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ON THE INTERMEDIATE VALUE THEOREM OVER A NON-ARCHIMEDEAN FIELD

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The paper investigates general properties of the power series over a non-Archimedean ordered field, extending to the set of algebraic power series the intermediate value theorem and Rolle's theorem and proving that an algebraic series attains its maximum and its minimum in every closed interval.

The paper also investigates a few properties concerning the convergence of power series, Taylor's expansion around a point and the order of a zero.

1. Introduction

It is well-known that over $\mathbb{R}$, which is a complete Archimedean ordered field, the least upper bound property implies that the following basic results of Mathematical Analysis hold for every continuous function on a closed interval:

1. the intermediate value theorem

2. the boundedness theorem

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3. the Weierstrass theorem (on the attainment of the absolute maximum and minimum)

4. the mean value theorem

5. Rolle's theorem.

There are many different proofs of the above results. We want to point out that the dichotomic procedure, based on the nested cells property, which is equivalent to the existence of the least upper bound, can be used to prove 1 and 2, obtaining then 3 as a consequence of 2, and 4 and 5 as consequences of 3.

If $\mathbb{K}$ is a non-Archimedean complete ordered field, it lacks the least upper bound property (and so the dichotomic procedure). In this event not only the Archimedean proofs of the above properties do not work, but the properties fail to be true in general. It is nevertheless true that they hold for polynomial and rational functions (see [1]), but completeness is neither necessary nor sufficient, a different property is required (over $\mathbb{R}$ it is implied by completeness): the field must be maximal ordered (or real closed in Artin's terminology, see [1], §2, 5, [5], chap. XI, [13], chap. IX). Moreover proofs of the above results over a non-Archimedean maximal ordered field can be obtained with a quite different logical order. In fact the intermediate value theorem, whose proof is algebraic, can be used to obtain Rolle's and the mean value theorems and so to prove the attainment of the absolute maximum and minimum. It is worth observing that this approach, when polynomials and rational functions are considered, works both in the Archimedean and in the non-Archimedean case.

In this paper we want to investigate the intermediate value theorem and its consequences over a non-Archimedean field $\mathbb{K}$, so in lack of the dichotomic procedure. We are able to show that such properties, false in general for a continuous function, can be extended to any power series $y(X)$ which is algebraic over the field $\mathbb{K}(X)$ of rational functions, provided that $\mathbb{K}$ is maximal ordered (but not necessarily complete). The logical structure of our construction follows more or less what can be done for polynomials, i.e. the key tool is the intermediate value theorem, whose proof requires most of our efforts.

To the purpose of achieving such a result, we need some properties of power series over a maximal ordered or a Cauchy complete field $\mathbb{K}$, concerning the convergence domain, Taylor's expansion around a point and the order of a zero. A few among them come out to be unexpectedly quite different from what is true on the field of real numbers. It is for instance the case of a translation property, which is false over $\mathbb{R}$ and also over $\mathbb{C}$. To our knowledge some of these properties are new, while others are partially proved in [9].

We think that it is worth observing that the intermediate value theorem has also been investigated in other non-Archimedean cases, for instance in [7] and in [10]. Over a Levi-Civita field it has been investigated and proved (for a larger class of power series) in [11], while the mean value and the extreme value theorems are proved in [12].

2. General facts

Let $\mathbb{N} = \{0, 1, \cdots\}$ be the set of natural numbers. The Archimedean property for $\mathbb{N}$ ((AP) from now on) states what follows:

(AP) Given two natural numbers r, s, where $r \neq 0$, there is a natural number n such that $nr > s$.

(AP) can easily be extended to the set $\mathbb{Q}$ of rational numbers: given $r, s \in \mathbb{Q}^+, r \neq 0$, there is $n \in \mathbb{N}$ such that $nr > s$.

(AP) for $\mathbb{Q}$ can be equivalently stated as follows: the set $\mathbb{N}$ is not bounded above in $\mathbb{Q}$, i.e., for every rational number q, there is $n \in \mathbb{N}$ such that $n > q$.

(AP) is well-defined for every ordered field $\mathbb{K}$ (see [1], [4], [13] for general properties of ordered fields). In fact every ordered field $\mathbb{K}$ contains automatically $\mathbb{Q}$ (see [1], §2, 3, Example 1) and so also $\mathbb{N}$. $\mathbb{K}$ is called non-Archimedean if $\mathbb{N}$ is bounded above.

Notation.

$\mathbb{K}$ is a non-Archimedean ordered field, $\bar{\mathbb{K}}$ the ordered closure, i.e the largest algebraic extension that can be ordered (see [1], §2, 5, Theorem 2, and [13], chap. IX, 71, Theorem 8), $\hat{\mathbb{K}}$ the Cauchy completion (see [13], chap. IX, 67 and [4], §3) and $\mathbb{K}^c$ the algebraic closure.

An element $\varepsilon \in \mathbb{K}, \varepsilon \neq 0$ is infinitesimal with respect to $\mathbb{Q}$ if $|\varepsilon| < \frac{1}{n}, \forall n$ positive whole number, and an element $\omega \in \mathbb{K}$ is unlimited with respect to $\mathbb{Q}$ if $|\omega| > n, \forall n$ positive whole number (see [4], §3).

Over an ordered field $\mathbb{K}$ the notion of convergence of a sequence (a_n) and of a series $\sum u_n$ can be given as usual. For instance $\lim_{n\to\infty} a_n = l$ means that, given any positive $\varepsilon \in \mathbb{K}$, there is n such that, $\forall m > n, |a_m - l| < \varepsilon$.

We recall the following property (proved in [8]) which holds only in the non-Archimedean case:

Proposition 2.1. *Let $S = \sum_{n=0}^{\infty} b_n$ be any series over a non-Archimedean field $\mathbb{K}$; then S is convergent in $\hat{\mathbb{K}}$ if and only if* $\lim_{n\to\infty} b_n = 0$. *As a consequence a series is convergent in $\hat{\mathbb{K}}$ if and only if it is absolutely convergent.*

The following example will be useful to produce counterexamples to the statements that depend on (AP).

Example 2.2. Let $\mathbb{K} = \mathbb{Q}(t)$ be the field of rational functions over $\mathbb{Q}$. We define a total order in $\mathbb{K}$ by setting, for every quotient of polynomials: $\frac{f(t)}{g(t)} > 0$ if the function of t defined by $\frac{f(t)}{g(t)}$ is positive for t large enough.

It is easy to show that we obtain an ordered field and that, given $t \in \mathbb{K}$, there is no $n \in \mathbb{N}$ such that $n > t$, because $n - t$ will be negative for t large enough, notwithstanding which n we choose. Therefore $\mathbb{K}$ is a non-Archimedean ordered field. The following sets: $U_n = \{x \in \mathbb{K} : |x| < t^{-n}, n \in \mathbb{N}\}$ form a countable basis for the neighbourhoods of 0 (see [6], chap. I, p. 50).

Of course we obtain a similar example if we consider the field $\mathbb{L} = \mathbb{R}(t)$.

We want to show that $\mathbb{Q}(t)$ is not complete.

Let us consider the following power series $\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} t^{-n}$ and put: $s_n =$ sum of the first $n+1$ terms (from 0 to n). Such a series converges to some element $a \in \widehat{Q(t)}$ (see the above Proposition 2.1). It can be shown that s_n^2 converges to $1 + \frac{1}{t}$ and so that s_n converges to $\sqrt{1 + \frac{1}{t}}$, which is easily seen to be outside of $\mathbb{Q}(t)$.

