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DIFFERENTIAL SANDWICH THEOREMS FOR p-VALENT FUNCTIONS ASSOCIATED WITH GENERALIZED MULTIPLIER TRANSFORMATIONS

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In this paper, we obtain some applications of theory of differential subordination, superordination and sandwich results involving the operator $\mathcal{J}_p^m(\lambda,\ell)$.

1. Introduction

Let $H(U)$ denote the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and $H[a, p]$ denote the subclass of functions $f \in H(U)$ of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \dots \quad (a \in \mathbb{C};\ p \in \mathbb{N} = \{1, 2, \dots\}).$$

Also, let $\mathcal{A}(p)$ denote the subclass of functions $f \in H(U)$ of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \ \ (p \in \mathbb{N}). \tag{1}$$

Also, let $\mathcal{A}(1) = \mathcal{A}$.

If f and g are analytic function in U, we say that f is subordinate to g, written $f \prec g$ if there exists a Schwarz function w, which is analytic in U with

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$w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$. Furthermore, if the function g is univalent in U, then we have the following equivalence (see [11] and [19]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For $k,h \in H(U)$, let $\varphi(r,s,t;z) : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If $k(z)$ satisfies the second order differential subordination

$$\varphi(k(z), zk'(z), z^2k''(z); z) \prec h(z), \tag{2}$$

then $k(z)$ is a solution of the differential subordination (2). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination, if $k(z) \prec q(z)$ for all the functions $k(z)$ satisfying (2). A dominant $\widetilde{q}(z)$ is said to be the best dominant of (2) if $\widetilde{q}(z) \prec q(z)$ for all dominants $q(z)$. If $k(z)$ and $\varphi(k(z), zk'(z), z^2k''(z); z)$ are univalent functions in U and if $k(z)$ satisfies the second order differential superordination

$$h(z) \prec \varphi(k(z), zk'(z), z^2k''(z); z), \tag{3}$$

then $k(z)$ is a solution of the differential superordination (3). The univalent function $q(z)$ is called a subordinant of the solutions of the differential superordination, if $q(z) \prec k(z)$ for all the functions $k(z)$ satisfying (3). A subordinant $\widetilde{q}(z)$ is said to be the best subordinant of (3) if $q(z) \prec \widetilde{q}(z)$ for all the subordinants $q(z)$. Recently Miller and Mocanu [20] obtained conditions on the functions h, q and φ for which the following implication holds:

$$h(z) \prec \varphi(k(z), zk'(z), z^2k''(z); z) \Rightarrow q(z) \prec k(z).$$

Using the results of Miller and Mocanu [20], Bulboacă [10] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [9]. Ali et al. [1], have used the results of Bulboacă [10] (see also [3] and [4]) to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are univalent functions in U with $q_1(0) = q_2(0) = 1$.

Prajapat [24] defined a generalized multiplier transformation operator , as follows:

$$\mathcal{J}_p^m(\lambda,\ell) : \mathcal{A}(p) \to \mathcal{A}(p)$$
$$\mathcal{J}_p^m(\lambda,\ell) f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+\ell+\lambda(k-p)}{p+\ell}\right)^m a_k z^k$$

$$(\lambda \geq 0;\ \ell > -p;\ p \in \mathbb{N};\ m \in \mathbb{Z} = \{0, \pm 1, \dots\};\ z \in U). \tag{4}$$

It is readily verified from (4) that

$$\lambda z \left(\mathcal{J}_p^m (\lambda, \ell) f(z) \right)' \\ = (\ell + p)\, \mathcal{J}_p^{m+1} (\lambda, \ell) f(z) - [\ell + p(1-\lambda)]\, \mathcal{J}_p^m (\lambda, \ell) f(z) \quad (\lambda > 0). \tag{5}$$

By specializing the parameters m, λ, ℓ and p, we obtain the following operators studied by various authors:

(i) $\mathcal{J}_p^m (\lambda, \ell) f(z) = I_p^m(\lambda, \ell) f(z)$ $\quad (\ell \geq 0,\ p \in \mathbb{N},\ \lambda \geq 0$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ (see [[12]]);

(ii) $\mathcal{J}_p^m (1, \ell) f(z) = I_p(m, \ell) f(z)$ $\quad (\ell \geq 0,\ p \in \mathbb{N}$ and $m \in \mathbb{N}_0)$ (see [18] and [29]);

(iii) $\mathcal{J}_p^m (\lambda, 0) f(z) = D_{\lambda,p}^m f(z)$ $\quad (\lambda \geq 0,\ p \in \mathbb{N}$ and $m \in \mathbb{N}_0)$ (see [5]);

(iv) $\mathcal{J}_p^m (1, 0) f(z) = D_p^m f(z)$ $\quad (m \in \mathbb{N}_0$ and $p \in \mathbb{N})$ (see [6], [17] and [21]);

