideals.

Let A be a 0-dimensional ring, as in the last part of paragraph 1 and $(0) = \mathcal{Q}_1 \cap ... \cap \mathcal{Q}_t$ a primary decomposition of (0), where $\mathcal{M}_i = \sqrt{\mathcal{Q}_i}$, $i = 1, ..., t$, are the maximal ideals of A.

Lemma 2.1. *Let s be any element of A and s^i its canonical image in $A_{\mathcal{M}_i}$. Then s belongs to $Ann(\mathcal{M}_i)$ iff the following conditions are satisfied:*

$i)$ $s \in \mathcal{Q}_j$, $j \neq i$.

$ii)$ $s^i \in Ann\ \mathcal{M}_i A_{\mathcal{M}_i}$.

As a consequence, any $s \in Ann(\mathcal{M}_i)$ lies in $\mathcal{M}_j$, $j \neq i$; it lies also in $\mathcal{M}_i$ iff $dim_K A_{\mathcal{M}_i} > 1$.

Proof. Let us consider the canonical isomorphism

$$\phi : A \longrightarrow A_{\mathcal{M}_1} \oplus \dots \oplus A_{\mathcal{M}_t},$$

defined by:

$$\phi(s) = (s^1, \dots, s^t).$$

As $\phi(\mathcal{M}_j) = A_{\mathcal{M}_1} \oplus \dots \oplus \mathcal{M}_j A_{\mathcal{M}_j} \oplus \dots \oplus A_{\mathcal{M}_t}$, we have that $\phi(s)\phi(\mathcal{M}_i) = 0$ iff

$$(*) \qquad s^j = 0,\ j \neq i, \quad s^i \mathcal{M}_i A_{\mathcal{M}_i} = 0.$$

Taking into account that $A_{\mathcal{M}_j} \simeq A_j = A/\mathcal{Q}_j$, these conditions are the same as $i)$ and $ii)$.

Now, let us observe that:

$s \in \mathcal{Q}_j \Rightarrow s \in \mathcal{M}_j$, so that $s \in Ann(\mathcal{M}_i)$ implies $s \in \mathcal{M}_j$, $j \neq i$.

Finally: $s^i \mathcal{M}_i A_{\mathcal{M}_i} = 0 \Rightarrow s^i \in \mathcal{M}_i A_{\mathcal{M}_i}$ iff $A_{\mathcal{M}_i} \neq K$.

In fact, $s^i \notin \mathcal{M}_i A_{\mathcal{M}_i}$ implies s^i invertible in $A_{\mathcal{M}_i}$, so that the equality holds iff $\mathcal{M}_i A_{\mathcal{M}_i} = 0$, which is equivalent to say that $A_{\mathcal{M}_i}$ is a field. Viceversa, if $A_{\mathcal{M}_i} = K$, $s^i \mathcal{M}_i A_{\mathcal{M}_i} = 0$ is true for any s^i and not only for $s^i = 0$. Moreover, $s^i \in \mathcal{M}_i A_{\mathcal{M}_i}$ is equivalent to $s \in \mathcal{M}_i$. □

Remark 2.2. The equivalence between $s \in Ann(\mathcal{M}_i)$ and condition $(*)$ implies:

$Ann(\mathcal{M}_i A) \cap Ann(\mathcal{M}_j A) = (0),\ i \neq j$.

Theorem 2.3. *For $s \in A$, the following facts are equivalent:*

$i)$ $dim_K(A/(s)A) = dim_K A - 1$;

$ii)$ $(s)A = (s)K,\ s \neq 0$;

$iii)$ there exists a maximal ideal $\mathcal{M}_i \subset A$ such that $s \in Ann(\mathcal{M}_i) - \{0\}$;

$iv)$ $Ann((s)A)$ is a maximal ideal of A.

Proof. $i) \Leftrightarrow ii)$: $(s)K \neq (s)A \Leftrightarrow dim_K(s)A > 1 \Leftrightarrow dim_K A/(s)A < dim_K A - 1$.

To prove the equivalence between $ii)$ and $iii)$, we will reduce the problem to the local case.

As a consequence of the isomorphism ϕ considered in Lemma 2.1, we can restate $ii)$ and $iii)$ equivalently as follows.

$ii)$: $dim_K(s)A = 1 \Leftrightarrow \exists i, (s^i)A_{\mathcal{M}_i} = (s^i)K,\ (s^j)A_{\mathcal{M}_j} = 0,\ j \neq i \Leftrightarrow \exists i, (s^i)A_{\mathcal{M}_i} = (s^i)K,\ s^j = 0, j \neq i$;

$iii)$: $s \neq 0,\ s\mathcal{M}_i = (0) \Leftrightarrow \phi(s) \neq 0,\ \phi(s)\phi(\mathcal{M}_i) = (0) \Leftrightarrow s^j = 0, j \neq i,\ s^i \in Ann(\mathcal{M}_i A_{\mathcal{M}_i}),\ s^i \neq 0$. In other words: $Ann\ \phi(\mathcal{M}_i) = (\oplus_{j=1}^{i-1}(0_j) \oplus Ann(\mathcal{M}_i A_{\mathcal{M}_i}) \oplus_{j=i+1}^{t} (0_j)$.

So the proof of the equivalence between $ii)$ and $iii)$ is reduced to the local case and we suppose: $A = K[x_1, \dots, x_n]$ to be a local ring of dimension zero and $\mathcal{M} = (x_1, \dots, x_n)$.

$iii) \Longrightarrow ii)$ Let $s \in Ann\mathcal{M} - \{0\}$, $a = a_0 + \sum x_i p_i, a_0 \in K$ any element of A. Then $sa = sa_0$, so that $(s)A = (s)K$.

$ii) \Longrightarrow iii)$ It is enough to prove that $sx_i = 0, i = 1, ..., n$. The hypothesis implies $sx_i = a_i s,\ a_i \in K,\ i = 1, ..., n$ and, as a consequence, $s(x_i - a_i) = 0$. If $a_i \neq 0$, then $x_i - a_i$ is invertible and $s = 0$, a contradiction.

$ii) \Longrightarrow iv)$ Let $b \in A - Ann(s)$. Such a condition and $ii)$ say that $bs = \lambda s$, for a suitable $\lambda \in K - \{0\}$; moreover, for any $c \in A$, there exists $\mu \in K$ such that $sc = \mu s = \mu\lambda^{-1}bs$; this implies: $c - \mu\lambda^{-1}b \in Ann(s)$, so that $c \in (Ann(s), b)$. As a consequence, $Ann(s)$ is a maximal ideal.

$iv) \Longrightarrow ii)$ Any $c \in A$ can be written as: $c = n + \lambda$, $n \in Ann(s), \lambda \in K$, as K is supposed algebraically closed. As a consequence: $sc = s(n + \lambda) = s\lambda$. □

Using Remark 2.2, we immediately see that the maximal ideal in $iii)$ is necessarily unique. So, we give the following:

Definition 2.4. *An element s, satisfying the equivalent conditions of Theorem 2.3, will be called a separator of A or, more precisely, an $\mathcal{M}_i$ - separator of A, to point out that it is a separator with respect to the maximal ideal $\mathcal{M}_i$. Its corresponding principal ideal $(s) = (s)A = sK$ will be called separating ideal. Moreover, we call degree of (s) and denote $D(s)$ the degree Ds of s see Definition 1.1, which turns out to be independent of the generator chosen in the ideal.*

If A is the coordinate ring of a scheme **X**, *then s will also be called a separator of* **X** *and (s) a separating ideal of* **X**.

Lemma 2.1 can now be restated as follows:

Proposition 2.5. *$i)$ Let $\mathcal{M}_1, \cdots, \mathcal{M}_t$ be the maximal ideals of A. An element $s \in A$ is a separator in A iff there exists a maximal ideal $\mathcal{M}_h$ such that s becomes a separator s^h in $A_{\mathcal{M}_h}$, while it becomes zero in $A_{\mathcal{M}_j} = A/\mathcal{Q}_j$, $j \neq h$.*

$ii)$ If s is an $\mathcal{M}_i$-separator, then $s \in \mathcal{M}_j$, $j \neq i$. Moreover, $s \notin \mathcal{M}_i$ iff $dim_K A_{\mathcal{M}_i} = 1$.

Proposition 2.6. *Every ring $A = K[x_1, \ldots, x_n]$, with Krull dimension zero, has at least one separator.*

Proof. We have: $Ann(\mathcal{M}_i A) \neq (0)$, for any maximal ideal $\mathcal{M}_i$, as the hypothesis on the dimension implies $(0) = \cap \mathcal{Q}_i$ where $\sqrt{\mathcal{Q}_i} = \mathcal{M}_i$; so there exists $s \neq 0$ such that $s\mathcal{M}_i = (0)$.

