LE MATEMATICHE
Vol. LXVIII (2013) – Fasc. II, pp. 249–266
doi: 10.4418/2013.68.2.18

DIFFERENTIAL SANDWICH THEOREMS OF SYMMETRIC POINTS ASSOCIATED WITH DZIOK-SRIVASTAVA OPERATOR

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In this paper we obtain some applications of theory of differential subordination, superordination and sandwich results for the classes of symmetric points associated with Dziok-Srivastava operator.

1. Introduction

Let $H(U)$ denote the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and $H[a,1]$ denote the subclass of functions $f \in H(U)$ of the form:

$$f(z) = a + a_1 z + a_2 z^2 + \dots (a \in \mathbb{C}).$$

Also, let $\mathcal{A}$ denote the subclass of functions $f \in H(U)$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

If f and g are analytic function in U, we say that f is subordinate to g, written $f \prec g$ if there exists a Schwarz function w, which is analytic in U with

Entrato in redazione: 24 dicembre 2012
AMS 2010 Subject Classification: 30C45.
Keywords: Symmetric points, Differential subordination, Superordination, Sandwich theorems, Dziok-Srivastava operator.

$w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$. Furthermore, if the function g is univalent in U, then we have the following equivalence (see [6] and [12]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For $k, h \in H(U)$, let $\varphi : \mathbb{C}^2 \times U \to \mathbb{C}$ and let h be univalent in U. If $k(z)$ satisfies the first order differential subordination

$$\varphi(k(z), zk'(z); z) \prec h(z), \tag{1.2}$$

then $k(z)$ is a solution of the differential subordination (1.2). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (1.2), if $k(z) \prec q(z)$ for all the functions $k(z)$ satisfying (1.2). A univalent dominant $\widetilde{q}(z)$ is said to be the best dominant of (1.2) if $\widetilde{q}(z) \prec q(z)$ for all dominants $q(z)$. If $k(z)$ and $\varphi(k(z), zk'(z); z)$ are univalent functions in U and if $k(z)$ satisfies the first order differential superordination

$$h(z) \prec \varphi(k(z), zk'(z); z), \tag{1.3}$$

then $k(z)$ is a solution of the differential superordination (1.3). The univalent function $q(z)$ is called a subordinant of the solutions of the differential superordination, if $q(z) \prec k(z)$ for all the functions $k(z)$ satisfying (1.3). A subordinant $\widetilde{q}(z)$ is said to be the best subordinant of (1.3) if $q(z) \prec \widetilde{q}(z)$ for all the subordinants $q(z)$. Using the results of Miller and Mocanu [13], Bulboaca [5] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [4]. Ali et al. [1], have used the results of Bulboaca [5] (see also [2, 3]) to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are univalent functions in U with $q_1(0) = q_2(0) = 1$.

Sakaguchi [20] introduced a class S_s^* of functions starlike with respect to symmetric points, which consists of functions $f(z) \in \mathcal{A}$ satisfying the inequality

$$\Re\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\} > 0 \ (z \in U). \tag{1.4}$$

Obviously, it forms a subclass of close-to-convex functions and hence univalent. Moreover, this class includes the class of convex functions and odd starlike functions with respect to the origin (see [20]).

For real parameters $\alpha_1,\ldots,\alpha_q$ and $\beta_1,\ldots,\beta_s$ ($\beta_j \notin \mathbb{Z}_0^- = \{0,-1,-2,\ldots\}$; $j=1,2,\ldots,s$), we now define the generalized hypergeometric function ${}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)$ by (see for example, [22, p.19])

$$ {}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_q)_k}{(\beta_1)_k \ldots (\beta_s)_k} \cdot \frac{z^k}{k!} $$

$$ (q \le s+1; q,s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U), $$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$ (\theta)_\nu = \frac{\Gamma(\theta+\nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu=0;\ \theta \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}), \\ \theta(\theta+1)\ldots(\theta+\nu-1) & (\nu \in \mathbb{N};\ \theta \in \mathbb{C}). \end{cases} \tag{1.5} $$