(v) $\mathcal{J}_p^{-m} (\lambda, \ell) f(z) = J_p^m(\lambda, \ell) f(z)$ $\quad (\ell \geq 0,\ \lambda \geq 0,\ p \in \mathbb{N}$ and $m \in \mathbb{N}_0)$ (see [7], [15] and [28]);

(vi) $\mathcal{J}_p^{-m} (1, 1) f(z) = D^m f(z)$ $\quad (m \in \mathbb{Z})$ (see [23]);

(vii) $\mathcal{J}_1^m (1, \ell) f(z) = I_\ell^m f(z)$ $\quad (\ell \geq 0$ and $m \in \mathbb{N}_0)$ (see [13] and [14]);

(viii) $\mathcal{J}_1^m (\lambda, 0) f(z) = D_\lambda^m f(z)$ $\quad (\lambda \geq 0$ and $m \in \mathbb{N}_0)$ (see [2]);

(ix) $\mathcal{J}_1^m (1, 0) f(z) = D^m f(z)$ $\quad (m \in \mathbb{N}_0)$ (see [26]);

(x) $\mathcal{J}_1^{-m} (\lambda, 0) f(z) = I_\lambda^{-m} f(z)$ $\quad (\lambda \geq 0$ and $m \in \mathbb{N}_0)$ (see [22] and [8]);

(xi) $\mathcal{J}_1^{-m} (1, 1) f(z) = I^m f(z)$ $\quad (m \in \mathbb{N}_0)$ (see [16]).

2. Definitions and preliminaries

In order to prove our results, we shall need the following definition and lemmas.

Definition 2.1 ([20])**.** Let $\mathcal{Q}$ be the set of all functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where $E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$ and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Lemma 2.2 ([19]). *Let q be univalent in the unit disc U and let θ and ϕ be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$ when $w \in q(U)$. Set*

$$Q(z) = zq'(z)\phi(q(z)) \text{ and } h(z) = \theta(q(z)) + Q(z), \tag{6}$$

suppose that
(i) Q is a starlike function in U,
(ii) $\Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$, $z \in U$.
If k is analytic in U with $k(0) = q(0)$, $k(U) \subseteq D$ and

$$\theta(k(z)) + zk'(z)\phi(k(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{7}$$

then $k(z) \prec q(z)$ and q is the best dominant of (7).

Lemma 2.3 ([27]). *Let $\xi, \varphi \in \mathbb{C}$ with $\varphi \neq 0$ and let q be a convex function in U with*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\{0; -\Re\frac{\xi}{\varphi}\}.$$

If k is analytic in U and

$$\xi k(z) + \varphi zk'(z) \prec \xi q(z) + \varphi zq'(z), \tag{8}$$

then $k \prec q$ and q is the best dominant of (8).

Lemma 2.4 ([11]). *Let q be a univalent function in U and let θ and φ be analytic in a domain D containing $q(U)$. Suppose that*
(i) $\Re\left\{\frac{\theta'(q(z))}{\varphi(q(z))}\right\} > 0$ for $z \in U$,
(ii) $Q(z) = zq'(z)\varphi(q(z))$ is starlike univalent in U.
If $k \in H[q(0),1] \cap \mathcal{Q}$, with $k(U) \subseteq D$, $\theta(k(z)) + zk'(z)\varphi(k(z))$ is univalent in U and

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(k(z)) + zk'(z)\varphi(k(z)), \tag{9}$$

then $q(z) \prec k(z)$ and q is the best subordinant of (9).

Lemma 2.5 ([20]). *Let q be convex univalent in U and let $\beta \in \mathbb{C}$, with $\Re\{\beta\} > 0$. If $k \in H[q(0),1] \cap \mathcal{Q}$, $k(z) + \beta zk'(z)$ is univalent in U and*

$$q(z) + \beta zq'(z) \prec k(z) + \beta zk'(z), \tag{10}$$

then $q \prec k$ and q is the best subordinant of (10).

Lemma 2.6 ([25]). *The function $q(z) = (1-z)^{-2ab}$ $(a,b \in \mathbb{C}^*)$ is univalent in U if and only if $|2ab-1| \leq 1$ or $|2ab+1| \leq 1$.*

3. Subordinant results

Unless otherwise mentioned, we shall assume in the reminder of this paper that $\lambda \geq 0,\ \ell > -p,\ p \in \mathbb{N},\ \alpha \in \mathbb{C}^* = \mathbb{C}\setminus\{0\},\ m \in \mathbb{Z}$ and $z \in U$ and the powers are understood as principle values.