When the scheme **X** is reduced, or, equivalently, A is a regular ring, the concept of separator has a more precise geometrical meaning: a separator is the image, in A, of a polynomial representing a hypersurface passing through all the points, but one; it corresponds to a hypersurface separating that point from all the others. In this situation, Theorem 2.3 can be completed as follows:

Proposition 2.7. *Let A be a regular ring of Krull dimension zero, with $dim_K A = r$, and $\mathbf{X} = \{P_1, \dots, P_r\}$ its corresponding scheme. Then $s \in A$ is a separator iff one of the following equivalent conditions holds:*

iii') $(s)A$ is the annihilator of a maximal ideal of A;

v) $\exists i \in \{1, ..., r\}$ such that $s \in \cap_{j\neq i}\mathcal{M}_j$, $s \notin \mathcal{M}_i$;

vi) $(s)A = \mathcal{I}(P_1, \dots, \check{P}_i, \dots, P_r)A$ for some i.

Proof. Coming back to the proof of Lemma 2.1, we observe that, in the canonical isomorphism ϕ, the codomain is $A_{\mathcal{M}_1} \oplus \dots \oplus A_{\mathcal{M}_r} \simeq \oplus_{i=1}^{r} K_i$, $K_i \simeq K$, so that

$$\phi((s)A) = \oplus_{j=1}^{r}(s^j)K_j, \qquad \phi(\mathcal{M}_i) = \oplus_{u=1}^{i-1}K_u \oplus (0) \oplus_{v=i+1}^{r} K_v.$$

As a consequence, $Ann\ \phi(\mathcal{M}_i) = \oplus_{u=1}^{i-1}(0_u) \oplus K_i \oplus_{v=i+1}^{r} (0_v)$ turns out to be a principal ideal, generated by any of its non zero elements. So, the equivalence between (ii) and (iii) in Theorem 2.3 gives rise to the equivalence between ii) and iii').

v) $\Longrightarrow$ i) The maximal ideals of $A/(s)A$ are the images of the maximal ideals of A, but $\mathcal{M}_i$, so their number is $r - 1$.

i') $\Longrightarrow$ v) The hypothesis is equivalent to say that the number of the maximal ideals of $A/(s)A$ is $r - 1$; as a consequence, s is contained in all but one maximal ideals of A.

vi) $\Leftrightarrow$ v) It is enough to observe that

$$\mathcal{I}(P_1, \dots, \check{P}_i, \dots, P_r)A = \bigcap_{j\neq i}\mathcal{M}_j \simeq \bigcap_{j\neq i}\phi(\mathcal{M}_j) = Ann\phi(\mathcal{M}_i) \simeq Ann\mathcal{M}_i.$$

□

Remark 2.8. There is a $1 - 1$ correspondence between the points of the reduced scheme $\mathbf{X}$ and the separating ideals of A. Such a correspondence arises as follows:

$$P_i \leftrightarrow \mathcal{M}_i \leftrightarrow Ann(\mathcal{M}_i) = (s)K.$$

This fact allows us to replace a point by a separating ideal, that is by a principal ideal, which is also a 1-dimensional K-space; the generator of the ideal is a separator of A, defined by the point up to a multiplicative constant.

A separating ideal (s) corresponding to P_i will also be called a $P_i - separator$.

Taking into account Definition 1.1, we get:

Remark 2.9. If (s) is a P_i-separator of the reduced scheme $\mathbf{X}$, then $D(s)$ is the minimum degree of the hypersurfaces containing all the points of $\mathbf{X}$, but P_i.

Example 2.10.

1. Let us consider $A = K[X, Y]/(XY, Y(Y-1), X(X-1)(X-2)) = K[x, y]$.

A is the affine ring of the reduced scheme $\mathbf{X}$, consisting of the points:

$$P_1(0,0),\ P_2(1,0),\ P_3(2,0),\ P_4(0,1).$$

Indeed its maximal ideals are: $\mathcal{M}_1 = (x, y), \mathcal{M}_2 = (x-1, y), \mathcal{M}_3 = (x-2, y), \mathcal{M}_4 = (x, y-1)$. It is immediate to check that a K-basis for A is $\mathcal{B} = (1, x, y, x^2)$.

Let us compute $Ann\mathcal{M}_1$.

$(a + bx + cy + dx^2)x = 0 \Leftrightarrow ax + bx^2 + d(3x^2 - 2x) = 0 \Leftrightarrow a = 2d, b = -3d$

$(2d - 3dx + cy + dx^2)y = 0 \Leftrightarrow 2dy + cy^2 = 0 \Leftrightarrow 2dy + cy = 0 \Leftrightarrow c = -2d$

As a consequence: $Ann(\mathcal{M}_1) = (2 - 3x - 2y + x^2) = ((x + y - 1)(x - 2))$.

Analogously, we see that:

$Ann\mathcal{M}_2 = (x(x-2)), \quad Ann\mathcal{M}_3 = (x(x-1)), \quad Ann\mathcal{M}_4 = (y)$.

So, there is a $1 - 1$ correspondence between points and separating ideals: $P_i \longleftrightarrow Ann(\mathcal{M}_i) = (s_i)$. Moreover, $D(s_1) = D(s_2) = D(s_3) = 2,\ D(s_4) = 1$ means that each $P_i, i = 1, 2, 3$, is separated by the others three points by a conic, while P_4 is separated by a line.

Let us observe that, in this example, A is a ring of four points, three on a line; so $Ann(\mathcal{M}_i)$ can be computed more directly, as the principal ideal corresponding to any curve of minimal degree through the three points different from P_i. The previous computation points out a technique utilizable in any 0-dimensional, possibly non reduced, ring.

2. Let us consider the ring of a scheme of degree 2, supported at one point.

$A = K[X]/(X^2) = K[x]$ has as a K-basis $\mathcal{B} = (1, x)$. Its unique maximal ideal is $\mathcal{M} = (x)$ and $Ann\mathcal{M} = (x)$. So, there is only one separating ideal and its degree is 1.

3. Let us consider two different rings associated to a scheme of degree

3, supported at one point.

$A_1 = K[X,Y]/(X^2, XY, Y^2) = K[x,y]$ has as a K-basis $\mathcal{B} = (1, x, y)$ and its unique maximal ideal is $\mathcal{M} = (x, y)$. In this case, $Ann\mathcal{M} = (x, y)$. So, there are infinitely many separating ideals, all of degree 1.

$A_2 = K[X]/(X^3) = K[x]$ has a K-basis $\mathcal{B} = (1, x, x^2)$ and its unique maximal ideal is $\mathcal{M} = (x)$. Moreover, $Ann(\mathcal{M}) = (x^2)$. So, there is only one separating ideal and its degree is 2.

Now, let us pass to the projective situation.

Definition 2.11. A form $f \in A^*$ is a separator of A^* iff $f = \psi(s)$, where s is a separator of A.

Theorem 2.12. *Let $f \in A^*_d$. The following facts are equivalent:*

i) f is a separator

ii) 1) $\exists\, t_0 \geq d,\ dim_K((f)A^*)_{t_0} = 1$,
2) the regular $1-$form y_0 does not divide f

iii) 1') $dim_K((f)A^*)_t = 1,\ t \geq d$
2') any regular form g of A^*, with positive degree, cannot divide f

iv) $f \in Ann\mathcal{P}^*_i \cap (A^* - (y_0)A)$, where $\mathcal{P}^*_i$ is a prime ideal associated to (0).

Proof. $i) \Rightarrow ii)$ Let $f = \psi(s),\ s = \widetilde{p} + I$, so that $f = \widetilde{p}^* + I^*$ and let $g = q + I^*$ be any form of degree δ in A^*. Then fg is a form of degree $d+\delta$ in A^* and $fg = (\widetilde{p}^* + I^*)(q + I^*) = \widetilde{p}^* q + I^*$. By the hypothesis on s, $(\widetilde{p}^* q)_* = \widetilde{p} q_*$ is of the form $\lambda\widetilde{p} + i,\ \lambda \in K, i \in I$. As a consequence, $\widetilde{p}^* q = (\lambda\widetilde{p} + i)^* {Y_0}^{d+\delta-deg(\widetilde{\lambda p+i})}$.

The choice of $\widetilde{p}$ as an element of minimal degree in its coset implies that:

$$(\lambda\widetilde{p} + i)^* = \lambda\widetilde{p}^* + {Y_0}^{deg\widetilde{p} - deg i} i^*, \text{ if } deg\ i \leq deg\ \widetilde{p};$$

$$(\lambda\widetilde{p} + i)^* = \lambda\widetilde{p}^* {Y_0}^{deg i - deg\widetilde{p}} + i^*, \text{ if } deg\ i > deg\ \widetilde{p}.$$

As a consequence: $\widetilde{p}^* q + I^* = \lambda\widetilde{p}^* {Y_0}^h + I^*$, for some h; that means: $fg = \lambda f x_0^h$.

Indeed, we proved $iii)$ 1'), which is stronger than $ii)$ 1).

Finally, the regular 1-form y_0 does not divide f, thanks to Lemma 1.2 $i)$.

$ii) \Rightarrow iii)$ Let us prove that 1) + 2) $\Rightarrow$ 1').

The hypothesis says that, for any form g of degree $\delta \leq\ h = t_0 - d$, we have: $f((y_0)^{t_0-\delta-d} g) = c{y_0}^{t_0-\delta} f, c \in K$; by the regularity of y_0, we conclude $fg = c{y_0}^d f$. Now, to deal with the case $t = t_0 + 1$, it is enough to consider the products of f with any monomial of degree $h + 1$. We have: $f y_{i_1} \dots y_{i_h} y_{i_{h+1}} = c\ f{y_0}^h y_{i_{h+1}} = c\ c' f{y_0}^{h+1}$. To conclude, we use induction on $t - t_0 \geq 1$.