Corresponding to the function $H_{q,s}(\alpha_1)f(z) = h(\alpha_1,\ldots\alpha_q;\beta_1\ldots,\beta_s;z)f(z)$, defined by (see [9] and [10])

$$ \begin{aligned} H_{q,s}(\alpha_1)f(z) &= [z{}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)] * f(z) \\ &= z + \sum_{k=2}^{\infty} \Gamma_k(\alpha_1) a_k z^k, \end{aligned} \tag{1.6} $$

where

$$ \Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k-1} \cdot \ldots \cdot (\alpha_q)_{k-1}}{(\beta_1)_{k-1} \ldots (\beta_s)_{k-1} (k-1)!}. \tag{1.7} $$

From (1.6), we have:

$$ z\left(H_{q,s}(\alpha_1)f(z)\right)' = \alpha_1 H_{q,s}(\alpha_1+1)f(z) - (\alpha_1-1)\, H_{q,s}(\alpha_1)f(z). \tag{1.8} $$

We note that:

(i) $H_{2,1}(\alpha_1,\alpha_2;\beta_1)f(z) = \mathcal{F}(\alpha_1,\alpha_2;\beta_1)f(z)$ (see Hohlov [11]);

(ii) $H_{2,1}(a,1;c)f(z) = \mathcal{L}(a;c)f(z)$ $(a,c>0)$ (see Carlson-Shaffer [7]);

(iii) $H_{2,1}(n+1,1;1)f(z) = \mathcal{D}^n f(z)$ $(n>-1)$ (see Ruscheweyh [19]);

(iv) $H_{2,1}(\lambda+1,c;a)f(z) = I^\lambda(a,c)f(z)$ $(a,c \in \mathbb{R}\backslash\mathbb{Z}_0^-;\ \lambda > -1)$ (see Cho et al. [8]);

(v) $H_{2,1}(2,1;n+1)f(z) = I_n f(z)$ $(n>-1)$ (see Noor [16] and Noor and Noor [17]).

2. Definitions and preliminaries

In order to prove our results, we shall need the following definition and lemmas.

Definition 2.1. [13] Let $\mathcal{Q}$ be the set of all functions f that are analytic and injective on $\overline{U}\setminus E(f)$, where $E(f)=\{\zeta\in\partial U:\lim\limits_{z\to\zeta}f(z)=\infty\}$ and are such that $f'(\zeta)\neq 0$ for $\zeta\in\partial U\setminus E(f)$.

Lemma 2.2. *[12] Let q be univalent in the unit disc U and let θ and ϕ be analytic in a domain D containing $q(U)$, with $\phi(w)\neq 0$ when $w\in q(U)$. Set*

$$Q(z)=zq'(z)\phi(q(z)) \text{ and } h(z)=\theta(q(z))+Q(z), \tag{2.1}$$

suppose that
(i) Q is a starlike function in U,
(ii) $\Re\left\{\dfrac{zh'(z)}{Q(z)}\right\}>0,\ z\in U.$
If k is analytic in U with $k(0)=q(0)$, $k(U)\subseteq D$ and

$$\theta(k(z))+zk'(z)\phi(k(z))\prec\theta(q(z))+zq'(z)\phi(q(z)), \tag{2.2}$$

then $k(z)\prec q(z)$ and q is the best dominant of (2.2).

Lemma 2.3. *[21] Let $\xi,\varphi\in\mathbb{C}$ with $\varphi\neq 0$ and let q be a convex function in U with*

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\}>\max\{0;-\Re\frac{\xi}{\varphi}\}.$$

If k is analytic in U and

$$\xi k(z)+\varphi zk'(z)\prec\xi q(z)+\varphi zq'(z), \tag{2.3}$$

then $k\prec q$ and q is the best dominant of (2.3).