Theorem 3.1. *Let $q(z)$ be univalent in U, with $q(0) = 1$ and suppose that*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0; -\frac{p(p+\ell)}{\lambda}\Re\left(\frac{1}{\alpha}\right)\right\}. \tag{11}$$

If $f(z) \in \mathcal{A}(p)$ such that $\frac{z^p}{\mathcal{J}_p^m(\lambda,\ell)f(z)} \neq 0$ and satisfies the subordination

$$\frac{(p+\alpha)}{p}\left(\frac{z^p}{\mathcal{J}_p^m(\lambda,\ell)f(z)}\right) - \frac{\alpha}{p}\left(\frac{z^p\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)}{\left(\mathcal{J}_p^m(\lambda,\ell)f(z)\right)^2}\right) \prec q(z) + \frac{\lambda\alpha zq'(z)}{p(p+\ell)}, \tag{12}$$

then

$$\frac{z^p}{\mathcal{J}_p^m(\lambda,\ell)f(z)} \prec q(z)$$

and q is the best dominant of (12).

Proof. Define a function $k(z)$ by

$$k(z) = \frac{z^p}{\mathcal{J}_p^m(\lambda,\ell)f(z)} \quad (z \in U), \tag{13}$$

where $k(z)$ is analytic in U with $k(0) = 1$. By differentiating (13) logarithmically with respect to z, we obtain that

$$\frac{zk'(z)}{k(z)} = p - \frac{z\left(\mathcal{J}_p^m(\lambda,\ell)f(z)\right)'}{\mathcal{J}_p^m(\lambda,\ell)f(z)}. \tag{14}$$

From (14) and (5), a simple computation shows that

$$\frac{(p+\alpha)}{p}\left(\frac{z^p}{\mathcal{J}_p^m(\lambda,\ell)f(z)}\right) - \frac{\alpha}{p}\left(\frac{z^p\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)}{\left(\mathcal{J}_p^m(\lambda,\ell)f(z)\right)^2}\right) = k(z) + \frac{\lambda\alpha zk'(z)}{p(p+\ell)},$$

hence the subordination (12) is equivalent to

$$k(z) + \frac{\lambda\alpha zk'(z)}{p(p+\ell)} \prec q(z) + \frac{\lambda\alpha zq'(z)}{p(p+\ell)}.$$

Now, applying Lemma 2.3, with $\varphi = \frac{\lambda\alpha}{p(p+\ell)}$ and $\xi = 1$, the proof is completed. □

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem 3.1, the condition (11) reduces to

$$\Re\left\{\frac{1-Bz}{1+Bz}\right\} > \max\left\{0; -\frac{p(p+\ell)}{\lambda}\Re\left(\frac{1}{\alpha}\right)\right\}. \tag{15}$$

It is easy to check that the function $\psi(\zeta) = \frac{1-\zeta}{1+\zeta}$, $|\zeta| < |B|$, is convex in U and since $\psi(\overline{\zeta}) = \overline{\psi(\zeta)}$ for all $|\zeta| < |B|$, it follows that the image $\psi(U)$ is convex domain symmetric with respect to the real axis, hence

$$\inf\left\{\Re\left(\frac{1-Bz}{1+Bz}\right)\right\} = \frac{1-|B|}{1+|B|} > 0. \tag{16}$$

Then the inequality (15) is equivalent to $\frac{|B|-1}{|B|+1} \le \frac{p(p+\ell)}{\lambda}\Re\left(\frac{1}{\alpha}\right)$, hence, we obtain the following corollary.

Corollary 3.2. *Let $f(z) \in \mathcal{A}(p)$, $-1 \le B < A \le 1$ and*

$$\max\left\{0; -\frac{p(p+\ell)}{\lambda}\Re\left(\frac{1}{\alpha}\right)\right\} \le \frac{1-|B|}{1+|B|},$$

then

$$\frac{(p+\alpha)}{p}\left(\frac{z^p}{\mathcal{J}_p^m(\lambda,\ell)f(z)}\right) - \frac{\alpha}{p}\left(\frac{z^p\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)}{\left(\mathcal{J}_p^m(\lambda,\ell)f(z)\right)^2}\right) \prec \frac{1+Az}{1+Bz} + \frac{\lambda\alpha}{p(p+\ell)}\frac{(A-B)z}{(1+Bz)^2}, \tag{17}$$

implies

$$\frac{z^p}{\mathcal{J}_p^m(\lambda,\ell)f(z)} \prec \frac{1+Az}{1+Bz}$$

and $\dfrac{1+Az}{1+Bz}$ *is the best dominant of (17).*

Taking $q(z) = \frac{1+z}{1-z}$ in Theorem 3.1 (or putting $A = 1$ and $B = -1$ in Corollary 3.2), the condition (11) reduces to

$$\frac{p(p+\ell)}{\lambda}\Re\left(\frac{1}{\alpha}\right) \ge 0, \tag{18}$$

hence, we obtain the following corollary.