Finally, we need to prove that $1) + 2) \Rightarrow 2')$.

Let us denote $g \in A^*_\delta$, $\delta > 0$ a regular form such that $f = gq$. Condition 1) says that, for every form $r \in A^*_\tau$, $\tau \geq 0$, we have: $gqr = \lambda gq{y_0}^\tau$, $\lambda \in K$. As g is regular, that relation implies $qr = \lambda q{y_0}^\tau$, $\lambda \in K$. As a consequence, choosing $r = g$, we get $f = qg = \lambda q{y_0}^\delta$, a contradiction.

$iii) \Rightarrow ii)$ Obvious.

$iii) \Rightarrow i)$ Condition $2')$ and Lemma 1.2 guarantee that there exists $s \in A$ such that $f = \psi(s)$; now we prove that s is a separator in A. If $t \in A$, Lemma 1.2 says that: $\psi(st) = \psi(s)\psi(t){y_0}^{-h} = f\psi(t){y_0}^{-h}$. Now, condition $iii)$ 1') implies that $f\psi(t) = \lambda f{y_0}^\mu$, $\lambda \in K$; moreover, $\psi(st)$ is not divisible by y_0, so that we conclude $\psi(st) = \lambda f$. By the injectivity of ψ, we can conclude $st = \psi^{-1}(\lambda f) = \lambda\psi^{-1}(f) = \lambda s$.

$iv) \Leftrightarrow i)$ Condition $iii)$ of Theorem 2.3 means that f is a separator iff $f = \psi(s)$, $s \in Ann\mathcal{M}_i$, where $\mathcal{M}_i$ is a prime ideal associated to $(0) \subset A$. Lemma 1.2 $iv_2)$ says that if $(0) = \mathcal{Q}_1 \cap \dots \cap \mathcal{Q}_t$, $\sqrt{\mathcal{Q}_i} = \mathcal{M}_i$, is a primary decomposition of $(0) \subset A$, then $(0) = \mathcal{Q}^*_1 \cap \dots \cap \mathcal{Q}^*_t$, $\sqrt{\mathcal{Q}^*_i} = \mathcal{M}^*_i = \mathcal{P}_i$, is a primary decomposition of $(0) \subset A^*$. As a consequence, $(Ann\mathcal{M}_i)^* = Ann(\mathcal{P}^*_i)$, thanks to Lemma 1.2 $iv_4)$. □

Remark 2.13.

1. Condition 1) of $ii)$ and $1')$ of $iii)$ can be restated in terms of Hilbert functions (see [10], [8], [13], [15],) as follows:

1) $\exists\ t_0 \geq d$, $H(A^*/(f), t_0) = H(A^*, t_0) - 1$,

$1')$ $H(A^*/(f), t) = H(A^*, t) - 1$, if $t \geq d$.

2. Let us suppose that condition 1) of $ii)$ holds, but condition 2) is not necessarily satisfied. If we set $f = F + I^*$, then $f_* = F_* + I$ is still a separator in A, but it may happen that F_* is not an element of minimal degree in f_*; in such a case, we have $f = {y_0}^h (f_*)^*$, where h is the maximal degree of y_0 dividing $f \in A^*$ and $(f_*)^*$ is a separator.

3. Conditions $iii)$ shows that the notion of separator in A^* is independent from the regular element y_0 used to produce A.

The principal ideal generated by a separator will still be called *separating ideal.*

4. Translating Remark 2.8 into the projective situation, we can say that, if the scheme $\mathbf{X}$ is reduced, there is a $1-1$ correspondence between points of $\mathbf{X}$ and separating ideals.

Proposition 2.14. *If f is a non invertible separator in A^*, then (f) is a saturated ideal of A^*.*

Proof. It is enough to observe that $f = (f_*)^*$, so that $(f) = \Psi(f_*)$ and use Lemma 1.2 $iv_3)$. □

Remark 2.15. The map ψ induces a $1-1$-correspondence between separators in A and separators in A^*, that preserves the degree (see Lemma 1.2 $iii)$).

The following proposition unifies the notion of separator in the affine situation and the analogous notion in the projective one.

Proposition 2.16. *Let C be either a 0-dimensional, finitely generated K-algebra (affine situation) or a graded 1-dimensional, arithmetically Cohen-Macaulay, finitely generated K-algebra (projective situation) and let $(0) = \mathcal{Q}_1 \cap \ldots \cap \mathcal{Q}_t$, $\sqrt{\mathcal{Q}_h} = \mathcal{P}_h$, be the primary decomposition of* (0).

If s is a separator of C, there exists $h \in \{1, ..., t\}$ such that:

$$(*) \qquad s\mathcal{P}_h = (0).$$

Moreover, such a condition implies $s \in \mathcal{Q}_j,\ j \neq h$.

In the affine situation, a separator is completely characterized by condition $(*)$. In the projective situation, s is a separator iff condition $(*)$ and, in addition, one of the following (equivalent) conditions holds:

$i)$ any regular linear form cannot divide s;

$i')$ there exists a regular linear form not dividing s.

Proof. In the affine case, Theorem 2.3 $iii)$ says that s is a separator iff $s \neq 0$ and $\exists h \in \{1, ..., t\}$ such that $s\mathcal{P}_h = (0)$. That clearly implies $s \in \mathcal{Q}_j,\ j \neq h$.

To deal with the projective case, we observe that, if L is any regular linear form, by dehomogenizing and then homogenizing (through ψ) with respect to L the primary decomposition of (0), the property of annihilating a prime associated to (0) and the property of being a separator are preserved. So, it is enough to use the equivalence between $i)$ and $iv)$ of Theorem 2.12. □

3. Separating sequences.

Let

$$(s_1)B \subseteq (s_1, s_2)B \subseteq (s_1, s_2, s_3)B \subseteq \ldots \subseteq (s_1, \ldots, s_i)B$$

be a chain of ideals in a commutative ring B.

We will denote it by means of the sequence: $S = (s_1, s_2, \ldots, s_i)$.

Let us observe that S is not uniquely defined, as each s_j can be replaced by $\sum_{k=1}^{j} \lambda_k s_k$, λ_j invertible in B.

In the case in which $s_j \in (s_1, \ldots, s_{j-1})$, we agree to replace it by 0, chosen as a privileged representative.

Remark 3.1. The relation "$S \sim T$ *iff S and T arise from the same chain of ideals*" is clearly an equivalence relation. From now on, we will work with the equivalence class of S, more then with S itself and S will denote any element of its class.

We go on using the notation of paragraph 1.

Starting from the concept of separator, we give the following

Definition 3.2. $S = (s_1, \ldots, s_i)$ is an *i-separating sequence* in A, or equivalently,

$$(s_1)A \subseteq (s_1, s_2)A \subseteq (s_1, s_2, s_3)A \subseteq \ldots \subseteq (s_1, \ldots, s_i)A$$

is an *i-separating chain*, iff:

$i)$ s_1 is a separator in A

$ii)$ $\bar{s}_j$, $1 < j \leq i$ is a separator in $A_j = A/(s_1, \ldots, s_{j-1})A$.

Obviously, if $S \sim T$, then S is an i-separating sequence iff T is so.

If $dim_K A = r$, an r-separating sequence (chain) shall also be called *A-separating sequence (chain)*.

If we do not need to point out the number of elements of S, we simply say: *separating sequence (chain)*.

If A is the affine coordinate ring of **X**, we will also say that S is a separating sequence of **X**.

It is immediate to verify that:

Proposition 3.3. $S = (s_1, \ldots, s_i)$ *is an i-separating sequence in A iff*

$$dim_K(s_1, \ldots, s_j)A = j, \; j = 1, ..., i.$$

Analogous definitions can be given in the projective situation, where the sequence $S = (f_1, \dots, f_i)$ consists of forms. More explicitly:

Definition 3.4. The sequence of forms $(f_1, \dots, f_i)$ is an i-separating sequence in A^* iff:
$i)$ f_1 is a separator in A^*;
$ii)$ $\bar{f_j}$ is a separator in $A^*/(f_1, \dots, f_{j-1})A^*$, $j = 2, \dots, i$.

Let us observe that, if y_0 is regular for A^* and f is a separator, then y_0 is regular also for $A^*/(f)$, so that the given definition is consistent. In fact, the primes associated to (f, I^*) are some of the primes associated to I^*, so that Y_0 is still outside their union.

Proposition 3.5. *The sequence $(f_1, \dots, f_i)$ is an i-separating sequence in A^* iff:*
$i)$ $dim_K((f_1, \dots, f_j)A^*)_d = j,\ d >> 0,\ j = 1, ..., i$
$ii)$ $f_1 \notin (y_0)$*, and* y_0 *does not divide any element of the coset* $(f_j + (f_1, \dots, f_{j-1})A^*),\ j = 2, ..., i.$

Proof. If $i = 1$, the two conditions become those stated in Theorem 2.12 $ii)$; then we go on by induction on i. □

Let us observe that the definition of equivalence given in the affine situation extends naturally to the projective one. Moreover, $S \sim T$ still implies that S is an i-separating sequence iff T is so.

