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# DIFFERENTIAL INEQUALITIES FOR MEROMORPHIC p-VALENT FUNCTIONS ASSOCIATED WITH GENERALIZED INTEGRAL OPERATOR

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In this paper the author introduced a new generalized integral operator for meromorphic p-valent functions in $U^* = \{z : z \in C \text{ and } 0 < |z| < 1\}$. The object of this paper is to give an application of this operator to the differential inequalities.

## 1. Introduction

Let $\Sigma_p$ be the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p} \qquad (p \in \mathbb{N} = \{1, 2, \dots\}), \tag{1}$$

which are analytic and p-valent in the punctured unit disc $U^* = \{z : z \in \mathbb{C}$ and $0 < |z| < 1\} = U \backslash \{0\}$.
For a function $f(z) \in \Sigma_p$, we define the integral operator

$$\mathfrak{I}_{p,\lambda}^{\beta,m}(\alpha,\mu) f(z) \ (\alpha, \beta, \mu, \lambda \geq 0, \alpha+\beta \neq 0, \mu+\lambda \neq 0)$$

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as follows:

$$\Im_{p,\lambda}^{\beta,0}(\alpha,\mu)f(z)=f(z),$$

$$\Im_{p,\lambda}^{\beta,1}(\alpha,\mu)f(z)=\left(\tfrac{\alpha+\beta}{\mu+\lambda}\right)z^{-p-\left(\frac{\alpha+\beta}{\mu+\lambda}\right)}\int_0^z t^{\left(\frac{\alpha+\beta}{\mu+\lambda}+p-1\right)}f(t)dt$$

$$(f\in\Sigma_p;z\in U^*),$$

$$\Im_{p,\lambda}^{\beta,2}(\alpha,\mu)f(z)=\left(\tfrac{\alpha+\beta}{\mu+\lambda}\right)z^{-p-\left(\frac{\alpha+\beta}{\mu+\lambda}\right)}\int_0^z t^{\left(\frac{\alpha+\beta}{\mu+\lambda}+p-1\right)}\Im_{p,\lambda}^{\beta,1}(\alpha,\mu)f(t)dt$$

$$(f\in\Sigma_p;z\in U^*),$$

and, in general,

$$\begin{aligned}&\Im_{p,\lambda}^{\beta,m}(\alpha,\beta,\mu)f(z)\\&=\left(\tfrac{\alpha+\beta}{\mu+\lambda}\right)z^{-p-\left(\frac{\alpha+\beta}{\mu+\lambda}\right)}\int_0^z t^{\left(\frac{\alpha+\beta}{\mu+\lambda}+p-1\right)}\Im_{p,\lambda}^{\beta,m-1}(\alpha,\mu)f(t)dt\\&=\underbrace{\Im_{p,\lambda}^{\beta,1}(\alpha,\mu)\left(\tfrac{1}{z^p(1-z)}\right)*\cdots*\Im_{p,\lambda}^{\beta,1}(\alpha,\mu)\left(\tfrac{1}{z^p(1-z)}\right)}_{m-times}*f(z)\end{aligned}\qquad(2)$$

We note that if $f(z)\in\Sigma_p$, then from (1) and (2), we have

$$\Im_{p,\lambda}^{\beta,m}(\alpha,\mu)f(z)=\frac{1}{z^p}+\sum_{k=1}^{\infty}\left[\frac{\alpha+\beta}{\alpha+\beta+(\mu+\lambda)k}\right]^m a_k z^{k-p}$$

$$(\alpha,\beta,\mu,\lambda\geq0,\alpha+\beta\neq0,\mu+\lambda\neq0;p\in\mathbb{N};m\in\mathbb{N}_0=\mathbb{N}\cup\{0\};z\in U^*).\qquad(3)$$