Corollary 3.3. *Let $f(z) \in \mathcal{A}(p)$, assume that (18) holds true and*

$$\frac{(p+\alpha)}{p}\left(\frac{z^p}{\mathcal{J}_p^m(\lambda,\ell)f(z)}\right)-\frac{\alpha}{p}\left(\frac{z^p\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)}{\left(\mathcal{J}_p^m(\lambda,\ell)f(z)\right)^2}\right) \prec \frac{1+z}{1-z}+\frac{\lambda\alpha}{p(p+\ell)}\frac{2z}{(1-z)^2}, \quad (19)$$

then

$$\frac{z^p}{\mathcal{J}_p^m(\lambda,\ell)f(z)} \prec \frac{1+z}{1-z}$$

and $\frac{1+z}{1-z}$ *is the best dominant of (19).*

Theorem 3.4. *Let $q(z)$ be univalent in U, with $q(0)=1$ and $q(z)\neq 0$ for all $z\in U$, $\eta,\zeta\in\mathbb{C}^*$, $\rho,\tau\in\mathbb{C}$, with $\rho+\tau\neq 0$, $f(z)\in\mathcal{A}(p)$ and suppose that f and q satisfy the next conditions:*

$$\frac{(\rho+\tau)z^p}{\rho\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)+\tau\mathcal{J}_p^m(\lambda,\ell)f(z)}\neq 0 \quad (z\in U) \quad (20)$$

and

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\}>0 \quad (z\in U). \quad (21)$$

If

$$1+\zeta\eta\left\{p-\frac{\rho z\left(\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)\right)'+\tau z\left(\mathcal{J}_p^m(\lambda,\ell)f(z)\right)'}{\rho\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)+\tau\mathcal{J}_p^m(\lambda,\ell)f(z)}\right\}\prec 1+\eta\frac{zq'(z)}{q(z)}, \quad (22)$$

then

$$\left(\frac{(\rho+\tau)z^p}{\rho\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)+\tau\mathcal{J}_p^m(\lambda,\ell)f(z)}\right)^{\zeta}\prec q(z)$$

and q is the best dominant of (22).

Proof. Let

$$g(z)=\left(\frac{(\rho+\tau)z^p}{\rho\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)+\tau\mathcal{J}_p^m(\lambda,\ell)f(z)}\right)^{\zeta} \quad (z\in U), \quad (23)$$

then $g(z)$ is analytic in U, differentiating $g(z)$ logarithmically with respect to z, we obtain

$$\frac{zg'(z)}{g(z)} = \zeta \left\{ p - \frac{\rho z \left(\mathcal{J}_p^{m+1}(\lambda,\ell) f(z)\right)' + \tau z \left(\mathcal{J}_p^{m}(\lambda,\ell) f(z)\right)'}{\rho \mathcal{J}_p^{m+1}(\lambda,\ell) f(z) + \tau \mathcal{J}_p^{m}(\lambda,\ell) f(z)} \right\}. \tag{24}$$

Now, using Lemma 2.2 with $\theta(w) = 1$ and $\phi(w) = \frac{\eta}{w}$, then θ is analytic in $\mathbb{C}$ and $\phi(w) \neq 0$ is analytic in $\mathbb{C}^*$. Also if we let

$$Q(z) = zq'(z)\phi(q(z)) = \eta \frac{zq'(z)}{q(z)},$$

and

$$h(z) = \theta(q(z)) + Q(z) = 1 + \eta \frac{zq'(z)}{q(z)},$$

then, $Q(0) = 0$ and $Q'(0) \neq 0$, and the assumption (3.11) yields that Q is a starlike function in U. From (21) we have

$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\} = \Re\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0 \ (z \in U),$$

then, by using Lemma 2.2, we deduce that the assumption (22) implies $g(z) \prec q(z)$ and the function q is the best dominant of (22). □

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$, $\rho = 0$ and $\tau = \eta = 1$ in Theorem 3.4, the condition (21) reduces to

$$\left\{1 - \frac{2Bz}{1+Bz} - \frac{(A-B)z}{(1+Az)(1+Bz)}\right\} > 0, \tag{25}$$

hence, we obtain the following corollary.

Corollary 3.5. *Let $f(z) \in \mathcal{A}(p)$, assume that (25) holds true, $-1 \leq B < A \leq 1$ and suppose that $\frac{\mathcal{J}_p^m(\lambda,\ell) f(z)}{z^p} \neq 0$ $(z \in U)$. If*

$$1 + \zeta \left\{ p - \frac{z\left(\mathcal{J}_p^m(\lambda,\ell) f(z)\right)'}{\mathcal{J}_p^m(\lambda,\ell) f(z)} \right\} \prec 1 + \frac{(A-B)z}{(1+Az)(1+Bz)}, \tag{26}$$

then

$$\left(\frac{z^p}{\mathcal{J}_p^m(\lambda,\ell) f(z)}\right)^{\zeta} \prec \frac{1+Az}{1+Bz}, \tag{27}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant of (26).