Also the series $\sum_{n=0}^{\infty} (n!t^n)^{-1} = e^{\frac{1}{t}}$ is not algebraic over $\mathbb{Q}(t)$ or even over $\mathbb{R}(t)$ (see [8], p. 340).

We now shortly recall a few theorems concerning polynomials and rational functions over a **maximal ordered** field $\mathbb{K} = \bar{\mathbb{K}}$.

In what follows $P(X) = a_0 + \cdots + a_m X^m$ is a polynomial and $a, b, a < b$, two elements of $\mathbb{K}$. The derivative of a polynomial is defined as usual and the rules of differentiation for the sum, the product and the quotient of polynomials can be easily seen to hold true.

The following results are contained (more or less explicitly) in [1] as exercises without proofs, with the exception of the intermediate value theorem, whose proof is given in §2, 5, Proposition 5. Actually the arguments of our proofs for algebraic series contained in section 4 (see Theorems 4.10, 4.11, 4.14) work for the properties below (from 2 to 4).

1. (The intermediate value theorem - [1], §2, Prop. 5). Let $P(X)$ be a polynomial over $\mathbb{K}$. If $P(a)P(b) < 0$, then there is $x \in \mathbb{K}, a < x < b$ such that $P(x) = 0$.

2. (Rolle's theorem - [1], ex. 12, p. 57) Assume that the polynomial $P(X)$ vanishes both at a and at b and nowhere else between a and b. Let $\frac{h(X}{k(X)}$ be a rational function such that $k(x) \neq 0, \forall x \in [a,b]$. Then $F(X) = P(X)h(X) + k(X)P'(X)$ vanishes at least at one point $x \in]a,b[$. In particular, if $h(X) = \frac{h_1(X)}{h_2(X)}$ is a rational function such that $h_2(x) \neq 0, \forall x, a < x < b$ and $h(a) = h(b) = 0$, then the equation $h'(X) = 0$ has at least one root $x \in \mathbb{K}, a < x < b$.

3. (The mean value theorem, [1], Exercise 13, p. 57). If $h(X) = \frac{h_1(X)}{h_2(X)}$ is a rational function such that $h_2(x) \neq 0, \forall x, a \leq x \leq b$, then there is $c \in]a,b[$

such that $h(b) - h(a) = (b-a)h'(c)$.

4. (Monotonic functions, [1], Exercise 13, p. 57)

   (i) The rational function $h(X)$ is strictly increasing (decreasing) at $x \in [a,b]$ if $h'(x) > 0$ ($h'(x) < 0$), where the derivative at a (b) is the right-hand (left-hand) derivative.

   (ii) If the rational function $h(X)$ is increasing (decreasing) at $x \in [a,b]$, then $h'(x) \geq 0$ ($h'(x) \leq 0$).

   (iii) The rational function $h(X)$ is increasing (decreasing) in $[a,b]$ if and only if $h'(x) \geq 0$ $(h'(x) \leq 0), \forall x, a \leq x \leq b$.

   (iv) At a point $x \in]a,b[$ where $h(X)$ attains a local maximum or a local minimum the derivative $h'(x)$ vanishes.

   (The arguments to prove (i), (ii), (iv) do not depend upon the intermediate value theorem, they depend on the definition of the derivative as a limit, while (iii) requires the mean value theorem which follows from the intermediate value theorem).

5. Let $h(X)$ be a rational function defined on the closed interval $[a,b]$. Then

   (i) $h(X)$ is bounded above and below,

   (ii) $h(X)$ attains both its absolute maximum and its absolute minimum.

   This result is not stated explicitly in [1], but it can be easily obtained by using 4. We give here a proof since we use it in section 4 of the present paper.

   Proof of 5.

   Let $A = \{x \in]a,b[: h'(x) = 0\} = \{x_1, \cdots, x_r\}$ and put $B = A \cup \{a,b\}$. We consider the following cases.

   1. $A = \emptyset$. In this event $y'(X)$ is everywhere strictly positive or negative, so the absolute maximum and the absolute minimum are attained at the endpoints.

   2. $A \neq \emptyset$. Then the absolute maximum and the absolute minimum have to be looked for among the values at the endpoints and the local maxima and minima.

   Let in fact $z \notin B$. Then z lies between two elements of B, say for instance x_i, x_{i+1}. Since $h'(X) > 0$ or < 0 in such interval, the maximum and the minimum in it are the endpoints, so not z. Therefore the absolute maximum and the absolute minimum are attained at points belonging to B.

Remark 2.3. On a complete Archimedean field (in fact $\mathbb{R}$), property 5 above, for every continuous function $f(X)$, can be proved in two steps:

step 1: $f(X)$ is bounded above and below and so it has the least upper bound M and the greatest lower bound m,

step 2: if there is no absolute maximum (or minimum), also $\frac{1}{M-f(X)}$ $(\frac{1}{f(X)-m})$ is continuous and so bounded, which leads to a contradiction.

When we deal with a non-Archimedean field, where a bounded set can lack the least upper bound and the greatest lower bound, the absolute maximum and the absolute minimum (for a polynomial) are proved directly to be taken on, obtaining as a consequence the boundedness result.

Remark 2.4. If $h(X)$ is a rational function, then $|h(X)|$ attains the absolute maximum and the absolute minimum in any closed interval $[a,b]$. If either $h(X) \leq 0$ or $h(X) \geq 0$ anywhere it is obvious. Otherwise, let $M = h(c) \geq 0$ be the absolute maximum and $m = h(d) \leq 0$ be the absolute minimum. Then $\max(M, -m)$ is the absolute maximum of $|h(X)|$, while 0 is the absolute minimum (thanks to the intermediate value theorem applied to $h(X)$).

The same property holds for every function $f(X)$ attaining its maximum and minimum and satisfying the intermediate value theorem.

Remark 2.5. When $\mathbb{K}$ is not maximal ordered, the intermediate value theorem holds true in $\bar{\mathbb{K}}$ but not necessarily in $\hat{\mathbb{K}}$, which might not contain $\bar{\mathbb{K}}$. This is a key difference between the Archimedean and the non-Archimedean case since, if $\mathbb{K}$ is Archimedean, $\bar{\mathbb{K}} \subset \hat{\mathbb{K}} = \mathbb{R}$.

For instance, on $\mathbb{K} = \mathbb{R}(t)$, the polynomial $P(X) = X^2 - t$ is positive at t and negative at 1, so $\bar{\mathbb{K}}$ contains a square root of t. However it is easy to see that $\sqrt{t}$ is isolated in $\bar{\mathbb{K}}$, so that it does not belong to the completion $\hat{\mathbb{K}}$. In fact, assume that $\sqrt{t} = \lim_{n\to\infty} \frac{P_n(t)}{Q_n(t)}$, where $\deg(P_n) = p_n$, $\deg(Q_n) = q_n$. Then, given any $r \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that, $\forall n > N, |\frac{P_n^2(t)}{Q_n^2(t)} - t| < t^{-r}$, i.e. $|P_n^2(t)t^r - t^{r+1}Q_n^2(t)| < Q_n^2(t)$. We point out that the degree on the right side is $2q_n$ and the degree on the left side is higher, unless $2p_n + r = r + 1 + 2q_n$, and this is impossible.

When $\mathbb{K}$ is a non-Archimedean field, it is easy to produce a continuous function (with respect to the topology of the order) that takes on positive and negative values but vanishes nowhere.