Remark 3.6. If $(f_1, \dots, f_i)$ is an i-separating sequence in A^*, then $((f_1)_*, \dots, (f_i)_*)$ is an i-separating sequence in A. However, if $(s_1, \dots, s_i)$ is an i-separating sequence in A, $(s_1^*, \dots, s_i^*)$ is not necessarily an i-separating sequence in A^*, because condition $ii)$ of Proposition 3.5 could fail (see example 3.9,1.).

Definition 3.7. To any i- separating sequence $S = (s_1, \dots, s_i)$ in A (resp. $S = (f_1, \dots, f_i)$ in A^*) we associate a numerical sequence $\delta(S) = (\delta_1, \dots, \delta_i)$, where $\delta_1 = Ds_1$, $\delta_j = D\bar{s}_j$, $\bar{s}_j \in A/(s_1, \dots, s_{j-1})A$, $j > 1$ (resp. $\delta_j = \deg \bar{f_j} \in A^*/(f_1, \dots, f_{j-1})A^*$).

If S is an A (resp. A^*)- separating sequence, defined in the coordinate ring of $\mathbf{X}$, we will say that the corresponding numerical sequence $\delta(S)$ is a *realizable sequence* of $\mathbf{X}$ and that S realizes $\delta(S)$.

Remark 3.8. If S is a separating sequence of a reduced scheme $\mathbf{X}$, the corresponding numerical sequence $\delta(S)$ is a realizable sequence of $\mathbf{X}$, according to the definition given in [5], ordered in the opposite way. In fact (see Remarks 2.8 and 2.13,4) S gives rise to an ordering of the points of $\mathbf{X}$, and the corresponding $\delta(S)$ contains, at each step, the minimal degree of the hypersurfaces separating a point from the following ones. If we reverse the order of the points and take, at each step, the minimal degree of the hypersurfaces separating a point from the preceding ones, we get the sequence defined in [5].

Examples 3.9. Let us consider again the rings already introduced in 2.10.

1. Up to equivalence, the separating sequences of this ring are 4!, as they correspond to the permutations of the four points. However, the corresponding numerical sequences are only two: $(2, 1, 1, 0)$ and $(1, 2, 1, 0)$; in fact $\mathbf{X}$ is a k-configuration ([7]), so that its realizable numerical sequences, with the reverse order, are just the ones obtained by direct transpositions from the sequence M_Γ, $\Gamma = (1\ 2\ 1)$ ([4]).

Let us incidentally observe that the same Castelnuovo function arises also from just another kind of schemes, that is the complete intersection of two conics. In this case, the sequence $(1, 2, 1, 0)$ is not allowed, as we can see either directly or taking into account the Caley-Bacharach property ([9]).

Coming back to the scheme $\mathbf{X}$, we see that the numerical sequence $(2, 1, 1, 0)$ arises, for instance, from :

$$S = (2 - 3x - 2y + x^2, x(x - 2), x(x - 1), y),$$

or, equivalently from :

$$S' = (2 - 3x - 2y + x^2, x + 2y - 2, x, 1),$$

while $(1, 2, 1, 0)$ comes from:

$$T = (y, 2 - 3x + x^2, x(x - 2), x(x - 1))$$

or, equivalently, from

$$T' = (y, 2 - 3x + x^2, x - 2, 1).$$

Now let us consider the projective coordinate ring of $\mathbf{X}$,

$$A^* = K[X, Y, Z]/(XY, Y(Y - Z), X(X - Z)(X - 2Z)) = K[x, y, z].$$

Homogenizing the separators of A, we produce all the separators of A^*; they are, up to a multiplicative factor in K:

$$f_1 = 2z^2 - 3xz - 2yz + x^2, \quad f_2 = x(x-2z), \quad f_3 = x(x-z), \quad f_4 = y.$$

The sequence corresponding to S is

$$S^* = (2z^2 - 3xz - 2yz + x^2, x + 2y - 2z, x, 1).$$

Let us observe that S^* can be obtained by homogenizing the elements of S', but not the ones of S. For instance, $X(X-2)$ is an element of minimal degree in the coset $X(X-2)+I$, but it is no more so in $X(X-2)+(I, 2-3X-2Y+X^2)$; as a consequence, in $A/(f_1)$ the element $x(x-2z)$ is a multiple of z, so that it is not a separator.

2. The only separating chain of this ring is $((x), (1))$ and the corresponding numerical sequence is $(1, 0)$.

3. Let us consider A_1. Its separating sequences are infinitely many, as they are all the ones of the form $(L_1, L_2, 1)$, where L_1, L_2 are two independent linear forms; however, they all give rise to the same numerical sequence $(1, 1, 0)$.

Let us consider A_2. The only separating chain is $((x^2), (x), (1))$, with numerical sequence $(2, 1, 0)$.

Theorem 3.10. *Let* $\mathbf{X}$ *be any* 0*-dimensional projective scheme and* $(\delta_1, \ldots, \delta_r)$ *any realizable sequence of* $\mathbf{X}$*. The set* $\mathcal{D} = \{\delta_1, \ldots, \delta_r\}$ *is completely determinate by the Hilbert function* $H(A^*, \cdot)$ *or, equivalently, by its first difference* $\Gamma(A^*, \cdot)$*. In fact,* $\mathcal{D}$ *contains each natural number* t*, repeated* $\Gamma(A^*, t)$ *times.*

Proof. Let $S = (f_1, \ldots, f_r)$ be any A^*-separating sequence in A^*, realizing the numerical sequence $(\delta_1, \ldots, \delta_r)$. We remind that $\delta_i = deg\ \bar{f_i}$, $\bar{f_i} \in A^*/(f_1, \ldots, f_{i-1})A^*$ and $(\bar{f_i})_t = K(\bar{y}_0)^{(t-\delta_i)}\bar{f_i}$, $t \geq \delta_i$.

If $r = 1$, the theorem is true, as $A^* = K[Y_0]$ and, as a consequence, $S = (1)$, $\delta(S) = (0)$, $\Gamma(A^*, 0) = 1$, $\Gamma(A^*, t) = 0$, $t \neq 0$.

Going on by induction on r, let us suppose the theorem true until $r-1$, so that we have:

$\Gamma(A^*/(f_1), t) =$ number of entries of the sequence $(\delta_2, \ldots, \delta_r)$ which are equal to t. Moreover, the exact sequence:

$0 \longrightarrow (f_1) \longrightarrow A^* \longrightarrow A^*/(f_1) \longrightarrow 0$

and the condition that f_1 is a separator, give rise to the relations:

$H(A^*, t) = H(A^*/(f_1), t)$, if $t < \delta_1$,

$H(A^*, t) = H(A^*/(f_1), t) + 1$, if $t \geq \delta_1$,
or, equivalently:

$$(**) \qquad \begin{aligned} \Gamma(A^*, t) &= \Gamma(A^*/(f_1), t), \text{ if } t \neq \delta_1 \\ \Gamma(A^*, \delta_1) &= \Gamma(A^*/(f_1), \delta_1) + 1. \end{aligned}$$

Now, passing from $(f_2, \dots, f_r)$ to $(f_1, f_2, \dots, f_r)$, we add δ_1 to the numerical sequence and $(**)$ says that we increase by 1 the value of Γ at $t = \delta_1$, leaving unchanged the other values. □

Remark 3.11. If **X** is reduced, Th.3.10 becomes Th.2.1 of [4]. We already observed in Remark 3.8 that the numerical sequence $(\delta_1, \dots, \delta_r)$, with the old notation, corresponds to $(d_1, \dots, d_r)$, where $d_j = \delta_{r-j+1}$.

4. Separating sequences of localizations and quotients.

The final aim of this section is to state a relation between the separating sequences of A and the separating sequences of its localizations with respect to its maximal ideals $\mathcal{M}_1, \dots, \mathcal{M}_t$. For that purpose, we will use the canonical isomorphism ϕ of the Artin-rings structure theorem, Proposition 2.5 and a preliminary investigation on the separating sequences of a quotient.