Putting $p \in \mathbb{N}$, $n = \rho = 0, \tau = \eta = 1$ and $q(z) = (1+Bz)^{\frac{\zeta(A-B)}{B}}$ ($\zeta \in \mathbb{C}^*$, $-1 \leq B < A \leq 1$, $B \neq 0$) in Theorem 3.4 and using Lemma 2.6, it is easy to check that the assumption (21) holds, hence we obtain the next corollary:

Corollary 3.6. *Let $f \in \mathcal{A}(p)$, $\zeta \in \mathbb{C}^*$, $-1 \leq B < A \leq 1$, with $B \neq 0$ and suppose that $\left|\frac{\zeta(A-B)}{B} - 1\right| \leq 1$ or $\left|\frac{\zeta(A-B)}{B} + 1\right| \leq 1$. If*

$$1 + \zeta\left(p - \frac{zf'(z)}{f(z)}\right) \prec \frac{1 + [B + \zeta(A-B)]z}{1+Bz}, \tag{28}$$

then

$$\left(\frac{z^p}{f(z)}\right)^{\zeta} \prec (1+Bz)^{\frac{\zeta(A-B)}{B}}$$

and $(1+Bz)^{\frac{\zeta(A-B)}{B}}$ is the best dominant of (28).

Putting $m = \rho = 0$, $\tau = 1$, $\eta = \frac{1}{ab}$ $(a, b \in \mathbb{C}^*)$, $\zeta = a$, and $q(z) = (1-z)^{-2ab}$ in Theorem 3.4, hence combining this togther with Lemma 2.6, we obtain the following corollary.

Corollary 3.7. *Let $f(z) \in \mathcal{A}(p)$, assume that (21) holds true and $a, b \in \mathbb{C}^*$ such that $|2ab - 1| \leq 1$ or $|2ab + 1| \leq 1$. If*

$$1 + \frac{1}{b}\left(p - \frac{zf'(z)}{f(z)}\right) \prec \frac{1+z}{1-z}, \tag{29}$$

then

$$\left(\frac{z^p}{f(z)}\right)^{a} \prec (1-z)^{-2ab}$$

and $(1-z)^{-2ab}$ is the best dominant of (29).

Theorem 3.8. *Let $q(z)$ be univalent in U, with $q(0) = 1$, $\eta, \zeta \in \mathbb{C}^*$, $\rho, \tau, \sigma, \varkappa \in \mathbb{C}$, with $\rho + \tau \neq 0$ and $f(z) \in \mathcal{A}(p)$. Suppose that f and q satisfy the next two conditions:*

$$\frac{(\rho+\tau)z^p}{\rho\mathcal{J}_p^{m+1}(\lambda,\ell)f(z) + \tau\mathcal{J}_p^{m}(\lambda,\ell)f(z)} \neq 0 \ (z \in U) \tag{30}$$

and

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\{0; -\Re\left(\tfrac{\sigma}{\eta}\right)\} \ (z \in U). \tag{31}$$

If

$$\mathcal{F}(z) = \left(\frac{(\rho+\tau)z^p}{\rho\mathcal{J}_p^{m+1}(\lambda,\ell)f(z) + \tau\mathcal{J}_p^{m}(\lambda,\ell)f(z)}\right)^{\zeta}.$$

$$\cdot\left[\sigma+\zeta\eta\left(p-\frac{\rho z\left(\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)\right)^{'}+\tau z\left(\mathcal{J}_p^{m}(\lambda,\ell)f(z)\right)^{'}}{\rho\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)+\tau\mathcal{J}_p^{m}(\lambda,\ell)f(z)}\right)\right]+\varkappa \quad (32)$$

and

$$\mathcal{F}(z)\prec\sigma q(z)+\eta zq^{'}(z)+\varkappa, \quad (33)$$

then

$$\left(\frac{(\rho+\tau)z^p}{\rho\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)+\tau\mathcal{J}_p^{m}(\lambda,\ell)f(z)}\right)^{\zeta}\prec q(z) \quad (34)$$

and q is the best dominant of (34).

Proof. Let $g(z)$ defined by (23), we see that (24) holds and

$$zg'(z)=\zeta g(z)\left\{p-\frac{\rho z\left(\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)\right)^{'}+\tau z\left(\mathcal{J}_p^{m}(\lambda,\ell)f(z)\right)^{'}}{\rho\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)+\tau\mathcal{J}_p^{m}(\lambda,\ell)f(z)}\right\}. \quad (35)$$

Now, Let us consider $\theta(w)=\sigma w+\varkappa$ and $\phi(w)=\eta$, then θ and $\phi(w)\neq 0$ are analytic in $\mathbb{C}$. Also if we let

$$Q(z)=zq'(z)\phi(q(z))=\eta zq'(z),$$

and

$$h(z)=\theta(q(z))+Q(z)=\sigma q(z)+\eta zq'(z)+\varkappa$$

then the assumption (31) yields that Q is a starlike function in U and that

$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\}=\Re\left\{\frac{\sigma}{\eta}+1+\frac{zq^{''}(z)}{q^{'}(z)}\right\}>0\ (z\in U).$$

The proof follows by applying Lemma 2.2. □

Taking $q(z)=\frac{1+Az}{1+Bz}$ $(-1\leq B<A\leq 1)$ and using (16), the condition (31) reduces to

$$\max\left\{0;-\Re\frac{\sigma}{\eta}\right\}\leq\frac{1-|B|}{1+|B|}, \quad (36)$$

hence, putting $\eta=\rho=1$ and $\tau=0$ in Theorem 3.8, we obtain the following corollary.