Example 2.6. Let $\mathbb{K}$ be $\mathbb{R}(t)$ as introduced in Example 2.2. Let $f(x)$ be defined as follows:

$f(x) = -1$ if there is $a \in \mathbb{R}, x \leq a, f(x) = 1$ otherwise.

Then f is a continuous function on $\mathbb{K}$ taking on positive and negative values, but vanishing nowhere.

If $\mathbb{K}$ is not maximal ordered, a rational function, and even a polynomial, can lack both the absolute maximum and the absolute minimum and also the greatest lower bound and the least upper bound on a closed interval (this is well-known over $\mathbb{Q}$).

The following is a non-Archimedean example.

Example 2.7. Over $\mathbb{K} = \mathbb{R}(t)$ let us consider $P(X) = X^3 - 3tX + 1$, whose derivative vanishes both at $x = \sqrt{t}$ and at $y = -x$. In $\bar{\mathbb{K}}$ consider for instance the closed interval $[0,t]$. The polynomial attains its minimum value at x. But $x \notin \mathbb{R}(t)$ and therefore the polynomial has no minimum on $\mathbb{R}(t)$ and lacks also the greatest lower bound. Similarly we can find an interval where there is no least upper bound.

3. Power series over a non-Archimedean field

Notation.

In what follows $\mathbb{K}[[X]]$ denotes the ring of formal power series in one variable X over $\mathbb{K}$. The domain of convergence of the formal power series $S(X) = \sum a_n X^n$ is the set D_S of those $x \in \mathbb{K}$ such that $\sum_{i=0}^{n} a_i x^i$ is a Cauchy (or fundamental) sequence of elements of $\mathbb{K}$ (see [13], chap. IX, 67 and [4], §3); the value $S(x)$ at $x \in D_S$ belongs to $\hat{\mathbb{K}}$ and not necessarily to $\mathbb{K}$.

Hence a power series $S(X)$ can be considered a function $D_S \to \hat{\mathbb{K}}$.

We point out that, for many properties of power series (see [8]), a countable basis (see [6], chap. 1, p. 50) is necessary. This means that there is a sequence of positive numbers (ε_n) converging to 0, so that $U_n = \{x : |x| < \varepsilon_n\}$ is a basis of the neighbourhoods of 0. In this event two cases can occur (see also [9] for a slightly different approach):

1. there is an infinitesimal positive element $\varepsilon \in \mathbb{K}$ such that $\lim_{n\to\infty} \varepsilon^n = 0$; in this case one can choose $\varepsilon_n = \varepsilon^n$;

2. if the preceding property does not hold, a subsequence $(\bar{\varepsilon}_n)$ can be extracted, such that $\bar{\varepsilon}_{n+1} < \bar{\varepsilon}_n^r, \forall r$. In fact we can replace the original sequence by a decreasing sequence starting with an element less than 1; then we choose $\bar{\varepsilon}_0 = \varepsilon_0, \bar{\varepsilon}_1 = \varepsilon_j$, where ε_j is the first element of the sequence less than every power $\varepsilon_0^i, i \in \mathbb{N}$, and, by recursion, $\bar{\varepsilon}_{n+1} = \varepsilon_h$, where ε_h is the first element of the sequence less than $\bar{\varepsilon}_n^i, i \in \mathbb{N}$.

It is worth observing that the second case (which we do not exclude), gives rise to odd properties for the power series over $\mathbb{K}$, i.e. all power series converge either everywhere or only at 0.

In fact, let us assume that the power series $S(X) = \sum_0^\infty a_n X^n$ is convergent at some $x \neq 0$. Then $\lim_{n\to\infty} |a_n x^n| = 0$. As by hypothesis it is not true that $\lim_{n\to\infty}|x^n| = 0$, and the sequence $(|x^n|)$ is monotonic, we conclude that $\lim_{n\to\infty} a_n = 0$.

If $y \neq 0$ is another element, it is not true that $\lim_{n\to\infty}|y^n| = \infty$, because this would imply $\lim_{n\to\infty}|(y)^{-n}| = 0$. Due to the fact that the sequence $(|y^n|)$ is monotonic, the conclusion is that it is bounded above. Then $\lim_{n\to\infty}|a_n y^n| = 0$, and $S(X)$ is convergent at y.

We must observe that, if there is no countable basis (for instance when hyperreals are considered, see [2]), a power series different from a polynomial converges only at 0.

Remark 3.1. Let $A(X)$ and $B(X)$ be two formal power series, with convergence domains D_A and D_B. The series $C(X) = A(X) + B(X)$ and $D(X) = A(X)B(X)$ are defined by the usual operations on the coefficients. If a value d belongs both to D_A and to D_B, the equalities $C(d) = A(d) + B(d)$ and $D(d) = A(d)B(d)$ hold true, the former being obvious and the latter depending upon the following proposition, whose proof is given in [3], §168 and does not depend on the Archimedean property.

Proposition 3.2. *Let $\sum_{k=0}^{\infty} a_k = A, \sum_{k=0}^{\infty} b_k = B$ two absolutely convergent series. Then*

$$\sum_{k=0}^{\infty}\Big(\sum_{h=0}^{k} a_h b_{k-h}\Big)$$

is convergent and its value is AB.

We recall a few results concerning power series over a non-Archimedean field $\mathbb{K}$ that also appear in [9]. We think that it is useful to the reader to find here short proofs of the statements contained in Propositions 3.3 and 3.4 below.

Proposition 3.3. *Let $S(X) = \sum_{n=0}^{\infty} a_n X^n$ be any power series with coefficients in $\mathbb{K}$.*

1. If $S(X)$ is convergent at x, then it is so also at $-x$ and at any x' between x and $-x$; moreover $S(X)$ is uniformly convergent in $[-|x|,|x|]$.

2. If $S(X)$ is convergent both at x and at y, then it is convergent also at $x+y$.

3. If $S(X)$ is convergent at $x \neq 0$, then there is $\delta > 0$ such that $S(X)$ is convergent in the open interval $]x-\delta, x+\delta[$.

4. From 1. and 2. it follows that D_S is a subgroup of $\mathbb{K}$.

Proof. 1. It is enough to observe that $\forall z \in [-|x|,|x|], S(z)$ is bounded above by $\sum_{n=0}^{\infty}|a_n x^n|$.

2. Assume for the sake of simplicity that $0 < x \leq y$, the other cases being very similar. It is enough to observe that $S(X)$ is convergent at $2y \geq x+y$, because $\lim a_n y^n = 0$ implies $\lim a_n (2y)^n = 0$.

3. We can choose $\delta = |y|$, for any $y \in D_S$.
4. Obvious (since $0 \in D_S$).

□

The derivative of a power series at a point $x \in D_S$ is defined as

$$\lim_{h \to 0} \frac{S(x+h) - S(x)}{h}$$

(provided that such limit exists) and fulfils all the usual rules for the sum, the product and the quotient of series, as it can easily be seen.

Proposition 3.4. *Let $S(X) = \sum_{n=0}^{\infty} a_n X^n$ be a power series converging at every point of the closed interval $[a,b]$. Then the following hold true:*
1. $S(X)$ is a continuous function at every $x \in [a,b]$,
2. the series $\bar{S}(X) = \sum_{n=1}^{\infty} n a_n X^{n-1}$ converges in $[a,b]$.
3. $\forall x \in [a,b], \bar{S}(x) = S'(x)$,

Proof. 1. Put: $S_n(X) = \sum_{i=0}^{n} b_i X^i$ and observe that $S_n(X)$ is a polynomial, hence a continuous function; then it is enough to notice that, if $h \in \mathbb{K}$ is such that $S(x+h)$ is defined, then we have:

$$|S(x+h) - S(x)| \leq |S(x+h) - S_n(x+h)| + |S_n(x+h) - S_n(x)| + |S_n(x) - S(x)|$$

and that $S_n(X)$ converges uniformly to $S(X)$ and is everywhere continuous.