Proposition 4.1. *Let $s \in A$ be an $\mathcal{M}_h$ - separator and $\mathcal{B}$ any ideal of A. Then the image $\bar{s}$ of s in $\bar{A} = A/\mathcal{B}$ is either zero or an $\overline{\mathcal{M}_h}$ - separator of $\bar{A}$. In particular, $\bar{s} = 0$ if $\mathcal{B} \not\subset \mathcal{M}_h$.*

Proof. According to condition $iii)$ of Theorem 2.3, $s\mathcal{M}_h = (0)$. As a consequence, $\bar{s}\overline{\mathcal{M}_h} = (\bar{0})$, so that $\bar{s}$ is a separator in $\bar{A}$, iff $\bar{s} \neq 0$; if $\mathcal{B} \not\subset \mathcal{M}_h$, then $\bar{s}\bar{A} = (\bar{0})$, so that $\bar{s} = (\bar{0})$. □

Corollary 4.2. *If s and t are separators and $(s) \neq (t)$, then (s, t) and (t, s) are 2-separating sequences.*

Proof. It is enough to observe that $\bar{t} \neq 0$ in $A/(s)A$, as (s) and (t) are two different K-spaces of dimension 1. □

Corollary 4.3. *Let $S = (s_1, \dots, s_i)$ be a separating sequence in A and $\mathcal{B}$ any ideal. The image $\pi(S) = (\bar{s}_1, \dots, \bar{s}_i)$ of S in the quotient $\bar{A} = A/\mathcal{B}$, after deleting $\bar{s}_j$ if $\bar{s}_j \in (\bar{s}_1, \dots, \bar{s}_{j-1})$, is a separating sequence $\bar{S}$ in $\bar{A}$. In particular, if S is an A-separating sequence, then $\bar{S}$ is an $\bar{A}$-separating sequence.*

Remark 4.4. The condition $\bar{s}_j \in (\bar{s}_1, \ldots, \bar{s}_{j-1})$ is equivalent to say that $\bar{s}_j$ is zero as an element of $\bar{A}/(\bar{s}_1, \ldots, \bar{s}_{j-1})$, so that $\bar{s}_j$ can be replaced by $\bar{0}$, by changing representative in the class of $\pi(S)$. So, we can say that $\pi(S)$ becomes a separating sequence $\bar{S}$, after deleting its zero elements.

Remark 4.5. If $\bar{s} \neq 0$, then $D\,\bar{s} \leq D\,s$.

Taking into account the isomorphism $A_{\mathcal{M}_j} \simeq A_j$ of Proposition 1.4, we can use the previous results in the localization of A with respect to its maximal ideals.

Let $S = (s_1, \ldots, s_i)$ be an i-separating sequence in A. Then, using repeatedly Theorem 2.3 iii), we see that each s_j has this characteristic property: there is a maximal ideal $\mathcal{M}_h$ of A such that $\bar{s}_j \in Ann(\overline{\mathcal{M}_h}) - \{0\}$, where " - " denotes the quotient mod $(s_1, \ldots, s_{j-1})A$. So, it is natural to produce an order preserving partition of S into subsequences S_h, where S_h contains all the elements of S linked to the maximal ideal $\mathcal{M}_h$.

If $\mathcal{M}_1, \ldots, \mathcal{M}_r$ are the maximal ideals involved, let us denote $\mathcal{P} = \{S_1, \ldots, S_r\}$ such a partition and call it *canonical partition* of S. Moreover, let us denote S^h the image $SA_{\mathcal{M}_h}$, $h = 1, ..., r$, of S in $A_{\mathcal{M}_h}$, free of its zero elements.

Proposition 4.6. *A sequence $S = (s_1, \ldots, s_i)$ is an i-separating sequence iff there is a partition of S in subsequences S_h, $h = 1, ..., r$, $r \leq i$, S_h of cardinality n_h, such that each S_h gives rise to a n_h-separating sequence $S_hA_{\mathcal{M}_h}$ in $A_{\mathcal{M}_h} \simeq A_h$, that coincides with S^h, and to the empty set in $A_{\mathcal{M}_j} \simeq A_j$, $j \neq h$; moreover, $\sum n_h = i$.*

Proof. Let $S = (s_1, \ldots, s_i)$ be an i-separating sequence in A and $\mathcal{P} = \{S_1, \ldots, S_r\}$ its canonical partition. Repeatedly using Proposition 2.5, we see that S_h becomes the n_h-separating sequence S^h in $A_{\mathcal{M}_h}$ and the empty set in $A_{\mathcal{M}_j}$, $h \neq j$; as a consequence: $\sum n_h = i$.

Viceversa, let $S = (s_1, \ldots, s_i)$ admit a partition $\mathcal{P} = \{S_1, \ldots, S_r\}$, such that S_h becomes a separating sequence S^h in $A_{\mathcal{M}_h}$ and the zero-sequence in $A_{\mathcal{M}_j}$, $h \neq j$. We use induction on i. If $i = 1$, the statement coincides with Proposition 2.5. So, let us suppose it true until $i - 1$ and prove it for i. As we can suppose $s_1^1 \in Ann(\mathcal{M}_1 A_{\mathcal{M}_1})$, Proposition 2.5 says that s_1 is a separator in A. So, it is enough to verify that $\bar{S} = (\bar{s}_2, \ldots, \bar{s}_i)$ is a $(i-1)$ - separating sequence in $A/(s_1)A$. To this aim, we observe that $\mathcal{P}$ induces a partition $\overline{\mathcal{P}} = \{\bar{S}_1, \ldots, \bar{S}_r\}$, where each $\bar{S}_h$, $h > 1$, becomes a

n_h - separating sequence in $(A/(s_1)A)_{\overline{\mathcal{M}}_h}$ (as $s_1 = 0$ in $A_{\mathcal{M}_h}$), while $\bar{S}_1$ is either empty or an $n_1 - 1$-separating sequence. By inductive hypothesis, $\bar{S}$ is a $(i-1)$-sequence in $A/(s_1)A$. □

Lemma 4.7. *Let $S = (s_1, ..., s_i)$ and $T = (t_1, ..., t_i)$ be equivalent i-separating sequences in A and $\mathcal{P}(S) = \{S_1, \dots, S_r\}$, $\mathcal{P}(T) = \{T_1, \dots, T_r\}$ their canonical partitions; then T^h is equivalent to S^h, $h = 1, ...r$.*

Proof. Let us prove: $S \sim T \Longrightarrow S^h \sim T^h$, $h = 1, ..., r$, by induction on i.

If $i = 1$, the statement is obvious; let us suppose it true until $i-1$ and prove it for i. The equivalence between S and T implies that $t_1 \in s_1 K$, so that we can suppose $t_1 = ks_1 \in Ann\mathcal{M}_1$. In $A/(s_1)A$, S and T produce two equivalent $i-1$ -separating sequences $\bar{S}$ and $\bar{T}$; by induction, $\bar{S}$ and $\bar{T}$ give rise to $\mathcal{P}(\bar{S}) = \{\bar{S}_1, \dots, \bar{S}_r\}$, $\mathcal{P}(\bar{T}) = \{\bar{T}_1, \dots, \bar{T}_r\}$, where $\bar{S}^h \sim \bar{T}^h$, $h = 1, ..., r$ (and it may happen: $\bar{T}_1 = \bar{S}_1 = \emptyset$). Let us observe that S_h and T_h are a lifting of $\bar{S}_h$ and, respectively, $\bar{T}_h$. Now, if $h \neq 1$, then $s_1 = 0$ in $A_{\mathcal{M}_h}$, so that the quotient $mod(s_1)$ in $A_{\mathcal{M}_h}$ is inessential; if $h = 1$, then $\bar{S}^1 \sim \bar{T}^1$ iff $S^1 \sim T^1$, by definition of equivalence. □

With the notation of Lemma 2.1, we state the following:

Proposition 4.8. *Let $S = (s_1, ..., s_i)$ be an i- separating sequence. There is a separating sequence $T = (t_1, ..., t_i)$, equivalent to S, such that, if $t_h \in Ann\mathcal{M}_{u(h)}, mod(t_1, ..., t_h - 1)$, then $t_h \in \mathcal{Q}_j$, $j \neq u(h)$. As a consequence, all the elements of its corresponding canonical partition $\mathcal{P} = \{T_1, \dots, T_r\}$ are separating sequences in A.*

Proof. We use induction on i. If $i = 1$, we can obviously choose $t_1 = s_1$, thanks to Proposition 2.5. So, let us suppose the statement true until $i-1$ and prove it for i; in other words, we suppose that there exists $T'(t_1, ..., t_{i-1})$ equivalent to $S' = (s_1, ..., s_{i-1})$ and satisfying the required condition and produce t_i. Let us suppose $s_i \in Ann\mathcal{M}_1, mod(s_1, ..., s_{i-1})$, where $(s_1, ..., s_{i-1})A = (t_1, ..., t_{i-1})A$, and consider the canonical isomorphism of Proposition 1.4:

$$\phi : A \longrightarrow A/\mathcal{Q}_1 \oplus ... \oplus A/\mathcal{Q}_r.$$

If $\phi(s_i) = (s_i^1, s_i^2, ..., s_i^r)$, the required t_i is: $t_i = \phi^{-1}(s_i^1, 0, ..., 0)$. In fact, $t_i^j = 0$, $j \neq 1$, means that $t_i \in \mathcal{Q}_j$, $j \neq 1$. Moreover, passing to the quotient $\bar{A} = A/(t_1, ..., t_{i-1})A$, we get the canonical isomorphism:

$$\bar{\phi} : \bar{A} \longrightarrow \bar{A}/\bar{\mathcal{Q}}_1 \oplus ... \oplus \bar{A}/\bar{\mathcal{Q}}_r,$$

where some summand may be zero, and $\bar{\phi}(\bar{t}_i) = \bar{\phi}(\bar{s}_i) = (\bar{s}_i^1, \bar{0}, ..., \bar{0})$, as $\bar{s}_i$ is a separator in $\bar{A}$.