From (3), it is easy to verify that

$$(\mu+\lambda)z(\Im_{p,\lambda}^{\beta,m+1}(\alpha,\mu)f(z))'=(\alpha+\beta)\Im_{p,\lambda}^{\beta,m}(\alpha,\mu)f(z)$$

$$-[p(\mu+\lambda)+(\alpha+\beta)]\Im_{p,\lambda}^{\beta,m+1}(\alpha,\mu)f(z).\qquad(4)$$

We note that:

(i) $\Im_{p,0}^{0,\gamma}(1,1)f(z)=P_p^{\gamma}f(z)(\gamma>0)$ (see Aqlan et al. [1]);

(ii) $\Im_{1,0}^{0,m}(\alpha,\mu)f(z)=\mathcal{L}^m(\alpha,\mu)f(z)(\alpha>0,\mu\geq0,m\in\mathbb{N}_0)$ (see Bulboaca et al. [2]);

(iii) $\Im_{1,0}^{0,\gamma}(1,\mu)f(z)=P_{\mu}^{\gamma}f(z)(\gamma>0,\mu>0)$ (see Lashin [3]).

Also we note that

(i) $\Im_{p,1}^{0,m}(\alpha,1)f(z)=\Im_{p}^{m}(\alpha)f(z)(\alpha>0,m\in\mathbb{N}_0)$,
where $\Im_{p}^{m}(\alpha)f(z)=\frac{1}{z^p}+\sum_{k=1}^{\infty}\left(\frac{\alpha}{\alpha+2k}\right)^m a_kz^{k-p}$;

(ii) $\Im_{p,\lambda}^{0,m}(1,0)f(z)=\Im_{p,\lambda}^{m}f(z)(\lambda\geq 0,m\in\mathbb{N}_0)$,
where $\Im_{p,\lambda}^{m}f(z)=\frac{1}{z^p}+\sum_{k=1}^{\infty}\left(\frac{1}{1+\lambda k}\right)^m a_kz^{k-p}$.

For our purpose, we introduce.

**Definition 1.1.** Let $H$ be the set of complex-valued function $h(r,s,t):\mathbb{C}^3\to\mathbb{C}$ ($\mathbb{C}$ is the complex plane) such that:

(i) $h(r,s,t)$ is continuous in a domain $D\subset\mathbb{C}^3$;

(ii) $(1,1,1)\in D$ and $|h(1,1,1)|<1$;

(iii) $$\left|h\left(e^{i\theta},e^{i\theta}+\left(\frac{\mu+\lambda}{\alpha+\beta}\right)\zeta e^{i\theta},\frac{\left(\frac{\alpha+\beta}{\mu+\lambda}\right)^2e^{i\theta}+\left(2\left(\frac{\alpha+\beta}{\mu+\lambda}\right)+1\right)\zeta e^{i\theta}+L}{\left(\frac{\alpha+\beta}{\mu+\lambda}\right)^2}\right)\right|\geq 1$$

whenever

$$\left(e^{i\theta},e^{i\theta}+\left(\frac{\mu+\lambda}{\alpha+\beta}\right)\zeta e^{i\theta},\frac{\left(\frac{\alpha+\beta}{\mu+\lambda}\right)^2e^{i\theta}+\left(2\left(\frac{\alpha+\beta}{\mu+\lambda}\right)+1\right)\zeta e^{i\theta}+L}{\left(\frac{\alpha+\beta}{\mu+\lambda}\right)^2}\right)\in D$$

with $\Re(e^{-i\theta}L)>\zeta(\zeta-1)$ for real $\theta$ and for $\zeta\geq 1$.