2. If $S(X)$ converges at some x, then $\lim_{n\to\infty} a_n x^n = 0$. In this event we also have: $\lim_{n\to\infty} n a_n x^{n-1} = 0$, because $|na_n| < \omega |a_n|$, where ω is any positive unlimited element, so the latter series converges at x, possibly to a sum $\bar{S}(x) \in \hat{\mathbb{K}}$.

3. We have:

$$R(h) = \frac{S(x+h) - S(x)}{h} - \sum_{1}^{\infty} n a_n x^{n-1} = \sum_{1}^{\infty} a_n \left(\frac{(x+h)^n - x^n}{h} - n x^{n-1}\right) =$$

$$h \sum_{2}^{\infty} a_n \left(\binom{n}{2} x^{n-2} + \binom{n}{3} x^{n-3} h + \cdots + h^{n-2}\right)$$

(converging series can be added and subtracted).

There is no loss of generality in assuming $|h| < |x|$, since we are considering $\lim_{h\to 0} R(h)$. The case $x = 0$ is excluded but trivial.

We have

$$|R_h| \leq |h| \omega \sum |a_n| |x^{n-2}| = |h| \omega \frac{T(x)}{|x|^2},$$

where ω is any positive unlimited element and $T(x) = \sum |a_n x^n|$.
Therefore we have: $\lim_{h\to 0} R_h = 0$.

□

The following two lemmas hold both in the Archimedean and in the non-Archimedean case. The former case is covered by [3], §169. We give here direct proofs which work in all cases.

Lemma 3.5. *Let* $\{\lambda_{ij} \geq 0, i = 0 \to \infty, j \in \mathbb{N}\}$ *be any set of elements in* $\mathbb{K}$. *Assume that the following limits exist:*

$$\lim_{n\to\infty} \sum_{i=0}^{n} (\sum_{j=0}^{i} \lambda_{ij}) = A = \sum_{i=0}^{\infty} (\sum_{j=0}^{i} \lambda_{ij}) \tag{1}$$

and

$$B_j = \lim_{k\to\infty} \sum_{i=j}^{k} \lambda_{ij} = \sum_{i=j}^{\infty} \lambda_{ij} \quad (\forall j \geq 0). \tag{2}$$

Then $B = \lim_{n\to\infty} \sum_{j=0}^{n} B_j = \lim_{n\to\infty} \sum_{j=0}^{n} (\sum_{i=j}^{\infty} \lambda_{ij}) = \sum_{j=0}^{\infty} (\sum_{i=j}^{\infty} \lambda_{ij})$ *exists and moreover* $A = B$.

Proof. If $k > n$, then

$$\sum_{j=0}^{n} (\sum_{i=j}^{k} \lambda_{ij}) \leq \sum_{j=0}^{k} (\sum_{i=j}^{k} \lambda_{ij}) = \sum_{i=0}^{k} (\sum_{j=0}^{i} \lambda_{ij}) \leq A.$$

Thanks to (2) we can take the limit when k tends to ∞:

$$\sum_{j=0}^{n} (\sum_{i=j}^{\infty} \lambda_{ij}) \leq A. \tag{3}$$

Moreover we have:

$$\sum_{j=0}^{n} (\sum_{i=j}^{\infty} \lambda_{ij}) \geq \sum_{j=0}^{n} (\sum_{i=j}^{n} \lambda_{ij}) = \sum_{i=0}^{n} (\sum_{j=0}^{i} \lambda_{ij}).$$

Now, thanks to (1), given $\varepsilon > 0$, we have that, for n large enough,

$$\sum_{j=0}^{n} (\sum_{i=j}^{\infty} \lambda_{ij}) \geq A - \varepsilon.$$

Now we use (3) and see that $\lim_{n\to\infty} \sum_{j=0}^{n} (\sum_{i=j}^{\infty} \lambda_{ij}) = A$. □

Lemma 3.6. *Let* $\{\lambda_{ij}, i, j \in \mathbb{N}\}$ *be any set of elements in* $\mathbb{K}$ *and assume that the following limits exist:*

$$\bar{A} = \lim_{n\to\infty} \sum_{i=0}^{n} (\sum_{j=0}^{i} |\lambda_{ij}|)$$

and

$$\bar{B}_j = \lim_{k\to\infty} \sum_{i=j}^{k} |\lambda_{ij}| \quad (\forall j \geq 0).$$

Then $B_j = \sum_{i=j}^{\infty} \lambda_{ij}, B = \sum_{j=0}^{\infty} B_j = \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} \lambda_{ij}$ *and* $A = \sum_{i=0}^{\infty}(\sum_{j=0}^{i} \lambda_{ij})$ *also exist and moreover* $B = A$.

Proof. It is enough to apply the previous lemma to the non-negative numbers $\lambda_{ij}^{(+)} = \frac{|\lambda_{ij}|+\lambda_{ij}}{2}$ and $\lambda_{ij}^{(-)} = \frac{|\lambda_{ij}|-\lambda_{ij}}{2}$ (the corresponding series converge thanks to 2.1). □

The following theorem, which states that a translation of the origin is allowed for a power series, improves a similar statement of [9] (2, property VI), giving the explicit computation of the coefficients of the translated series (thanks to the above lemmas).

Theorem 3.7. *Let* $S(X) = \sum_0^\infty a_i X^i$ *be a formal power series converging in a domain* D_S *and let* $d \in D_S$ *and* $y \in D_S$. *Then* $\sum_0^\infty a_i(y+d)^i = \sum_0^\infty b_j y^j$ *where* $b_j = \sum_{i=j}^{\infty} \binom{i}{j} a_i d^{i-j} = \frac{1}{j!} S^{(j)}(d)$, *the latter being a converging series for every* j.

Proof. By Propositions 2.1, 3.3, 3.4, $S(X) = \sum_0^\infty a_i X^i$ is absolutely convergent in D_S, along with all its derivatives, which can be computed by a term by term differentiation, just as in the Archimedean case. We obtain

$$S^{(j)}(X) = \sum_{i=j}^{\infty} \binom{i}{j} j! a_i X^{i-j}$$

Therefore the series

$$S^{(j)}(d) = \sum_{i=j}^{\infty} \binom{i}{j} j! a_i d^{i-j}$$

is absolutely convergent for each j, and so is also the following one

$$\frac{S^{(j)}(d) y^j}{j!} = \sum_{i=j}^{\infty} \binom{i}{j} a_i d^{i-j} y^j.$$

Moreover we have

$$\sum_{i=0}^{\infty} a_i (y+d)^i = \sum_{i=0}^{\infty} a_i \sum_{j=0}^{i} \binom{i}{j} d^{i-j} y^j$$

with absolute convergence.

Therefore Lemma 3.6 can be applied with

$$\lambda_{ij} = a_i \binom{i}{j} d^{i-j} y^j, A = \sum_{i=o}^{\infty} \sum_{j=0}^{i} a_i \binom{i}{j} d^{i-j} y^j, B_j = \sum_{i=j}^{\infty} a_i \binom{i}{j} d^{i-j} y^j = \frac{S^{(j)}(d)}{j!} y^j,$$

so that the claim follows.