This implies:

a) $\bar{\phi}(\bar{s}_i - \bar{t}_i) = \phi(\bar{s}_i) - \phi(\bar{t}_i) = \bar{0}$, so that $s_i - t_i \in (t_1, ..., t_{i-1})$, which implies that $T = (t_1, ..., t_i)$ is equivalent to S.

b) $\bar{t}_i^1 = \bar{s}_i^1 \in Ann\bar{\mathcal{M}}_1\bar{A}/\bar{\mathcal{Q}}_1$, which means that t_i^1 becomes a separator in $A/(\mathcal{Q}_1, t_1, ..., t_{i-1})$. Let us denote $\mathcal{T}$ the ideal generated by the t_h's, $h < i$, associated to $\mathcal{M}_1$. By the inductive hypothesis, we have that, if t_h is not associated to $\mathcal{M}_1$, then $t_h \in \mathcal{Q}_1$. As a consequence $A/(\mathcal{Q}_1, t_1, ..., t_{i-1}) = A/(\mathcal{Q}_1, \mathcal{T})$, so that t_i^1 becomes a separator also in $(A/\mathcal{T})/\mathcal{Q}_1$. This property, with the condition $t_i^j = 0$, $j \neq i$, says that t_i is a separator in $A/\mathcal{T}$; so, also the last assertion is proved. □

Proposition 4.9. *Let us consider a set $\{S^1, ..., S^r\}$, where $S^h = (s_1^h, ..., s_{n_h}^h)$ is a n_h -separating sequence in $A_{\mathcal{M}_h} \simeq A/\mathcal{Q}_h = A_h$. There is an i-separating sequence T in A, $i = \sum_{h=1}^r n_h$, with a canonical partition $\mathcal{P} = \{T_1, \dots, T_r\}$, such that $T = (T_1, \dots, T_r)$, $TA_{\mathcal{M}_h} = T_hA_{\mathcal{M}_h} = S^h$ and T_h is a separating sequence in A. Moreover, the i-separating sequences in A, giving rise (up to equivalence) to $\{S^h\}$ in $A_{\mathcal{M}_h}, h = 1, ..., r$, are (up to equivalence) the permutations of T, preserving the order of each subsequence T_h.*

Proof. To each $s_j^h \in S^h$ let us associate $\sigma_j^h = (0, \dots, 0, s_j^h, 0, \dots, 0) \in A_{\mathcal{M}_1} \oplus \dots \oplus A_{\mathcal{M}_j} \oplus \dots \oplus A_{\mathcal{M}_t}$, so producing the sequence $\Sigma = (\sigma_1^1, \dots, \sigma_{n_1}^1, \sigma_1^2, \dots, \sigma_{n_2}^2, \dots, \sigma_1^r, \dots, \sigma_{n_r}^r) = (\Sigma_1, \dots, \Sigma_r)$. Its corresponding sequence $T = \phi^{-1}(\Sigma) \subset A$ satisfies the conditions of the first part of the statement (see Proposition 4.6). Again thanks to Proposition 4.6, any permutation of T, preserving the order in each sequence of the partition, is still an i - separating sequence.

Now, let us suppose $U = (u_1, \dots, u_i)$ an i-separating sequence of A, giving rise to the set $\{S^1, \dots, S^r\}$. Thanks to Proposition 4.8, the i - separating sequence $\phi(U)$ is equivalent to a sequence U', whose elements have only one component different from zero. According to Lemma 4.7, U' gives rise to a set $\{U'^1, \dots, U'^r\}$, where U'^h is equivalent to S^h. Now it is enough to observe that U' is obtained from $(U'^1, \dots, U'^r)$ just as Σ is obtained from $(S^1, \dots, S^r)$, up to permutations preserving the order in each element of the partition. □

A separator $s \in A$ may have a degree different from that of its image s^j in $A_j = A/\mathcal{Q}_j \simeq A_{\mathcal{M}_j}$. In fact:

Proposition 4.10. *If s is a separator in A, annihilating $\mathcal{M}_j$, then $Ds \geq Ds^j$. More precisely, if $s^j = P + I_j$, $P \in R$, then Ds is the minimal degree of a polynomial in the set $(P + I_j) \cap (\cap_{t \neq j} I_t)$.*

Proof. Let $s = T + I$, with $Ds = deg\ T$. Then T is characterized by the condition $T = P + i_j \in \cap_{t\neq j} I_t$. So, T is chosen in $(P + I_j) \cap (\cap_{t\neq j} I_t)$, under the condition that its degree is the minimal possible in that set. □

Let us observe that $Ds - Ds^j$ depends not only on the maximal ideal $\mathcal{M}_j$ but also on the separator itself , as we can see in the following example.

Example 4.11. Let us consider the coordinate ring of an affine scheme **X** consisting of a triple point and a simple one:

$A = R/I = K[x, y]$, where $R = K[X, Y]$, $I = (X^2, XY, Y^2(Y-1))$.

The maximal ideals of A are $\mathcal{M}_1 = (x, y)$, $\mathcal{M}_2 = (x, y - 1)$.

Moreover, $I_1 = IA_{\mathcal{M}_1} \cap A = (X^2, XY, Y^2)$, $I_2 = IA_{\mathcal{M}_2} \cap A = (X, Y - 1)$ and $I = I_1 \cap I_2$.

We can choose as bases of the K-spaces A, $A_1 = K[X, Y]/I_1 = K[x, y]$ and $A_2 = K[X, Y]/I_2 \simeq K$ respectively:

$\mathcal{B} = (1, x, y, y^2)$, $\mathcal{B}_1 = (1, x, y)$, $\mathcal{B}_2 = (1)$.

The morphism ϕ of Proposition 1.4 acts as follows:

$\phi(a_0 + a_1x + a_2y + a_3y^2) = (a_0 + a_1x + a_2y, a_0 + a_2 + a_3)$.

We know (see 2.10, example 3) that the separators of A_1 are all the linear non zero forms $\alpha x + \beta y$, while the separators of A_2 are clearly all the elements of K different from zero. So, Proposition 4.9 allows us to produce all the separators of A, by lifting those of A_1 and A_2. More precisely,

$$\phi^{-1}(\alpha x + \beta y, 0) = \alpha x + \beta y - \beta y^2, \quad \phi^{-1}(0, c) = cy^2.$$

Let us observe that, passing from the local situation to the global one, the degree of a separator can increase. In fact $D(\alpha x + \beta y - \beta y^2) = 2$ in A, if $\beta \neq 0$, while $D(\alpha x + \beta y) = 1$ in A_1. However, if $\beta = 0$, the degree of the separator αx is 1, both in A and in A_1; so, $D\ s - D\ s^1$ depends on s, and not only on the maximal ideal $\mathcal{M}_1$.

Taking into account example 3 in 3.9 and Proposition 4.9, we can produce all the separating sequences of A. With the notation of Proposition 4.9, we obtain the sequence $T = (T_1, T_2)$, where:

$TA_{\mathcal{M}_1} = T_1 = (L_1, L_2, 1)$, $L_1 = \alpha_1 x + \beta_1 y, L_2 = \alpha_2 x + \beta_2 y$

independent linear forms, $TA_{\mathcal{M}_2} = T_2 = (1)$, so that, lifting to A, we get:

$$T = ((\alpha_1 x + \beta_1 y - \beta_1 y^2), (\alpha_2 x + \beta_2 y - \beta_2 y^2), 1 - y^2, y^2).$$

Let us compute the corresponding numerical sequence.

If $\beta_1 = 0$, then $\alpha_1 \neq 0, \beta_2 \neq 0$ and the sequence becomes, up to equivalence:

$$S = (x, y - y^2, 1 - y, 1);$$

the corresponding numerical sequence is $(1, 2, 1, 0)$.

If $\beta_1 \neq 0$, up to equivalence, the sequence becomes:

$$S = (\alpha x + y - y^2, L, 1 + L', 1),$$

where L and L' are independent linear forms. The corresponding numerical sequence is $(2, 1, 1, 0)$.

All the other separating sequences are, up to equivalence, the ones obtained from T by moving back y^2, from the fourth position to the third, second or first one. They do not produce new numerical sequences.

Remark 1. Let us observe that the map $\phi^{-1} : A_1 \oplus A_2 \longrightarrow A$ can be obtained directly, using the ideals I_1 and I_2. As it is also a morphism of A-modules, it is enough to write down $\phi^{-1}(1, 0)$ and $\phi^{-1}(0, 1)$. The equality $I_1 + I_2 = R$ implies $1 = i_1 + i_2$, so that $\phi^{-1}(1, 0) = i_2 + I$, $\phi^{-1}(0, 1) = i_1 + I$, and $\phi^{-1}(a, b) = (ai_2 + bi_1) + I$. In our case $i_1 = Y^2$, $i_2 = 1 - Y^2$, $\phi^{-1}(1, 0) = 1 - y^2$, $\phi^{-1}(0, 1) = y^2$, $\phi^{-1}(\alpha + \beta x + \gamma y, c) = (\alpha + \beta x + \gamma y)(1 - y^2) + cy^2 = \alpha + \beta x + \gamma y + (-\alpha - \gamma + c)y^2$.