Corollary 3.9. *Let $f(z)\in\mathcal{A}(p)$, $-1\leq B<A\leq 1$ and $\sigma\in\mathbb{C}$ with*

$$\max\{0;-\Re(\sigma)\}\leq\frac{1-|B|}{1+|B|},$$

suppose that $\frac{z^p}{\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)} \neq 0 \quad (z \in U)$ *and let* $\zeta \in \mathbb{C}^*$. *If*

$$\left(\frac{z^p}{\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)}\right)^{\zeta} \cdot \left[\sigma + \zeta\left(p - \frac{z\left(\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)\right)'}{\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)}\right)\right] + \varkappa$$
$$\prec \sigma\frac{1+Az}{1+Bz} + \frac{(A-B)z}{(1+Bz)^2} + \varkappa, \quad (37)$$

then

$$\left(\frac{z^p}{\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)}\right)^{\zeta} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant of (37).*

Putting $m = \rho = 0, \eta = \tau = 1,\ p \in \mathbb{N}$ and $q(z) = \frac{1+z}{1-z}$ in Theorem 3.8, we obtain the following corollary.

Corollary 3.10. *Let* $f(z) \in \mathcal{A}(p)$ *such that* $\frac{z^p}{f(z)} \neq 0$ *for all* $z \in U$ *and let* $\zeta \in \mathbb{C}^*$. *If*

$$\left(\frac{z^p}{f(z)}\right)^{\zeta} \cdot \left[\sigma + \zeta\left(p - \frac{zf'(z)}{f(z)}\right)\right] + \varkappa \prec \sigma\frac{1+z}{1-z} + \frac{2z}{(1-z)^2} + \chi, \quad (38)$$

then

$$\left(\frac{z^p}{f(z)}\right)^{\zeta} \prec \frac{1+z}{1-z}$$

and $\frac{1+z}{1-z}$ *is the best dominant of (38).*

4. Superordination and sandwich results

Theorem 4.1. *Let* $q(z)$ *be convex in* U, *with* $q(0) = 1$ *and*

$$\frac{\lambda}{p(p+\ell)}\Re(\alpha) > 0. \quad (39)$$

Let $f(z) \in \mathcal{A}(p)$ *and suppose that* $\frac{z^p}{\mathcal{J}_p^m(\lambda,\ell)f(z)} \in H[q(0),1] \cap \mathcal{Q}$. *If the function*

$$\frac{(p+\alpha)}{p}\left(\frac{z^p}{\mathcal{J}_p^m(\lambda,\ell)f(z)}\right) - \frac{\alpha}{p}\left(\frac{z^p\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)}{\left(\mathcal{J}_p^m(\lambda,\ell)f(z)\right)^2}\right),$$

is univalent in U and

$$q(z)+\frac{\lambda\alpha zq'(z)}{p(p+\ell)}\prec\frac{(p+\alpha)}{p}\left(\frac{z^p}{\mathcal{J}_p^m(\lambda,\ell)f(z)}\right)-\frac{\alpha}{p}\left(\frac{z^p\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)}{\left(\mathcal{J}_p^m(\lambda,\ell)f(z)\right)^2}\right), \tag{40}$$

then

$$q(z)\prec\frac{z^p}{\mathcal{J}_p^m(\lambda,\ell)f(z)}$$

and q is the best subordinant of (40).

Proof. Let $k(z)$ defined by (13), we see that (14) holds. After some computations, we obtain

$$\frac{(p+\alpha)}{p}\left(\frac{z^p}{\mathcal{J}_p^m(\lambda,\ell)f(z)}\right)-\frac{\alpha}{p}\left(\frac{z^p\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)}{\left(\mathcal{J}_p^m(\lambda,\ell)f(z)\right)^2}\right)=k(z)+\frac{\lambda\alpha zk'(z)}{p(p+\ell)} \tag{41}$$

and now, by using Lemma 2.5 we obtain the desired result. □

Taking $q(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$ in Theorem 4.1, we obtain the following corollary.