□

By using the above theorem we can prove the following

Proposition 3.8. *Let $S(X) = \sum_{n=0}^{\infty} a_n X^n$ be a formal power series converging in $[-x+d, x+d]$, for some $x, d \in \mathbb{K}, x > 0$. If $S(y) = 0, \forall y \in [-x+d, x+d]$, then $a_n = 0, \forall n$.*

Proof. Step 1.Let us first assume that $d = 0$; then $S(0) = 0$ implies $a_0 = 0$, hence $S(X) = X \sum_{n=1}^{\infty} a_n X^{n-1}$. Now consider the formal derivative

$$S'(X) = \sum_{n=1}^{\infty} n a_n X^{n-1}$$

and observe that, since $S(X)$ is the zero function in $[-x, x]$, also its derivative is the zero function. Then we have: $S'(0) = 0$, which implies $a_1 = 0$. Continuing in this way, we prove the claim in the case $d = 0$.

Step 2. As to the general case, we apply Theorem 3.7 and obtain:

$$0 = S(d+y) = \sum_{n=0}^{\infty} b_n y^n$$

which holds for all $y \in [-x, x]$. Now step 1 can be applied, obtaining $b_n = 0$. Using the formulas given in Theorem 3.7 the claim is proved. □

Corollary 3.9. *If two series take on the same values on an interval, then they coincide as formal power series.*

Remark 3.10. Theorem 3.7 above states that, if a series S has a convergence domain D_S, and $d \in D_S$, then the function given by the series has a Taylor expansion around d (a series of powers of $y = x - d$). The main difference from the Archimedean case is that the domain of convergence in the new variable y coincides with D_S. Therefore in the non-Archimedean case we cannot have a theory of the analytic continuation of a function outside of its domain.

Corollary 3.9 states that the well-known principle of identity for analytic functions holds also in our case.

Theorem 3.11. *Let $S(X) = \sum_0^{\infty} a_n X^n$ be a non-zero power series, $D_S \neq \{0\}$ its domain of convergence, and $d \in D_S$, such that $S(d) = 0$. Then there is a unique integer $s \neq 0$ such that*

$$S(X) = (X-d)^s q(X), \tag{4}$$

where $q(X)$ satisfies the following conditions:

(i) $q(X)$ is convergent in D_S,
(ii) $q(d) \neq 0$.
Moreover $S(x) = (x-d)^s q(x), \forall x \in D_S$.
(The number s is called multiplicity of d as a root of $S(X)$).

Proof. As $S(X)$ is convergent at d, Taylor's formula (Theorem 3.7) can be applied, so obtaining:

$$S(x) = \sum_0^\infty \frac{1}{n!} S^{(n)}(d)(x-d)^n, \forall x \in D_S. \tag{5}$$

If $S^{(s)}(d)$ is the first derivative not vanishing at d, then (5) can be written in the following form:

$$S(x) = (x-d)^s \sum_s^\infty \frac{1}{n!} S^{(n)}(d)(x-d)^{n-s}, \forall x \in D_S.$$

Thanks to Theorem 3.7 , this is equivalent to

$$S(x) = (x-d)^s \sum_{n=0}^\infty q_n x^n, \forall x \in D_S, \tag{6}$$

for suitable coefficients $q_n, n = 0, \cdots$. Corollary 3.9 states that (6) is equivalent to

$$S(X) = (X-d)^s \sum_{n=0}^\infty q_n X^n.$$

So (4) and (i) are satisfied with $q(X) = \sum_{n=0}^\infty q_n X^n$. As to (ii), if $q(d) = 0$, we can replace $S(X)$ by $q(X)$ and repeat our argument, so obtaining $q(X) = (X-d)^h q_1(X), h \geq 1$ and, as a consequence,

$$S(X) = (X-d)^{s+h} q_1(X). \tag{7}$$

But (7) implies $S^{(s+h)}(d) = 0, h \geq 1$, in contradiction with our choice of s.

Clearly the number s appearing in (4) is the order of the first derivative of $S(X)$ which does not vanish at d.

□

Remark 3.12. If $d \neq 0$, there is a unique power series $q(X) = \sum q_n X^n$ satisfying the following equality between formal power series: $S(X) = (X-d)^s q(X)$. In fact, if $s = 1$, there are recursive relations that define $q_n, \forall n$, and we can proceed by induction on s. However, if $S(d) \neq 0$, the equality of formal power series does not imply the equality of functions, because $q(X)$ does not converge at d (but it might converge in a subset of D_S not containing d). The above theorem states that the necessary condition $S(d) = 0$ is also sufficient for such equality.

4. The intermediate value theorem and the absolute maximum and minimum of algebraic series

In what follows $\mathbb{K} = \bar{\mathbb{K}}$ is a non-Archimedean maximal ordered field, $y = y(X) = \sum_0^\infty b_n X^n \in \mathbb{K}[[X]]$ is a power series, algebraic over $\mathbb{K}(X)$, i.e. there is an identical relation

$$a_0(X) + a_1(X)y + \cdots + a_m(X)y^m = 0, \tag{8}$$

where the $a_i^\prime s$ are polynomials and the degree m is chosen as small as possible. We can assume that $a_0(X)$ is monic and that there is no common factor to $a_0(X), \cdots, a_m(X)$ (divide if necessary by the common factors and by the leading coefficient of $a_0(X)$). In particular $a_0(X), \cdots, a_m(X)$ have no common root. Moreover we put: $y_n(X) = \sum_0^n b_r X^r$.

First of all we show that the intermediate value theorem is false for an algebraic continuous function (not a series) over a non-Archimedean field even if it is maximal ordered.

Example 4.1. Let $\mathbb{L}$ be any non-Archimedean field and consider the following algebraic function:

$y(X) = 1, \forall X$ positive unlimited (X larger than any integer)

$y(X) = -1$, elsewhere.

Then $y^2(X) = 1$, i.e. $y(X)$ is algebraic. If we choose any positive unlimited element $t \in \mathbb{L}$, then $y(0) = -1 < 0, y(t) = 1 > 0$, but $y(X)$ vanishes nowhere in $]0,t[$.

This example works over any non-Archimedean field, maximal ordered or not. We'll see that the intermediate value theorem holds for every algebraic series over a maximal ordered field, hence there is no series $S(X) = \sum_0^\infty b_n X^n \in \bar{\mathbb{L}}[[X]]$ such that $y(X) = S(X)$, at least on $[0,t]$ (but, as it is obvious, also on any closed interval $[a,b], a$ limited, b positive unlimited).

Now we give a few useful lemmas.

Lemma 4.2. *Let x be an element belonging to $\hat{\mathbb{K}}$ and algebraic over $\mathbb{K} = \bar{\mathbb{K}}$. Then $x \in \bar{\mathbb{K}}$.*

Proof. Observe that, since $\mathbb{K}[x] \subset \hat{\mathbb{K}}$, $\mathbb{K}[x]$ is an ordered algebraic extension of $\mathbb{K}$. □

Lemma 4.3. *Let $y(X) = \sum_{n=0}^\infty b_n X^n$ be an algebraic power series converging in $[a,b]$. Then $y(x) \in \mathbb{K}, \forall x \in [a,b]$*

Proof. If $y(X)$ is any series, it may happen that $y(x)$ exists, for some $x \in \mathbb{K}$, but belongs to $\hat{\mathbb{K}}$. Assume now that $y(X)$ is an algebraic series and let us plug any x (see 3.1) into the algebraic equation satisfied by $y(X)$:

$$a_0(x) + a_1(x)y(x) + \cdots + a_m(x)y(x)^m = 0.$$

This is an algebraic equation for $y(x)$ over $\mathbb{K}$, provided that the algebraic relation is not the identity $0 = 0$. But the coefficients $a_i's$ have no common root, so there is at least one i such that $a_i(x) \neq 0$. We conclude that $x \in \hat{\mathbb{K}} \cap \mathbb{K}^c$, hence $x \in \bar{\mathbb{K}} = \mathbb{K}$ by Lemma 4.2.