Remark 2. The set $\mathbf{S_X}$ of the numerical sequences linked to the scheme $\mathbf{X}$ is the same set linked to a reduced scheme $\mathbf{Y}$ consisting of four points, three on a line and one outside. As a matter of fact, we can think of $\mathbf{X}$ as a special element of a flat family (see [11]), whose generic scheme is reduced, with the same set $\mathbf{S}_X$ of allowed sequences. For instance, let us consider in $K[X, Y]$ the ideal

$$I(t) = (XY, X(X - t), Y(Y - 1)(Y - t)), \quad t \in K$$

which defines a flat family of schemes. If $t \neq 0$, the corresponding schemes consist of four distinct points, three on $X = 0$ and one outside; if $t = 0$, the corresponding ideal is I. □

Let us restate the results of this paragraph in the projective situation.

Proposition 4.12. *Let $f \in A^*$ be a separator and $\mathcal{B}^*$ any homogeneous, saturated, height zero ideal of A^*, that is of the form: $\mathcal{B}^* = \psi(\mathcal{B})A^*$, for* *some regular form y_0 (see Lemma 1.2 iv) . Then the image $\bar{f}$ of f in $A^*/\mathcal{B}^*$ is either zero or of the form $\bar{f} = \bar{y}_0^h\, \bar{g}$, where $\bar{g}$ is a separator in $A^*/\mathcal{B}^*$.*

Proof. Condition $ii)$ 1) of Theorem 2.12 is verified by $\bar{f}$ if it is verified by f, when $\bar{f} \neq 0$; condition $ii)$ 2) can fail, so that $\bar{f}$ needs to be replaced by $\bar{g} = (\bar{f}_*)^*$. □

Remark. Let us observe that, passing to the quotient, a separator f may give rise to a separator $\bar{g}$ of lower degree: the difference is $h = deg\ f - deg\ \bar{g}$, where $\bar{y}_0^h$ turns out to be the greatest power of $\bar{y}_0$ dividing $\bar{f}$.

Definition 4.13. Let $S = (f_1, \dots, f_i)$ be a separating sequence in A^* and $\mathcal{C}$ any ideal of A^*. We call " image of S in $\bar{A}^* = A^*/\mathcal{C}$ " and denote $\bar{S}$ the sequence obtained as follows. First we consider the sequence $(\bar{f}_1, \dots, \bar{f}_i)$ in $A^*/\mathcal{C}$; if $\bar{f}_1 = 0$ we delete it; if $\bar{f}_1 \neq 0$ and $\bar{f}_1 = \bar{y}_0^{h_1} \bar{f}_{1*}^{\,*}$, we replace $\bar{f}_1$ by $\bar{g}_1 = \bar{f}_{1*}^{\,*}$. We repeat the same procedure at each step, so that $\bar{f}_j$ is either deleted or replaced by $\bar{g}_j$, where $\bar{f}_j = \bar{y}_0^{h_j} \bar{g}_j$ in $\bar{A}^*/(g_1, \dots, g_{j-1})$, $\bar{g}_j = \bar{f}_{j*}^{\,*}$ and h_j is the maximal power of $\bar{y}_0$ dividing $\bar{f}_j$.

Let us observe that, in this procedure, we work inside the equivalence classes of S and $\bar{S}$, so that, in fact, the correspondence $S \longrightarrow \bar{S}$ is defined between their equivalence classes.

As an immediate consequence of Proposition 4.12, we get:

Corollary 4.14. *Let $S = (f_1, \dots, f_i)$ be an i-separating sequence in A^* and $\mathcal{B}^*$ a homogeneous, saturated, height zero ideal. The image $\bar{S}$ of S in the quotient $\bar{A}^* = A^*/\mathcal{B}^*$ is a separating sequence in $\bar{A}^*$.*

Now, let us point our attention on a special kind of quotients, that is, the rings $A_h^* = R^*/I_h^* \simeq A^*/\mathcal{Q}_h$ (see Proposition 1.6), corresponding, in the affine situation, to $A_h = R/I_h \simeq A_{\mathcal{M}_h}$.

If $S = (f_1, \dots, f_i)$ is an i-separating sequence in A^*, involving the primary ideals $\mathcal{Q}_1, \dots, \mathcal{Q}_r$, let us denote S^h the image of S in the quotient $A_h^* = A^*/\mathcal{Q}_h,\ h = 1, ..., r$; in particular, if f is a separator, we denote f^h its image in A_h^*.

Propositions 1.6 and 4.12 allow us to state the following:

Corollary 4.15. *Let $f \in A^*$ be a separator, such that $f^h \neq 0$ in A_h^*. Then $degf - degf^h \leq m_h$, where m_h is defined in Proposition 1.6.*

Using the injective morphism ϕ^* of Proposition 1.5, Proposition 4.6 can be restated in the projective situation as follows:

Proposition 4.16. *A sequence $S = (f_1, \ldots, f_i)$ is an i-separating sequence in A^* iff there is a partition of S into subsequences S_h, $h = 1, ..., r$, $r \leq t$, S_h of cardinality n_h, such that the image of S_h in A_j^*, $j \neq h$ is empty, while its image in A_h^* is an n_h-separating sequence; moreover, $\sum n_h = i$.*

Lemma 4.7 and Proposition 4.8 can be translated into the projective situation in an obvious way. We set explicitly the translation of Proposition 4.9.

Proposition 4.17. *Let us consider a set $\{S^1, \cdots, S^r\}$, where $S^h = (f_1^h, \cdots, f_{n_h}^h)$ is a n_h-separating sequence in $A^*/\mathcal{Q}_h$. There is an i-separating sequence T in A^*, $i = \sum n_h$, with a canonical partition $\mathcal{P} = \{T_1, \cdots, T_r\}$, such that: $T = (T_1, \cdots, T_r)$, where T_h is a separating sequence in A^*; for each h, the image of T in $A^*/\mathcal{Q}_h$ is equal to the image of T_h and both coincide with S^h. Moreover, the i-separating sequences in A^* giving rise (up to equivalence) to the set $\{S^h\}$, $h = 1, ..., r$, are (up to equivalence) the permutations of T, preserving the order of each subsequence T_h.*

Now, we consider again the scheme of example 4.11, as a subscheme of $\mathbf{P}^2$, and its projective coordinate ring.

Example 4.18. Let $A^* = K[X, Y, Z]/(X^2, XY, Y^2(Y - Z)) = K[x, y, z]$.

Let us observe that z is a regular linear form in A^* and that the ring A, considered in example 4.11, is the dehomogenization of A^* with respect to z.

We have the following primary decomposition:

$(0) = \mathcal{Q}_1 \cap \mathcal{Q}_2$, $\mathcal{Q}_1 = (y^2)$, $\mathcal{P}_1 = (x, y)$, $\mathcal{Q}_2 = (x, y - z)$, $\mathcal{P}_2 = \mathcal{Q}_2$.

The morphism ϕ^* of Proposition 1.5 acts as follows:

$$\phi^* : A^* \longrightarrow A_1^* \oplus A_2^*,$$

where: $A_1^* = A^*/\mathcal{Q}_1$, $A_2^* = A^*/\mathcal{Q}_2$,

$A^* =< 1 > \oplus < x, y, z > \oplus < y^2, yz, xz, z^2 > \oplus_{h=1}^{\infty} z^h < y^2, yz, xz, z^2 >$,

$A_1^* =< 1 > \oplus < x, y, z > \oplus_{h=1}^{\infty} z^h < x, y, z >$,

$A_2^* =< 1 > \oplus_{h=1}^{\infty} z^h < 1 >$.

We see that the degree of a separator cannot exceed 2 in A^*, 1 in A_1^*, 0 in A_2^*. In fact the Castelnuovo function $\Gamma(t)$ of A^*, A_1^*, A_2^* vanish respectively for $t \geq 3, t \geq 2, t \geq 1$; equivalently, we can recall that a separator is not a multiple of z. So, to compute the separators of the three rings, we need to point out that:

$\phi^*(ax + by + cz) = (ax + by + cz, (b + c)z)$

$\phi^*(ay^2 + byz + cxz + dz^2) = (byz + cxz + dz^2, (a + b + d)z^2)$

In A_1^*: $(Ann\bar{\mathcal{P}}_1)_1 =< x, y >$; its elements different from zero are exactly the separators of A_1^*.

In A_2^*: $Ann\bar{\mathcal{P}}_2 = K[z]$ gives rise to the set of all separators of A_2^*, that is to $K - \{0\}$.

The separators in A^* can be obtained either by a direct computation or by lifting those of A_1^* and A_2^*. By direct computation, we get:

$Ann\mathcal{P}_1 = (x, y^2 - yz)$ gives rise to the separators $\{ax,\ a \in K,\ a \neq 0\}$ in degree 1 and to $\{b(y^2 - yz) + dxz,\ b, d \in K,\ b \neq 0\}$ in degree 2.

$Ann\mathcal{P}_2 = (y^2)$; its elements of the form $ky^2,\ k \neq 0$, are the separators.

Lifting to A^* the separators of A_1^* and A_2^*, we get:

$\phi^*(ax + by + cz) = (\alpha x + \beta y, 0) \Leftrightarrow b = c = 0,\ a = \alpha$; so we find again the separators of degree 1.

$\phi^*(ax + by + cz) = (0, z)$ is impossible

$\phi^*(ay^2 + byz + cxz + dz^2) = (z(\alpha x + \beta y), 0) \Leftrightarrow a + b + d = 0, d = 0, b = \beta, c = \alpha$; so we find again the separators $< x >, < y^2 - yz >$.