**Definition 1.2.** Let $H$ be the set of complex-valued function $h(r,s,t)$:$\mathbb{C}^3\to\mathbb{C}$ ($\mathbb{C}$ is the complex plane) such that:

$(i)$ $h(r,s,t)$ is continuous in a domain $D\subset\mathbb{C}^3$;

$(ii)$ $(1,1,1)\in D$ and $|h(1,1,1)|<1$;

$(iii)$ $$\left|h\left(e^{i\theta},e^{i\theta}+\left(\frac{\mu+\lambda}{\alpha+\beta}\right)\zeta,\right.\right.$$
$$\left.\left.e^{i\theta}+\left(\frac{\mu+\lambda}{\alpha+\beta}\right)\zeta+\left(\frac{\mu+\lambda}{\alpha+\beta}\right)\frac{\left(\frac{\alpha+\beta}{\mu+\lambda}\right)\zeta e^{i\theta}+\zeta-\zeta^2+L}{\zeta+\left(\frac{\alpha+\beta}{\mu+\lambda}\right)e^{i\theta}}\right)\right|\geq 1$$

whenever

$$\left(e^{i\theta}, e^{i\theta}+\left(\tfrac{\mu+\lambda}{\alpha+\beta}\right)\zeta, e^{i\theta}+\left(\tfrac{\mu+\lambda}{\alpha+\beta}\right)\zeta+\left(\tfrac{\mu+\lambda}{\alpha+\beta}\right)\frac{\left(\tfrac{\alpha+\beta}{\mu+\lambda}\right)\zeta e^{i\theta}+\zeta-\zeta^2+L}{\zeta+\left(\tfrac{\alpha+\beta}{\mu+\lambda}\right)e^{i\theta}}\right)\in D$$

with $\Re(L)>\zeta(\zeta-1)$ for real $\theta$ and for $\zeta\geq 1$.

## 2. Main result

Unless otherwise mentioned, we assume throughout this paper that $\alpha,\beta,\mu,\lambda\geq 0, \alpha+\beta\neq 0, \mu+\lambda\neq 0$ and $m\geq 2$.
In proving our main result, we shall need the following lemma.

**Lemma 2.1.** [4] *Let $w(z)=a+w_nz^n+\ldots$, be analytic in $U$ with $w(z)\neq 0$ and $\zeta\geq 1$. If $z_0=r_0e^{i\theta}\,(0<r_0<1)$ and $|w(z_0)|=\max\limits_{|z|\leq r_0}|w(z)|$. Then*

$$z_0w'(z_0)=\zeta w(z_0) \tag{5}$$

*and*

$$\Re\left\{1+\frac{z_0w''(z_0)}{w'(z_0)}\right\}\geq\zeta, \tag{6}$$

*where $\zeta$ is a real number and*

$$\zeta\geq k\frac{|w(z_0)-a|^2}{|w(z_0)|^2-|a|^2}\geq k\frac{|w(z_0)|-|a|}{|w(z_0)|+|a|}.$$

**Theorem 2.2.** *Let $h(r,s,t)\in H$ and let $f\in\Sigma_p$ satisfy:*

$$\left(z^p\Im^{\beta,m}_{p,\lambda}(\alpha,\mu)f(z), z^p\Im^{\beta,m-1}_{p,\lambda}(\alpha,\mu)f(z), z^p\Im^{\beta,m-2}_{p,\lambda}(\alpha,\mu)f(z)\right)\in D\subset\mathbb{C}^3 \tag{7}$$

*and*

$$\left|h\left(z^p\Im^{\beta,m}_{p,\lambda}(\alpha,\mu)f(z), z^p\Im^{\beta,m-1}_{p,\lambda}(\alpha,\mu)f(z), z^p\Im^{\beta,m-2}_{p,\lambda}(\alpha,\mu)f(z)\right)\right|<1 \tag{8}$$