Corollary 4.2. *Let $q(z)$ be convex in U, with $q(0)=1$ and $\left[\frac{\lambda}{p(p+\ell)}\Re(\alpha)\right]>0$. Let $f(z)\in\mathcal{A}(p)$ and suppose that $\frac{\mathcal{J}_p^m(\lambda,\ell)f(z)}{z^p}\in H[q(0),1]\cap\mathcal{Q}$. If the function*

$$\frac{(p+\alpha)}{p}\left(\frac{z^p}{\mathcal{J}_p^m(\lambda,\ell)f(z)}\right)-\frac{\alpha}{p}\left(\frac{z^p\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)}{\left(\mathcal{J}_p^m(\lambda,\ell)f(z)\right)^2}\right),$$

is univalent in U and

$$\frac{1+Az}{1+Bz}+\frac{\lambda\alpha}{p(p+\ell)}\frac{(A-B)z}{(1+Bz)^2}\prec\frac{(p+\alpha)}{p}\left(\frac{z^p}{\mathcal{J}_p^m(\lambda,\ell)f(z)}\right)-\frac{\alpha}{p}\left(\frac{z^p\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)}{\left(\mathcal{J}_p^m(\lambda,\ell)f(z)\right)^2}\right), \tag{42}$$

then

$$\frac{1+Az}{1+Bz}\prec\frac{z^p}{\mathcal{J}_p^m(\lambda,\ell)f(z)}$$

and $\dfrac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$ is the best subordinant of (42).

The proof of the following theorem is similar to the proof of Theorem 4.1, so we state the theorem without proof.

Theorem 4.3. *Let $q(z)$ be convex in U, with $q(0)=1$, $\eta,\zeta\in\mathbb{C}^*$, $\rho,\tau\in\mathbb{C}$, with $\rho+\tau\neq 0$. Let $f(z)\in\mathcal{A}(p)$ and satisfy the next conditions:*

$$\frac{(\rho+\tau)z^p}{\rho\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)+\tau\mathcal{J}_p^m(\lambda,\ell)f(z)}\neq 0\ (z\in U)$$

and

$$\left(\frac{(\rho+\tau)z^p}{\rho\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)+\tau\mathcal{J}_p^m(\lambda,\ell)f(z)}\right)^{\zeta}\in H[q(0),1]\cap\mathcal{Q}.$$

If the function $1+\zeta\eta\left\{p-\frac{\rho z\left(\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)\right)'+\tau z\left(\mathcal{J}_p^m(\lambda,\ell)f(z)\right)'}{\rho\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)+\tau\mathcal{J}_p^m(\lambda,\ell)f(z)}\right\}$ is univalent in U and

$$1+\eta\frac{zq'(z)}{q(z)}\prec 1+\zeta\eta\left\{p-\frac{\rho z\left(\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)\right)'+\tau z\left(\mathcal{J}_p^m(\lambda,\ell)f(z)\right)'}{\rho\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)+\tau\mathcal{J}_p^m(\lambda,\ell)f(z)}\right\},$$

then

$$q(z)\prec\left(\frac{(\rho+\tau)z^p}{\rho\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)+\tau\mathcal{J}_p^m(\lambda,\ell)f(z)}\right)^{\zeta}\tag{43}$$

and q is the best subordinant of (43).

By applying Lemma 2.4, we obtain the following theorem.

Theorem 4.4. *Let $q(z)$ be convex in U, with $q(0)=1$, $\eta,\zeta\in\mathbb{C}^*$, $\rho,\tau,\sigma,\varkappa\in\mathbb{C}$, with $\rho+\tau\neq 0$ and $\Re\left(\frac{\sigma}{\eta}q'(z)\right)>0$. Let $f(z)\in\mathcal{A}(p)$ and satisfy the next conditions:*

$$\frac{(\rho+\tau)z^p}{\rho\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)+\tau\mathcal{J}_p^m(\lambda,\ell)f(z)}\neq 0\ (z\in U)$$

and

$$\left(\frac{(\rho+\tau)z^p}{\rho\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)+\tau\mathcal{J}_p^m(\lambda,\ell)f(z)}\right)^{\zeta}\in H[q(0),1]\cap\mathcal{Q}.$$

If the function $\mathcal{F}$ given by (32) is univalent in U and

$$\sigma q(z)+\eta zq'(z)+\varkappa\prec\mathcal{F}(z),\tag{44}$$

then

$$q(z)\prec\left(\frac{(\rho+\tau)z^p}{\rho\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)+\tau\mathcal{J}_p^m(\lambda,\ell)f(z)}\right)^{\zeta}$$

and q is the best subordinant of (44).

Combining Theorem 3.1 and Theorem 4.1, we obtain the following sandwich theorem.