Lemma 2.4. *[6] Let q be a univalent function in U and let θ and φ be analytic in a domain D containing $q(U)$. Suppose that*
(i) $\Re\left\{\dfrac{\theta'(q(z))}{\varphi(q(z))}\right\}>0$ for $z\in U$,
(ii) $Q(z)=zq'(z)\varphi(q(z))$ is starlike univalent in U.
If $k\in H[q(0),1]\cap\mathcal{Q}$, with $k(U)\subseteq D$, $\theta(k(z))+zk'(z)\varphi(k(z))$ is univalent in U and

$$\theta(q(z))+zq'(z)\varphi(q(z))\prec\theta(k(z))+zk'(z)\varphi(k(z)), \tag{2.4}$$

then $q(z)\prec k(z)$ and q is the best subordinant of (2.4).

Lemma 2.5. *[13] Let q be convex univalent in U and let $\beta \in \mathbb{C}$, with $\Re\{\beta\} > 0$. If $k \in H[q(0),1] \cap \mathcal{Q}$, $k(z)+\beta zk'(z)$ is univalent in U and*

$$q(z)+\beta zq'(z) \prec k(z)+\beta zk'(z), \tag{2.5}$$

then $q \prec k$ and q is the best subordinant of (2.5).

Lemma 2.6. *[18] The function $q(z) = (1-z)^{-2ab}$ $(a,b \in \mathbb{C}^*)$ is univalent in U if and only if $|2ab-1| \le 1$ or $|2ab+1| \le 1$.*

3. Subordinant results

Unless otherwise mentioned, we shall assume in the reminder of this paper that $\lambda \in \mathbb{C}^*$, $\alpha_1,...,\alpha_q \in \mathbb{R}$, $\alpha_1 \neq 0$, $\beta_1,...,\beta_s \in \mathbb{R}\backslash\mathbb{Z}_0^-$, $q,s \in \mathbb{N}_0$, $q \le s+1$, $z \in U$ and the powers are understood as principle values.

Theorem 3.1. *Let $q(z)$ be univalent in U, with $q(0) = 1$ and suppose that*

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\} > \max\left\{0; -\alpha_1 \cdot \Re\left(\frac{1}{\lambda}\right)\right\}. \tag{3.1}$$

If $f(z) \in \mathcal{A}$ satisfies the subordination

$$(1-\lambda)\left(\tfrac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z}\right)+\lambda\left(\tfrac{H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)}{2z}\right)$$
$$\prec q(z)+\frac{\lambda zq'(z)}{\alpha_1}, \tag{3.2}$$

then

$$\frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z} \prec q(z)$$

and q is the best dominant of (3.2).

Proof. Define a function $h(z)$ by

$$h(z) = \frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z} \quad (z \in U), \tag{3.3}$$

where $h(z)$ is analytic in U with $h(0) = 1$. By differentiating (3.3) with respect to z, we obtain that

$$zh'(z) = \tfrac{z\left(H_{q,s}(\alpha_1)f(z)\right)'-z\left(H_{q,s}(\alpha_1)f(-z)\right)'}{2z} - \tfrac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z}. \tag{3.4}$$

From (3.4) and (1.8), a simple computation shows that

$$zh'(z) = \alpha_1 \frac{H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)}{2z} - \alpha_1 \frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z},$$

hence the subordination (3.2) is equivalent to

$$h(z) + \frac{\lambda z h'(z)}{\alpha_1} \prec q(z) + \frac{\lambda z q'(z)}{\alpha_1}.$$

Now, applying Lemma 2.3, with $\varphi = \frac{\lambda}{\alpha_1}$ and $\xi = 1$, the proof is completed. □

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem 3.1, the condition (3.1) reduces to

$$\Re\left\{\frac{1-Bz}{1+Bz}\right\} > \max\left\{0; -\alpha_1 \cdot \Re\left(\frac{1}{\lambda}\right)\right\}. \tag{3.5}$$

It is easy to check that the function $\psi(\zeta) = \frac{1-\zeta}{1+\zeta}$, $|\zeta| < |B|$, is convex in U and since $\psi(\overline{\zeta}) = \overline{\psi(\zeta)}$ for all $|\zeta| < |B|$, it follows that the image $\psi(U)$ is convex domain symmetric with respect to the real axis, hence

$$\inf\left\{\Re\left(\frac{1-Bz}{1+Bz}\right)\right\} = \frac{1-|B|}{1+|B|} > 0. \tag{3.6}$$

Then the inequality (3.5) is equivalent to $\frac{|B|-1}{|B|+1} \le \alpha_1 \cdot \Re\left(\frac{1}{\lambda}\right)$, hence, we obtain the following corollary.