□

The following lemma improves Theorem 3.11 in the case of an algebraic series.

Lemma 4.4. *If $y(d) = 0$, then there is a largest integer $r \geq 1$ such that $y(X) = (X-d)^r q(X)$, where $q(X)$ is a power series converging to a sum $q(d) \in \mathbb{K}$. In this event $q(d) \neq 0$ and $q(X)$ is algebraic over $\mathbb{K}(X)$. Moreover $r \leq s$, where $s =$ multiplicity of d as a root of $a_0(X)$.*

Proof. Thanks to Theorem 3.11, it is enough to prove that $q(X)$ is algebraic with $q(d) \in \mathbb{K}$ and that $r \leq s$.

Let us now see that, whether $q(d) = 0$ or not, actually $q(d) \in \mathbb{K}$. To this purpose we observe that $q(X)$ satisfies the following algebraic equation:

$$a_0(X) + a_1(X)(X-d)^r q(X) + \cdots + a_m(X)(X-d)^{rm} q(X)^m = 0.$$

Now we divide the equation by the largest power of $(X-d)$ common to all coefficients, obtaining: $f_0(X) + f_1(X)q(X) + \cdots + f_m(X)q(X)^m = 0$, where at least one coefficient does not vanish at d. Thanks to Lemma 4.3, $q(d) \in \mathbb{K}$.

Let us now see the inequality $r \leq s$.

In fact, let s be the largest integer such that

$a_0(X) = (X-d)^s \bar{a}_0(X)$, with $\bar{a}_0(X) \in \mathbb{K}[X]$.

If $y(X) = (X-d)^{s+1} q_{s+1}(X)$,

with $q_{s+1}(X)$ converging at d to a sum $q_{s+1}(d) \in \mathbb{K}$, we have

$$\bar{a}_0(X)(X-d)^s + \cdots + a_m(X)(X-d)^{m(s+1)} q_{s+1}(X)^m = 0,$$

which implies the following equality:

$$\bar{a}_0(X) = -(X-d)(a_1(X)q_{s+1}(X) + \cdots + a_m(X)(X-d)^{(m-1(s+1)} q_{s+1}(X)^m,$$

where $q_{s+1}(X)$ is a power series converging at d. Now we replace X by d in the equation, and obtain $\bar{a}_0(d) = 0$, which is absurd.

In order to see that $q(X)$ is algebraic, we observe that $q(X) = \frac{y(X)}{(X-d)^r}$ belongs to $\mathbb{K}(X, y(X))$, which is an algebraic extension of $\mathbb{K}(X)$. □

Example 4.5. The algebraic equation $X^3 + (X^2 - X)y(X) - y^2(X) = 0$ is satisfied by $y(X) = X^2$ and we have: $a_0(X) = X^3, r = 2 < s = 3$.

Corollary 4.6. *Given any algebraic power series $y(X)$, then the set $S = \{x \in \mathbb{K}, y(x) = 0\}$ is finite.*

Proof. If c is a zero of $y(X)$, then $a_0(c) = 0$. Therefore $y(X)$ has at most $\deg(a_0(X))$ zeros. □

Lemma 4.7. *Let $y(X) = \sum_{n=0}^{\infty} b_n X^n$ be an algebraic power series converging in the closed interval $[a,b]$, then also $y'(X)$ is algebraic and, in this event, $y'(x) \in \mathbb{K}, \forall x \in [a,b]$.*

Proof. By hypothesis $y = y(X)$ satisfies equation (8). By differentiating we obtain:

$$a_0'(X) + a_1'(X)y(X) + \cdots + a_m'(X)y^m +$$
$$+ y'(a_1(X) + 2a_2(X)y + \cdots + ma_m(X)y^{m-1}) = 0,$$

which is an algebraic equation for $y'(X)$ over $\mathbb{K}(X,y)$. This equation cannot be an identity because there is no algebraic equation for $y(X)$ of degree less than m. Moreover $\mathbb{K}(X,y(X))$ is an algebraic extension of $\mathbb{K}(X)$ and so $\mathbb{K}(X,y(X),y'(X))$ is also an algebraic extension of $\mathbb{K}(X)$.

Now we apply Lemma 4.3. □

Theorem 4.8. *(The intermediate value theorem for algebraic series)*

Let $y(X) \in \mathbb{K}[[X]]$ be an algebraic power series defined over $[a,b]$ and such that $y(a)y(b) < 0$. Then $y(c) = 0$ for some $c \in]a,b[$.

Proof. If $c \in \mathbb{K}$ is such that $y(c) = 0$ and $y(X)$ satisfies equation (8), then $a_0(c) = 0$. We want to look for a root $c \in]a,b[$ of $a_0(X)$ such that the converse is also true, i.e. $y(c) = 0$.

First of all let us prove that there is a root c of $a_0(X)$ satisfying the relation $a \leq c \leq b$.

Let us set: $y_r(X) = a_0 + a_1 X + ... + a_r X^r$. Since the convergence of $y(X)$ is uniform on $[a,b], \forall \varepsilon > 0$ there is r_0 such that, if $r \geq r_0$, then $|y(x) - y_r(x)| < \varepsilon, \forall x \in [a,b]$. This means in particular (choose $\varepsilon \leq \inf\left(|\frac{y(a)}{2}|, |\frac{y(b)}{2}|\right)$), that there is a suitable $\bar{r}$ such that, $\forall r \geq \bar{r}, y_r(a)y_r(b) < 0$. Since $y_r(X)$ is a polynomial, the intermediate value theorem holds true, i.e. there is $c_r \in]a,b[$ such that $y_r(c_r) = 0$.

So, with the choice above, $\forall r \geq \bar{r}$, the following are simultaneously true:

A. $y_r(a)y_r(b) < 0$

B. there is $c_r \in [a,b]$ such that $y_r(c_r) = 0$

C. $|y(c_r) - y_r(c_r)| = |y(c_r)| < \varepsilon$.

Observe that B.,C. above hold for every root $c_r \in [a,b]$.

Moreover, we have (see Remark 3.1):

$$a_0(c_r) + a_1(c_r)y(c_r) + \cdots + a_m(c_r)y(c_r)^m = 0,$$

and this equation is not an identity because $a_i(c_r) \neq 0$, for some i (the $a_i's, i = 0,\cdots,m$, are supposed to have no common root). Set: $M = \max(|(a_1(x)|,\cdots,|a_m(x)|), x \in [a,b] \subset \bar{\mathbb{K}}$ (see Remark 2.4). Hence, if, moreover, $\varepsilon < 1$, we have

$$|a_0(c_r)| < Mm\varepsilon, \forall r \geq \bar{r}. \tag{9}$$

If in particular $a_0(c_r) = 0$, for some r, we have the required root of $a_0(X)$ lying between a and b. Otherwise, we proceed as follows.