Finally, $\phi^*(ay^2 + byz + cxz + dz^2) = (0, \alpha z^2) \Leftrightarrow b = c = d = 0, a = \alpha$; so, we recover the separators $< y^2 >$.

Now, all the separating sequences of A^* can be produced, as in the affine case, gluing together the separating sequences of A_1^* and A_2^*.

5. Geometric meaning of $\mathcal{S}_A$.

Let us consider the case in which **X** is reduced or, equivalently, its coordinate affine ring A is regular. We denote $\mathcal{S}_A$ the set of all A-separating sequences and $(\delta\mathcal{S})_A$ the set of the corresponding numerical sequences. In this situation, if $dim_K A = r$, we have exactly r separating ideals and, as a consequence, $r!$ A-separating sequences. In fact, as we already noticed, there is a bijection between the set of separating ideals and the set of points of X; moreover, an A-sequence corresponds to an ordering of the points. Clearly, different A-separating sequences can produce the same numerical sequence, but the way of obtaining $(\delta\mathcal{S})_A$ starting from A is to produce $\mathcal{S}_A$.

Let us remark that passing from $\mathcal{S}_A$ to $(\delta\mathcal{S})_A$ we loose some information. In fact, $(\delta\mathcal{S})_A$ points out the cardinality and the special position of all the subschemes of **X** not in general position (see [5], paragraph 2). For instance, the presence in **X** of six points on a conic is equivalent to the presence, in $(\delta\mathcal{S})_A$, of sequences ending with $(3, 2, 2, 1, 1, 0)$. However, looking at $(\delta\mathcal{S})_A$, we cannot decide how many subschemes consisting of six points on a conic are present in **X** and how they intersect each other; that information is contained in $\mathcal{S}_A$. To understand better the situation, let us look at the following

Example 5.1. Let us consider the ring

$A = K[X, Y]/I$, where $I = (X(X-1)(X+1), Y(Y-1)(Y+1), XY(X+1)(Y+1))$.

A has eight maximal ideals $\mathcal{M}_i = (X - a_i, Y - b_i)$, $i = 1, ..., 8$, corresponding to the eight points of $\mathbf{X} = SpecA$,that we denote: $B_1 = (-1, -1)$, $B_2 = (0, -1)$, $B_3 = (1, -1)$, $B_4 = (-1, 0)$, $B_5 = (0, 0)$, $B_6 = (1, 0)$, $B_7 = (-1, 1)$, $B_8 = (0, 1)$. Let us compute the separating ideal $(s_5)A$ corresponding to $\mathcal{M}_5$ or, equivalently, to the point $B_5 = (0, 0)$. As we already noticed, such an ideal can be computed either as the annihilator of $\mathcal{M}_5$, or as the lifting of the separating ideal (1) of $A_{\mathcal{M}_5}$. We will compute $Ann\mathcal{M}_5$. A basis of A, as a K-space, is $\mathcal{B} = (1, x, y, x^2, y^2, xy, x^2y, xy^2)$, with the relations: $x^3 = x$, $y^3 = y$, $x^2y^2 = -(x^2y + xy^2 + xy)$. As a consequence, it is easy to see that $s_5x = s_5y = 0 \Leftrightarrow s_5 = k(1 - x^2 - y^2 - xy - x^2y - xy^2)$, $k \in K$. So, $Ann\mathcal{M}_5 = (s_5)A$ and $D\, s_5 = 3$. Analogously, we can compute the separators $s_1, ..., s_8$, corresponding to the other points, and verify that all of them are of degree 3, as elements of A. So, we can produce all the 8! A-separating sequences. Two of them are the following ones:

$$S_1 = (s_6, s_3, s_8, s_5, s_2, s_7, s_4, s_1), \quad S_2 = (s_8, s_7, s_6, s_5, s_4, s_3, s_2, s_1).$$

The corresponding numerical sequences coincide, as $\delta(S_1) = \delta(S_2) = (3, 2, 3, 2, 1, 2, 1, 0) = \Delta$. The presence of Δ in $(\delta\mathcal{S})_A$ just says that there are two couples of three aligned points (number 2 in second position says that there are six points lying on a conic; number 1 in fifth position says that the conic is reduced, as three of them are on a line). Analyzing S_1 and S_2 we get the more precise information that there are two different reduced conics involved, intersecting in four points of **X**. In fact, B_1, B_4, B_7 are aligned, just as B_2, B_5, B_8, giving rise to the first conic. Analogously, B_4, B_5, B_6 and B_1, B_2, B_3 are aligned, giving rise to the second conic.

If the scheme $\mathbf{X}$ is not reduced, the situation is not so clear. Let us consider the case in which $\mathbf{X}$ consists of one point, with multiplicity $r+1,\ r > 0$; in other words, its affine ring $(A, \mathcal{M})$ is local, with $dim_K A = r+1$. In such a situation, the set of separators coincides with $Ann\mathcal{M}$, which is a K-space and, as a consequence, there are either only one separating ideal, if $dim_K Ann\mathcal{M} = 1$, or infinitely many, if $dim_K Ann\mathcal{M} \geq 2$. So, the equality between $dim_K A$ and the cardinality of the set of separating ideals fails and the separating ideals seem unfit for representing the "points" supported at $\mathcal{M}$. Looking at examples, we see that the separators of a multiple point seem to be the "limit" of the separators relative to the generic reduced scheme of some flat family (see [11]), possibly of all the ones having that multiple point as a special element.

Let us consider the easiest situations, that is $dim_K A = 2$ and $dim_K A = 3$.

If $dim_K A = 2$, then, up to isomorphism, $A = K[X]/(X^2) = K[x]$.

In this case, $Ann(x)A = (x)A$, so that there is only one separating ideal, with elements of degree 1: the only numerical sequence is $(1, 0)$.

Let us consider the flat family $\mathbf{X}_t$, defined, in $K[X, Y]$, by the ideal $I(t) = (Y, X(X - t)),\ t \in K$. Then $\mathbf{X}_t$ is not reduced iff $t = 0$. In this case, the corresponding ring is just A. In the generic case the separating ideals are $(x - t)A, (x)A$. For $t = 0$ they coincide with $(x)A$.

If $dim_K A = 3$, then, up to isomorphism, we have three different rings:

$i)$ $A = K[X]/X^3 = K[x]$.

In this case, $Ann(\mathcal{M} = (x^2)A$, so that there is only one separating ideal, with generator of degree 2.

As $A/(x^2)A \simeq K[X]/(X^2)$, the only numerical sequence is $(2, 1, 0)$, that is, the one characterizing a reduced scheme of three points on a line. Let us consider the flat family defined by: $I(t) = (y, x(x^2 - t^2))A$. For $t = 0$, we obtain $I(0) = (y, x^3)A$, so that $K[X, Y]/I(0) \simeq A$. The separating ideals of $A(t) = K[X, Y]/I(t)$ are, generically, $(x(x + t))A, (x(x - t)A), (x^2 - t^2)A$; when t=0, all of them coincide with $(x^2)A$.

Let us observe that A is isomorphic to the localization in $(x, y)A$ of the coordinate ring $K[x, y]$ of the scheme $\mathbf{X}$, obtained by intersecting a plane curve with a tangent at $O(0, 0)$, which is either a double point or an inflection point. This fact seems to justify the presence of the sequence $(2, 1, 0)$, characterizing the reduced schemes of three points on a line ([5], paragraph 2).

$ii)$ $A = K[X, Y]/(X^2, XY, Y^2) = K[x, y]$.

$Ann\mathcal{M} = (x, y)$, so that there are infinitely many separating ideals and infinitely many A-separating sequences, each starting with an element of degree 1; however, the only numerical sequence is $(1, 1, 0)$.

Let us consider the set of flat families: $A(k, t) = K[X, Y]/I(k, t)$, where $I(k, t) = (XY, X(X - t), Y(Y - kt)),\ k \neq 0$.

For $t = 0$, we have: $I(k, 0) = (XY, X^2, Y^2)$, so that $A(k, 0) = A$, for every $k \neq 0$.

The three separating ideals of $A(k, t)$, when $t \neq 0$, are $(x)A$, $(y)A$, $(kx + y - kt)A$; if $t = 0$, they become $(x)A$, $(y)A$, $(kx + y)A$; so, when k varies, they span the whole ideal $(x, y)A$.

$iii)$ $A = K[X, Y]/(X^2 - Y, XY, Y^2) = K[X, Y]/(X^2 - Y, XY) = K[x, y]$.

$Ann\mathcal{M} = (y)A$ and, as in the previous case, the only numerical sequence is $(1, 1, 0)$, but now there is only one separating ideal.

Let us consider the flat family defined by:

$$I(t) = (Y - X^2, Y(Y - t^2), X(Y - t^2)).$$

As $I(0) = (Y - X^2, Y^2, XY)$, we have: $A = K[X, Y]/I(0)$. The three different separating ideals of $A(t) = K[X, Y]/I(t)$ are $(y - t^2)$, $(tx + y)$, $(-tx + y)$ and, for $t = 0$, they all coincide with (y).

A similar ring was already considered in [2] and [9].

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