Corollary 3.2. *Let* $f(z) \in \mathcal{A}$, $-1 \le B < A \le 1$ *and* $\max\left\{0; -\alpha_1 \cdot \Re\left(\frac{1}{\lambda}\right)\right\} \le \frac{1-|B|}{1+|B|}$, *then*

$$(1-\lambda)\left(\frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z}\right) + \lambda\left(\frac{H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)}{2z}\right)$$
$$\prec \frac{1+Az}{1+Bz} + \frac{\lambda}{\alpha_1} \cdot \frac{(A-B)z}{(1+Bz)^2}, \tag{3.7}$$

implies

$$\frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant of (3.7)*.

Taking $q(z) = \frac{1+z}{1-z}$ in Theorem 1 (or putting $A = 1$ and $B = -1$ in Corollary 1), the condition (3.1) reduces to

$$\alpha_1 \cdot \Re\left(\frac{1}{\lambda}\right) \ge 0, \tag{3.8}$$

hence, we obtain the following corollary.

Corollary 3.3. *Let $f(z) \in \mathcal{A}$, assume that (3.8) holds true and*

$$(1-\lambda)\left(\frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z}\right)+\lambda\left(\frac{H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)}{2z}\right) \prec \frac{1+z}{1-z}+\frac{\lambda}{\alpha_1}\cdot\frac{2z}{(1-z)^2}, \tag{3.9}$$

then

$$\frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z} \prec \frac{1+z}{1-z}$$

and $\dfrac{1+z}{1-z}$ *is the best dominant of (3.9) .*

Theorem 3.4. *Let $q(z)$ be univalent in U, with $q(0)=1$ and $q(z)\neq 0$ for all $z\in U$, $\eta,\zeta\in\mathbb{C}^*$, $\rho,\tau\in\mathbb{C}$, with $\rho+\tau\neq 0$, $f(z)\in\mathcal{A}$ and suppose that f and q satisfy the next conditions:*

$$\frac{\rho\left(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\right)+\tau\left(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\right)}{2(\rho+\tau)z} \neq 0 \quad (z\in U) \tag{3.10}$$

and

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\}>0 \quad (z\in U). \tag{3.11}$$

If

$$1+\zeta\eta\left\{\frac{\rho z\left(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\right)'+\tau z\left(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\right)'}{\rho\left(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\right)+\tau\left(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\right)}-1\right\} \prec 1+\eta\frac{zq'(z)}{q(z)}, \tag{3.12}$$

then

$$\left(\frac{\rho\left(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\right)+\tau\left(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\right)}{2(\rho+\tau)z}\right)^{\zeta} \prec q(z)$$

and q is the best dominant of (3.12).

Proof. Let

$$g(z)=\left(\frac{\rho\left(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\right)+\tau\left(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\right)}{2(\rho+\tau)z}\right)^{\zeta}, \tag{3.13}$$

where $z \in U$, then $g(z)$ is analytic in U, differentiating $g(z)$ logarithmically with respect to z, we obtain

$$\frac{zg'(z)}{g(z)} = \zeta \left\{ \frac{\rho z\left(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\right)' + \tau z\left(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\right)'}{\rho\left(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\right)+\tau\left(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\right)} - 1 \right\}. \tag{3.14}$$

Now, using Lemma 2.2 with $\theta(w) = 1$ and $\phi(w) = \frac{\eta}{w}$, then θ is analytic in $\mathbb{C}$ and $\phi(w) \neq 0$ is analytic in $\mathbb{C}^*$. Also if we let

$$Q(z) = zq'(z)\phi(q(z)) = \eta \frac{zq'(z)}{q(z)},$$

and

$$h(z) = \theta(q(z)) + Q(z) = 1 + \eta \frac{zq'(z)}{q(z)},$$

then, $Q(0) = 0$ and $Q'(0) \neq 0$, and the assumption (3.11) yields that Q is a starlike function in U. From (3.11) we have

$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\} = \Re\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0 \; (z \in U),$$

then, by using Lemma 2.2, we deduce that the assumption (3.12) implies $g(z) \prec q(z)$ and the function q is the best dominant of (3.12). $\square$

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$, $\rho = 0$ and $\tau = \eta = 1$ in Theorem 3.4, the condition (3.11) reduces to

$$\left\{1 - \frac{2Bz}{1+Bz} - \frac{(A-B)z}{(1+Az)(1+Bz)}\right\} > 0, \tag{3.15}$$

hence, we obtain the following corollary.