*for all $z\in U$. Then we have*

$$\left|z^p\Im^{\beta,m}_{p,\lambda}(\alpha,\mu)f(z)\right|<1 \quad (z\in U)$$

*Proof.* We define the function $w(z)$ by

$$z^p \Im_{p,\lambda}^{\beta,m}(\alpha,\mu)f(z) = w(z),$$

for $f(z) \in \Sigma_p$. With the aid of the identity (4), we obtain

$$z^p \Im_{p,\lambda}^{\beta,m-1}(\alpha,\mu)f(z) = w(z) + \left(\tfrac{\mu+\lambda}{\alpha+\beta}\right) zw'(z)$$

and

$$z^p \Im_{p,\lambda}^{\beta,m-2}(\alpha,\mu)f(z) = \frac{\left(\frac{\alpha+\beta}{\mu+\lambda}\right)^2 w(z) + \left(2\left(\frac{\alpha+\beta}{\mu+\lambda}\right)+1\right) zw'(z) + z^2 w''(z)}{\left(\frac{\alpha+\beta}{\mu+\lambda}\right)^2}$$

we claim that $|w(z)| < 1$ for $z \in U$. Otherwise there exists a point $z_0 \in U$ such that $\max\limits_{|z|<|z_0|} |w(z)| = |w(z)| = 1$. Letting $w(z_0) = e^{i\theta}$ and using Lemma 2.1 we see that

$$z^p \Im_{p,\lambda}^{\beta,m}(\alpha,\mu)f(z_0) = w(z_0) = e^{i\theta},$$

$$z^p \Im_{p,\lambda}^{\beta,m-1}(\alpha,\mu)f(z_0) = e^{i\theta} + \left(\tfrac{\mu+\lambda}{\alpha+\beta}\right) \zeta e^{i\theta},$$

$$z^p \Im_{p,\lambda}^{\beta,m-2}(\alpha,\mu)f(z_0) = \frac{\left(\frac{\alpha+\beta}{\mu+\lambda}\right)^2 e^{i\theta} + \left(2\left(\frac{\alpha+\beta}{\mu+\lambda}\right)+1\right) \zeta e^{i\theta} + L}{\left(\frac{\alpha+\beta}{\mu+\lambda}\right)^2},$$

where $L = z_0^2 w''(z_0)$ and $\zeta \geq 1$.
Further, an application of (6) in Lemma 2.1 gives

$$\Re\left\{\frac{z_0 w''(z_0)}{w'(z_0)}\right\} = \Re\left\{\frac{z_0^2 w''(z_0)}{\zeta e^{i\theta}}\right\} \geq (\zeta - 1),$$

or

$$\Re(e^{-i\theta}L) \geq \zeta(\zeta - 1).$$

Since $h(r,s,t) \in H$, we have

$$\left|h\left(z^p \Im_{p,\lambda}^{\beta,m}(\alpha,\mu)f(z_0), z^p \Im_{p,\lambda}^{\beta,m-1}(\alpha,\mu)f(z_0), z^p \Im_{p,\lambda}^{\beta,m-2}(\alpha,\mu)f(z_0)\right)\right|$$

$$= \left| h\left( e^{i\theta}, e^{i\theta} + \left(\tfrac{\mu+\lambda}{\alpha+\beta}\right) \zeta e^{i\theta} \right.\right. ,$$

$$\left.\frac{\left(\frac{\alpha+\beta}{\mu+\lambda}\right)^{2}e^{i\theta}+\left(2\left(\frac{\alpha+\beta}{\mu+\lambda}\right)+1\right)\zeta e^{i\theta}+L}{\left(\frac{\alpha+\beta}{\mu+\lambda}\right)^{2}}\right)\right| \geq 1,$$

which contradicts the condition (8) of the theorem. Therefore we conclude that

$$\left|z^{p}\Im_{p,\lambda}^{\beta,m}(\alpha,\mu)f(z_0)\right| < 1 \quad (z\in U).$$

This complete the proof of Theorem 2.2. □

**Corollary 2.3.** *Let $h(r,s,t)=s$ and $f\in\Sigma_p$ satisfying the conditions* (7) *and* (8) *for $m\geq 2$. Then*

$$\left|z^{p}\Im_{p,\lambda}^{\beta,m+i}(\alpha,\mu)f(z)\right| < 1 \quad (i\geq 0; z\in U).$$