Set: $a_0(X) = (X-\alpha_1)\cdots(X-\alpha_h)(((X-b_1)^2+d_1^2)\cdots((X-b_v)^2+d_v^2)))$, where the α_i's are the roots in K (and possibly $\alpha_i = \alpha_j$ for some indices). exclude that $a_0(X)$ has no root at all. Assume now that $\alpha_i \notin [a,b], \forall i$ (either because the roots are outside or because no root belongs to $\mathbb{K}$). This means that $|c_r - \alpha_i| > m_i = \min\{|\alpha_i - a|, |b-\alpha_i|\}, \forall i$, hence $, \forall i, |c_r - \alpha_i| > m_0 = \min\{m_1,\cdots,m_h\} > 0$. In this event we have: $|a_0(c_r)| > m_0^h d_1^2 \cdots d_v^2$ (where we choose $m_0 = 1$ if there is no root in $\mathbb{K}$, and $d_1 = \cdots = d_v = 1$ if all the roots belong to $\mathbb{K}$). We find a contradiction if we choose $\varepsilon < \frac{m_0^h d_1^2\cdots d_v^2}{mM}$.

Therefore there is at least one root of $a_0(X)$ which belongs to $[a,b]$; our aim is to show that at least one of them is the limit of a sequence $\{c_{r_n}\}$ of zeros of $\{y_{r_n}\}$. Assume that $\alpha_1,\cdots,\alpha_u$ are the roots in $[a,b]$, while $\alpha_{u+1},\cdots,\alpha_h$ lie outside.

Let us now use any existing decreasing sequence of infinitesimal elements $(\varepsilon_n, n = 1,2,\cdots)$, such that $\lim_{n\to\infty}\varepsilon_n = 0$. We rewrite (9) explicitly, with ε replaced by ε_n and $\bar{r}$ by r_n, in the special case $r = r_n$, so obtaining that $\forall n$, there is $r(\varepsilon_n) = r_n$ such that $\forall r \geq r_n$ (in particular when $r = r_n$),

$$|a_0(c_{r_n})| =$$

$$= |(c_{r_n}-\alpha_1)\cdots(c_{r_n}-\alpha_h)(((c_{r_n}-b_1)^2+d_1^2)\cdots((c_{r_n}-b_v)^2+d_v^2)))| < Mm\varepsilon_n,$$

the inequality holding for any c_{r_n} in the finite set of the roots of $y_{r_n}(X)$ lying between a and b.

Put: $D = |d_1^2\cdots d_v^2\ m_{u+1}\cdots m_h|$. Then $|c_{r_n}-\alpha_1|\cdots|c_{r_n}-\alpha_u| < \frac{Mm\varepsilon_n}{D}$, which implies that, for each r_n there is at least one α_i such that $|c_{r_n}-\alpha_i| < \bar{\varepsilon}_n = (\frac{Mm\varepsilon_n}{D})^{\frac{1}{u}}$. However it may happen that the root α_i of $a_0(X)$ varies with the choice of the root c_{r_n} of y_{r_n}, i.e. that $|c_{r_n}-\alpha_i| < \bar{\varepsilon}_n$, for a choice of the root c_{r_n} and $|c_{r_n}-\alpha_j| < \bar{\varepsilon}_n, j \neq i$, for another choice.

Therefore, for each n, the finite set $C_{r_n} = \{c_{r_n}^{(1)},\cdots,c_{r_n}^{(h_n)}\}$ contains at least one element close to some α_i and possibly other elements close to another a_j.

Now observe that the set $A = \{\alpha_1,\cdots,\alpha_u\}$ is finite. This implies that there are infinitely many sets C_{r_n} containing at least one root close to the same fixed α_i.

Set: $c = \alpha_i =$ fixed root of $a_0(X)$ and choose, in those C_{r_n} containing an element close to c, the closest one (if there are two such elements, choose any of them).

So we obtain a subsequence $C = (c_{r_{n_s}} = d_s)$ with the following properties:

1. $y_{r_{n_s}}(d_s) = 0, \forall s$,
2. $|d_s - c| < \bar{\varepsilon}_{n_s}, \forall s$.

Since the sequence $(\bar{\varepsilon}_n)$ approaches 0 as n tends to ∞ (and so also $(\bar{\varepsilon}_{n_s})$ as s tends to ∞), we can conclude that $\lim_{s\to\infty} d_s = c$.

Since $y(X)$ is a continuous function, we have: $\lim_{s\to\infty} y(d_s) = y(c)$. Now observe that $y(X)$ is uniformly convergent in $[a,b]$, so that, given $\varepsilon > 0$, there is N such that, for all $n > N, |y(x) - y_n(x)| < \varepsilon$, independently on x. In particular we can choose $x = d_s$ and, $\forall s$ large enough, obtain that $|y(d_s) - y_{r_{n_s}}(d_s)| = |y(d_s) - 0| = |y(d_s)| < \varepsilon$. Therefore $\lim_{s\to\infty} y(d_s) = y(c) = 0$ and c is a root of $y(X)$ lying in $[a,b]$. But $c \neq a, c \neq b$ because of our hypothesis, and so $a < c < b$.
□

The following theorem and its corollary are consequences of 4.8, 3.11, 4.4.

Theorem 4.9. *Assume that the algebraic series $y(X)$ vanishes both at a and at b and nowhere else between a and b. Let $\frac{h(X)}{k(X)}$ be a quotient of power series converging in $[a,b]$ such that $k(x) \neq 0, \forall x \in [a,b]$. Then $F(X) = y(X)h(X) + k(X)y'(X)$ vanishes at least at one point $x \in]a,b[$.*

Proof. According to Lemma 4.4 let us choose the largest r and the largest s for which we can set: $y(X) = q(X)(X-a)^r(X-b)^s, q(X)$ being an algebraic power series that is convergent and does not vanish at every $x \in [a,b]$ (observe that, $\forall x \in]a,b[, q(x) = \frac{y(x)}{(x-a)^r(x-b)^s}$, so that $q(X)$ does not vanish on $]a,b[$). By the intermediate value theorem above, applied to both $q(X)$ and $k(X)$, it follows that: $q(a)q(b) > 0, k(a)k(b) > 0$. We have:

$F(X) = (X-a)^{r-1}(X-b)^{s-1}(q(X)[(X-a)(X-b)h(X) + rk(X)(X-b) + sk(X)(X-a)] + k(X)q'(X)(X-a)(X-b))) = (X-a)^{r-1}(X-b)^{s-1}G(X)$.

Since $G(a)G(b) = rsq(a)q(b)k(a)k(b)(a-b)(b-a) < 0$, we have: $G(x) = 0$ for some $x \in \mathbb{K}, a < x < b$ and, as a consequence, also $F(x) = 0$. □

Corollary 4.10. *If $y(X)$ is an algebraic series such that $y(a) = y(b) = 0$, then the equation $y'(X) = 0$ has at least one root $x, a < x < b$ (Rolle's theorem for algebraic series).*

Proof. Assume that $y(x) \neq 0, \forall x, a < x < b$. Then our claim follows from the above theorem, where we choose $h(X) = 0$. If there is $c, a < x < c$, such that $h(c) = 0$, choose c as close as possible to a (remember that y has finitely many zeros). Then we use the same argument, but in $[a,c]$. □

The following property has nothing to do with (AP), it is a formal consequence of Rolle's theorem, whatever argument is used to prove it (see [3], §7 and §8).