Corollary 3.5. *Let $f(z) \in \mathcal{A}$, assume that (3.15) holds true, $-1 \leq B < A \leq 1$ and suppose that $\frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z} \neq 0$ $(z \in U)$. If*

$$1 + \zeta\left\{\frac{z\left(H_{q,s}(\alpha_1)f(z) - H_{q,s}(\alpha_1)f(-z)\right)'}{H_{q,s}(\alpha_1)f(z) - H_{q,s}(\alpha_1)f(-z)} - 1\right\} \prec 1 + \frac{(A-B)z}{(1+Az)(1+Bz)}, \tag{3.16}$$

then

$$\left(\frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z}\right)^{\zeta} \prec \frac{1+Az}{1+Bz}, \tag{3.17}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant of (3.16).

Putting $\rho = 0, \tau = \eta = 1$ and $q(z) = (1+Bz)^{\frac{\zeta(A-B)}{B}}$ $(\zeta \in \mathbb{C}^*,\ -1 \leq B < A \leq 1, B \neq 0)$ in Theorem 3.4 and using Lemma 2.6, it is easy to check that the assumption (3.11) holds, hence we obtain the next corollary:

Corollary 3.6. *Let $f \in \mathcal{A}$, $\zeta \in \mathbb{C}^*$, $-1 \leq B < A \leq 1$, with $B \neq 0$ and suppose that $\left|\frac{\zeta(A-B)}{B} - 1\right| \leq 1$ or $\left|\frac{\zeta(A-B)}{B} + 1\right| \leq 1$. If*

$$1+\zeta\left(\frac{z\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)'}{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}-1\right) \prec \frac{1+[B+\zeta(A-B)]z}{1+Bz}, \tag{3.18}$$

then

$$\left(\frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z}\right)^{\zeta} \prec (1+Bz)^{\frac{\zeta(A-B)}{B}}$$

and $(1+Bz)^{\frac{\zeta(A-B)}{B}}$ is the best dominant of (3.18).

Putting $\rho = 0,\ \tau = 1,\ \eta = \frac{e^{i\lambda}}{ab\cos\lambda}\ \left(|\lambda| < \frac{\pi}{2},\ a,b \in \mathbb{C}^*\right),\ \zeta = a$, and $q(z) = \dfrac{1}{(1-z)^{2abe^{-i\lambda}\cos\lambda}}$ in Theorem 3.4, hence combining this togther with Lemma 2.6, we obtain the following corollary.

Corollary 3.7. *Let $f(z) \in \mathcal{A}$, assume that (3.11) holds true and $|\lambda| < \frac{\pi}{2}$, $a,b \in \mathbb{C}^*$ such that $\left|abe^{-i\lambda}\cos\lambda - 1\right| \leq 1$ or $\left|abe^{-i\lambda}\cos\lambda + 1\right| \leq 1$. If*

$$1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{z\left(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\right)'}{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}-1\right) \prec \frac{1+z}{1-z}, \tag{3.19}$$

then

$$\left(\frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z}\right)^{a} \prec \frac{1}{(1-z)^{2abe^{-i\lambda}\cos\lambda}}$$

and $\dfrac{1}{(1-z)^{2abe^{-i\lambda}\cos\lambda}}$ is the best dominant of (3.19).

Putting $\rho = 0,\ \tau = 1,\ \eta = \frac{1}{ab}\,(a,b \in \mathbb{C}^*),\ \zeta = a$, and $q(z) = (1-z)^{-2ab}$ in Theorem 3.4 (or putting $\lambda = 0$ in Corollary 3.7), hence combining this together with Lemma 2.6, we obtain the following corollary.