*Proof.* Since $h(r,s,t)=s\in H$, so with the aid of Theorem 2.2, we have

$$\left|z^{p}\Im_{p,\lambda}^{\beta,m-1}(\alpha,\mu)f(z)\right| < 1$$

$$\Longrightarrow \left|z^{p}\Im_{p,\lambda}^{\beta,m}(\alpha,\mu)f(z)\right| < 1 \quad (m\geq 2) \Longrightarrow \left|z^{p}\Im_{p,\lambda}^{\beta,m+i}(\alpha,\mu)f(z)\right| < 1 \; (i\geq 0).$$

□

**Theorem 2.4.** *Let $h(r,s,t)\in H$ and let $f\in\Sigma_p$ satisfy:*

$$\left(\frac{\Im_{p,\lambda}^{\beta,m}(\alpha,\mu)f(z)}{\Im_{p,\lambda}^{\beta,m+1}(\alpha,\mu)f(z)},\frac{\Im_{p,\lambda}^{\beta,m-1}(\alpha,\mu)f(z)}{\Im_{p,\lambda}^{\beta,m}(\alpha,\mu)f(z)},\frac{\Im_{p,\lambda}^{\beta,m-2}(\alpha,\mu)f(z)}{\Im_{p,\lambda}^{\beta,m-1}(\alpha,\mu)f(z)}\right)\in D\subset\mathbb{C}^{3} \quad (9)$$

*and*

$$\left|h\left(\frac{\Im_{p,\lambda}^{\beta,m}(\alpha,\mu)f(z)}{\Im_{p,\lambda}^{\beta,m+1}(\alpha,\mu)f(z)},\frac{\Im_{p,\lambda}^{\beta,m-1}(\alpha,\mu)f(z)}{\Im_{p,\lambda}^{\beta,m}(\alpha,\mu)f(z)},\frac{\Im_{p,\lambda}^{\beta,m-2}(\alpha,\mu)f(z)}{\Im_{p,\lambda}^{\beta,m-1}(\alpha,\mu)f(z)}\right)\right| < 1 \quad (10)$$

*for all $z\in U$. Then we have*

$$\left|\frac{\Im_{p,\lambda}^{\beta,m}(\alpha,\mu)f(z)}{\Im_{p,\lambda}^{\beta,m+1}(\alpha,\mu)f(z)}\right| < 1 \quad (z\in U).$$

*Proof.* Let

$$\frac{\Im_{p,\lambda}^{\beta,m}(\alpha,\mu)f(z)}{\Im_{p,\lambda}^{\beta,m+1}(\alpha,\mu)f(z)} = w(z).$$

Then it follows that $w(z)$ is either analytic or meromorphic in $U$, $w(0) = 1$ and $w(z) \neq 1$. With the aid of the identity (4), we obtain

$$\frac{\Im_{p,\lambda}^{\beta,m-1}(\alpha,\mu)f(z)}{\Im_{p,\lambda}^{\beta,m}(\alpha,\mu)f(z)} = \left\{ w(z) + \left(\frac{\mu+\lambda}{\alpha+\beta}\right)\frac{zw'(z)}{w(z)} \right\}$$

and

$$\frac{\Im_{p,\lambda}^{\beta,m-2}(\alpha,\mu)f(z)}{\Im_{p,\lambda}^{\beta,m-1}(\alpha,\mu)f(z)} = \left\{ w(z) + \left(\frac{\mu+\lambda}{\alpha+\beta}\right)\frac{zw'(z)}{w(z)} + \left(\frac{\mu+\lambda}{\alpha+\beta}\right)\frac{zw'(z) + \left(\frac{\mu+\lambda}{\alpha+\beta}\right)\left(\frac{zw'(z)}{w(z)} + \frac{z^2w''(z)}{w(z)} - \left(\frac{zw'(z)}{w(z)}\right)^2\right)}{w(z) + \left(\frac{\mu+\lambda}{\alpha+\beta}\right)\frac{zw'(z)}{w(z)}} \right\}$$