Theorem 4.11. *(The mean value theorem). If $y(X)$ is an algebraic series defined over $[a,b]$, then there is $c \in]a,b[$ such that $y(b)-y(a)=(b-a)y'(c)$.*

Proof. Consider the following function:

$$F(X) = det \begin{pmatrix} y(X) & X & 1 \\ y(a) & a & 1 \\ y(b) & b & 1 \end{pmatrix}.$$

It is an algebraic power series such that $F(a)=F(b)=0$. By Rolle's theorem, there is $c \in]a,b[$ such that $F'(c)=0$. But we have:

$$F'(c) = det \begin{pmatrix} y'(c) & 1 & 0 \\ y(a) & a & 1 \\ y(b) & b & 1 \end{pmatrix} = (a-b)y'(c)+y(b)-y(a),$$

so $F'(c)=0$ is exactly our claim. □

The following proposition depends only on the properties of the derivative $f'(X)$, not on (AP). As for the endpoints, monotonicity can also be proved, while the vanishing at a local maximum or minimum is false.

Proposition 4.12. *Let $f(X)$ be a differentiable function defined over $]a,b[$. Then $f'(X)>0$ $(<0), \forall x \in]a,b[$ implies that $f(X)$ is increasing (decreasing) at every point of $]a,b[$. If $f(X)$ has a local maximum or a local minimum at $x \in]a,b[$, then $f'(x)=0$.*

In particulare the statement holds true when $f(X)$ is a power series converging in $]a,b[$.

Proof. Assume that $f'(x) \neq 0$, for instance $f'(x)>0$. This means:

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} > 0.$$

Therefore, if h is small enough, $\frac{f(x+h)-f(x)}{h} > \frac{f'(x)}{2} > 0$. Hence $f(X)$ is an increasing function at x, and can have neither a maximum nor a minimum. □

The following property is a consequence of the mean value theorem, and so of the intermediate value and Rolle's theorems.

Theorem 4.13. *Let $y(X)$ be an algebraic power series such that $y'(x) \geq 0$ (≤ 0), $\forall x \in [a,b]$ and $y'(X)$ is not identically zero. Then $y(X)$ is (strictly) increasing (decreasing) in $[a,b]$.*

Proof. Let us assume that $x_1 < x_2 < \cdots < x_r$ are the (possibly multiple) finitely many roots of $y'(X)$ belonging to the open interval $]a,b[$. We split $[a,b]$ into the sub-intervals $[a,x_1],[x_1,x_2],\cdots,[x_r,b]$. Consider for instance the case $y'(x) > 0, \forall x \neq a,b,x_i, i = 1,\cdots,r$ and choose $x,\bar{x} \in [x_i,x_{i+1}], x < \bar{x}$. Then we have, by the mean value theorem, $y(\bar{x}) - y(x) = (\bar{x} - x)y'(u) > 0$, u being a suitable point such that $x < u < \bar{x}$, where $y'(u) > 0$. If on the contrary $x,\bar{x}$ belong to different sub-intervals, for instance $x_{i-1} < x < x_i < \bar{x} < x_{i+1}$, it is enough to observe that $y(x) < y(x_i) < y(\bar{x})$.

If $x,\bar{x}$ lie between a and x_1 or on the two sides of x_1, the above proof works, as it works around b.

The case $y'(x) < 0$ is similar. □

Theorem 4.14. *Let $y(X)$ be an algebraic power series converging in $[a,b]$. Then it attains both the absolute maximum and the absolute minimum. In particular $y(X)$ is bounded above and below.*

Proof. It is exactly the proof given in section 2 (proof of 5) for rational functions. □

Remark 4.15. Let $f(X,Y) = a_0(X) + a_1(X)Y + \cdots + a_m(X)Y^m = 0$ be a polynomial equation over $\mathbb{K} = \bar{\mathbb{K}}$. It is well-known that all solutions (defined over the algebraic closure $\mathbb{K}^c$ of $\mathbb{K}$) are elements of a field $K((X^{\frac{1}{n}}))$, for some $n \in \mathbb{N}$ (see [14], Theorem 3.1), i.e. every solution has the form:

$$y(X) = \frac{a_0 + a_1 X^{\frac{1}{n}} + \cdots + a_k X^{\frac{k}{n}} + \cdots}{X^{\frac{r}{n}}}$$

Then $y(X)X^{\frac{r}{n}} = u(Z) \in \mathbb{K}[[Z]]$ is algebraic over $\mathbb{K}(Z)$, where $Z = X^{\frac{1}{n}}$. If all coefficients of $y(X)$ belong to $\mathbb{K}$, then the intermediate value and the mean value theorems hold, as well as the extreme value theorem.

Remark 4.16. A classical proof of the intermediate value theorem for a continuous function $f : \mathbb{R} \to \mathbb{R}$ is based on the principle of the nested cells, which is equivalent to the claim that every set bounded above has the least upper bound. Our approach for an algebraic power series over a non-Archimedean field $\mathbb{K}$ cannot be based on the property of the nested cells, true for $\mathbb{R}$ but false for $\mathbb{K}$.

We meet the same obstacle when we want to prove that every algebraic power series $y(X)$ attains, in a closed interval $[a,b]$, its maximum and its minimum. We cannot use, as it is done in $\mathbb{R}$, the least upper bound of a set bounded above, to prove first that $y(X)$ is bounded above and to obtain, as a consequence, that it takes on its maximum (or minimum). Our approach requires a direct proof of the existence of the maximum (or minimum), and such a result yields the boundedness result as a consequence.

Example 4.17. Let $\mathbb{K} = \overline{\mathbb{R}(t)}, t$ being unlimited as usual. Then

$$y(X) = \sum_0^\infty \binom{-\frac{1}{2}}{n} (\frac{X}{t})^n$$

is an algebraic series. In fact $y(X)^2 = \frac{1}{1-\frac{X}{t}}$. The series has a sum when $X = x \in \mathbb{K}$ iff $\frac{x}{t}$ is infinitesimal. Within the closed interval $[0, \frac{1}{t}]$, we have: $y'(X) = \frac{1}{2t}(1 - \frac{X}{t})^{-\frac{3}{2}} > 0$ everywhere. Therefore $y(X)$ takes on its maximum value $t(t^2-1)^{-\frac{1}{2}}$ at $\frac{1}{t}$ and its minimum value 1 at 0.

Example 4.18. (With $\mathbb{K} = \mathbb{Q}$, $\mathbb{L}$ is the field that Hilbert used in his Grundlagen to introduce a non-Archimedean geometry). Let $\mathbb{L}$ be the overfield of $\mathbb{K}(X)$ obtained as follows: we start with the set of rational functions and then apply any finite sequence of the following operations on rational functions: addition, subtraction, product, quotient, $(1 + f(X)^2)^{\frac{1}{2}}$, where $f(X)$ is previously defined. Then $\mathbb{L} \subset \mathbb{K}(X)^c$ is an algebraic extension of $\mathbb{K}(X)$ and is contained as a subfield in $\overline{\mathbb{K}(X)}$. It is clear that if $f(X) \in \mathbb{L}$, then $f(X)$ is a quotient of an algebraic series and a power of X. Therefore the intermediate value theorem holds in $\mathbb{L}$.

Consider for instance $\mathbb{K} = \overline{R(t)}$ and choose $f(X) = \sqrt{1+X^2} + \sqrt{2+X^2} - \sqrt{1+\sqrt{t}X}$. Then we have: $f(n) < 0, \forall n \in \mathbb{N}, f(\sqrt{t}) > 0$. Therefore, by the intermediate value theorem, there is at least an element $x \in \mathbb{K}, x < \sqrt{t}, x > n, \forall n \in \mathbb{N}$ such that $f(x) = 0$.

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