Theorem 4.5. *Let q_1 and q_2 be two convex functions in U, such that $q_1(0) = q_2(0) = 1$ and $\left[\frac{\lambda}{p(p+\ell)}\Re(\alpha)\right] > 0$. Let $f(z) \in \mathcal{A}(p)$ and suppose that $\frac{\mathcal{J}_p^m(\lambda,\ell)f(z)}{z^p} \in H[q(0),1] \cap \mathcal{Q}$. If the function*

$$\frac{(p+\alpha)}{p}\left(\frac{z^p}{\mathcal{J}_p^m(\lambda,\ell)f(z)}\right) - \frac{\alpha}{p}\left(\frac{z^p\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)}{\left(\mathcal{J}_p^m(\lambda,\ell)f(z)\right)^2}\right)$$

is univalent in U and

$$q_1(z) + \frac{\lambda\alpha z q_1'(z)}{p(p+\ell)} \prec \frac{(p+\alpha)}{p}\left(\frac{z^p}{\mathcal{J}_p^m(\lambda,\ell)f(z)}\right) - \frac{\alpha}{p}\left(\frac{z^p\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)}{\left(\mathcal{J}_p^m(\lambda,\ell)f(z)\right)^2}\right) \prec q_2(z) + \frac{\lambda\alpha z q_2'(z)}{p(p+\ell)}, \quad (45)$$

then

$$q_1(z) \prec \frac{z^p}{\mathcal{J}_p^m(\lambda,\ell)f(z)} \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and dominant of (45).

Combining Theorem 3.4 and Theorem 4.3, we obtain the following sandwich theorem.

Theorem 4.6. *Let $q(z)$ be convex in U, with $q(0) = 1$, $\eta,\zeta \in \mathbb{C}^*$, $\rho,\tau \in \mathbb{C}$, with $\rho+\tau \neq 0$. Let $f(z) \in \mathcal{A}(p)$ and satisfy $\frac{(\rho+\tau)z^p}{\rho\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)+\tau\mathcal{J}_p^m(\lambda,\ell)f(z)} \neq 0$ $(z \in U)$ and $\left(\frac{(\rho+\tau)z^p}{\rho\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)+\tau\mathcal{J}_p^m(\lambda,\ell)f(z)}\right)^\zeta \in H[q(0),1] \cap \mathcal{Q}$. If the function*

$$1+\zeta\eta\left\{p - \frac{\rho z\left(\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)\right)' + \tau z\left(\mathcal{J}_p^m(\lambda,\ell)f(z)\right)'}{\rho\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)+\tau\mathcal{J}_p^m(\lambda,\ell)f(z)}\right\}$$

is univalent in U and

$$1+\eta\frac{zq_1'(z)}{q_1(z)} \prec 1+\zeta\eta\left\{p - \frac{\rho z\left(\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)\right)' + \tau z\left(\mathcal{J}_p^m(\lambda,\ell)f(z)\right)'}{\rho\mathcal{J}_p^{m+1}(\lambda,\ell)f(z)+\tau\mathcal{J}_p^m(\lambda,\ell)f(z)}\right\} \prec 1+\eta\frac{zq_2'(z)}{q_2(z)}, \quad (46)$$

then

$$q_1(z) \prec \left(\frac{(\rho+\tau)z^p}{\rho \mathcal{J}_p^{m+1}(\lambda,\ell) f(z) + \tau \mathcal{J}_p^m(\lambda,\ell) f(z)} \right)^\zeta \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and dominant of (46).

Combining Theorem 3.8 and Theorem 4.4, we obtain the following sandwich theorem.

Theorem 4.7. *Let q_1 and q_2 be two convex functions in U, with $q_1(0) = q_2(0) = 1$, let $\eta, \zeta \in \mathbb{C}^*$,$\rho, \tau, \sigma, \varkappa \in \mathbb{C}$, with $\rho + \tau \neq 0$ and $\Re\left(\frac{\sigma}{\eta} q'(z)\right) > 0$. Let $f(z) \in \mathcal{A}(p)$ satisfies* $\frac{(\rho+\tau)z^p}{\rho \mathcal{J}_p^{m+1}(\lambda,\ell) f(z) + \tau \mathcal{J}_p^m(\lambda,\ell) f(z)} \neq 0 \quad (z \in U)$ *and* $\left(\frac{(\rho+\tau)z^p}{\rho \mathcal{J}_p^{m+1}(\lambda,\ell) f(z) + \tau \mathcal{J}_p^m(\lambda,\ell) f(z)} \right)^\zeta \in H[q(0),1] \cap \mathcal{Q}$. *If the function $\mathcal{F}$ given by (32) is univalent in U and*

$$\sigma q_1(z) + \eta z q_1'(z) + \chi \prec \mathcal{F}(z) \prec \sigma q_2(z) + \eta z q_2'(z) + \varkappa, \tag{47}$$

then

$$q_1(z) \prec \left(\frac{(\rho+\tau)z^p}{\rho \mathcal{J}_p^{m+1}(\lambda,\ell) f(z) + \tau \mathcal{J}_p^m(\lambda,\ell) f(z)} \right)^\zeta \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and dominant of (47).

Remark 4.8. By Specializing λ, ℓ and m in the above results, we obtain the corresponding results for the operators $I_p^m(\lambda,\ell)$, $J_p^m(\lambda,\ell)$, $D_{\lambda,p}^m$ and $D_{\lambda,p}^m$, which are defined in introduction.

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