we claim that $|w(z)| < 1$ for $z \in U$. Otherwise there exists a point $z_0 \in U$ such that $\max_{|z|<|z_0|} |w(z)| = |w(z)| = 1$. Letting $w(z_0) = e^{i\theta}$ and using Lemma 2.1 with $a = k = 1$, we see that

$$\frac{\Im_{p,\lambda}^{\beta,m}(\alpha,\mu)f(z_0)}{\Im_{p,\lambda}^{\beta,m+1}(\alpha,\mu)f(z_0)} = e^{i\theta},$$

$$\frac{\Im_{p,\lambda}^{\beta,m-1}(\alpha,\mu)f(z_0)}{\Im_{p,\lambda}^{\beta,m}(\alpha,\mu)f(z_0)} = e^{i\theta} + \left(\frac{\mu+\lambda}{\alpha+\beta}\right)\zeta,$$

and

$$\frac{\Im_{p,\lambda}^{\beta,m-2}(\alpha,\mu)f(z_0)}{\Im_{p,\lambda}^{\beta,m-1}(\alpha,\mu)f(z_0)} = e^{i\theta} + \left(\frac{\mu+\lambda}{\alpha+\beta}\right)\zeta + \left(\frac{\mu+\lambda}{\alpha+\beta}\right)\frac{\left(\frac{\alpha+\beta}{\mu+\lambda}\right)\zeta e^{i\theta} + \zeta - \zeta^2 + L}{\left(\frac{\alpha+\beta}{\mu+\lambda}\right)e^{i\theta} + \zeta},$$

where $L = \dfrac{z_0^2 w''(z_0)}{w(z_0)}$ and $\zeta \geq 1$.

Further, an application of (6) in Lemma 2.1 gives

$$\Re(L) \geq \zeta(\zeta-1).$$

Since $h(r,s,t) \in H$, we have

$$\begin{aligned}
&\left| h\left( \frac{\Im_{p,\lambda}^{\beta,m}(\alpha,\mu)f(z_0)}{\Im_{p,\lambda}^{\beta,m+1}(\alpha,\mu)f(z_0)}, \frac{\Im_{p,\lambda}^{\beta,m-1}(\alpha,\mu)f(z_0)}{\Im_{p,\lambda}^{\beta,m}(\alpha,\mu)f(z_0)}, \frac{\Im_{p,\lambda}^{\beta,m-2}(\alpha,\mu)f(z_0)}{\Im_{p,\lambda}^{\beta,m-1}(\alpha,\mu)f(z_0)} \right) \right| \\
&\qquad = \left| h\left( e^{i\theta}, e^{i\theta} + \left(\tfrac{\mu+\lambda}{\alpha+\beta}\right)\zeta, \right.\right. \\
&\qquad\qquad \left.\left. e^{i\theta} + \left(\tfrac{\mu+\lambda}{\alpha+\beta}\right)\zeta + \left(\tfrac{\mu+\lambda}{\alpha+\beta}\right) \frac{\left(\tfrac{\alpha+\beta}{\mu+\lambda}\right)\zeta e^{i\theta} + \zeta - \zeta^2 + L}{\zeta + \left(\tfrac{\alpha+\beta}{\mu+\lambda}\right)e^{i\theta}} \right) \right| \geq 1,
\end{aligned}$$

which contradicts the condition (8) of the theorem. Therefore we conclude that

$$\left| \frac{\Im_{p,\lambda}^{\beta,m}(\alpha,\mu)f(z_0)}{\Im_{p,\lambda}^{\beta,m+1}(\alpha,\mu)f(z_0)} \right| < 1 \quad (z \in U).$$

This complete the proof of Theorem 2.4. □

**Remark 2.5.** By specializing the parameters $\alpha, \beta, \mu, \lambda$ and $m$, we obtain results corresponding to different operators defined in the introduction.

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