Corollary 3.8. *Let $f(z) \in \mathcal{A}$, assume that (3.11) holds true and $a,b \in \mathbb{C}^*$ such that $|2ab-1| \leq 1$ or $|2ab+1| \leq 1$. If*

$$1+\frac{1}{b}\left(\frac{z\left(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\right)'}{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}-1\right) \prec \frac{1+z}{1-z}, \tag{3.20}$$

then

$$\left(\frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z}\right)^a \prec (1-z)^{-2ab}$$

and $(1-z)^{-2ab}$ *is the best dominant of (3.20).*

Theorem 3.9. *Let* $q(z)$ *be univalent in* U, *with* $q(0)=1$, $\eta,\zeta\in\mathbb{C}^*$, $\rho,\tau,\sigma,\varkappa\in\mathbb{C}$, *with* $\rho+\tau\neq 0$ *and* $f(z)\in\mathcal{A}$.*Suppose that* f *and* q *satisfy the next two conditions:*

$$\frac{\rho\big(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\big)+\tau\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)}{2(\rho+\tau)z}\neq 0\ \ (z\in U) \quad (3.21)$$

and

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\}>\max\{0;-\Re\left(\frac{\sigma}{\eta}\right)\}\ \ (z\in U). \quad (3.22)$$

If

$$\mathcal{F}(z)=\left(\frac{\rho\big(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\big)+\tau\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)}{2(\rho+\tau)z}\right)^{\zeta}\times$$
$$\left[\sigma+\zeta\eta\left(\frac{\rho z\big(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\big)'+\tau z\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)'}{\rho\big(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\big)+\tau\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)}\right)-1\right]$$
$$+\varkappa \quad (3.23)$$

and

$$\mathcal{F}(z)\prec\sigma q(z)+\eta zq'(z)+\varkappa, \quad (3.24)$$

then

$$\left(\frac{\rho\big(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\big)+\tau\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)}{2(\rho+\tau)z}\right)^{\zeta}\prec q(z) \quad (3.25)$$

and q *is the best dominant of (3.25).*

Proof. Let $g(z)$ defined by (3.13), we see that (3.14) holds and

$$zg'(z)=\zeta g(z)$$
$$\times\left\{\left(\frac{\rho z\big(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\big)'+\tau z\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)'}{\rho\big(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\big)+\tau\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)}\right)-1\right\}. \quad (3.26)$$

Now, Let us consider $\theta(w)=\sigma w+\varkappa$ and $\phi(w)=\eta$, then θ and $\phi(w)\neq 0$ are analytic in $\mathbb{C}$. Also if we let

$$Q(z)=zq'(z)\phi(q(z))=\eta zq'(z),$$

and

$$h(z) = \theta(q(z)) + Q(z) = \sigma q(z) + \eta z q'(z) + \varkappa$$

then the assumption (3.22) yields that Q is a starlike function in U and that

$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\} = \Re\left\{\frac{\sigma}{\eta} + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0 \ (z \in U).$$

The proof follows by applying Lemma 2.2.

□

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ and using (3.6), the condition (3.22) reduces to

$$\max\left\{0; -\Re\frac{\sigma}{\eta}\right\} \le \frac{1-|B|}{1+|B|}, \tag{3.27}$$

hence, putting $\eta = \rho = 1$ and $\tau = 0$ in Theorem 3.9, we obtain the following corollary.

Corollary 3.10. *Let $f(z) \in \mathcal{A}$, $-1 \le B < A \le 1$ and $\sigma \in \mathbb{C}$ such that* $\max\{0;$ $-\Re(\sigma)\} \le \frac{1-|B|}{1+|B|}$, *suppose that* $\frac{\left(H_{q,s}(\alpha_1+1)f(z) - H_{q,s}(\alpha_1+1)f(-z)\right)}{2z} \neq 0 \ \ (z \in U)$ *and let $\zeta \in \mathbb{C}^*$. If*

$$\left(\frac{H_{q,s}(\alpha_1+1)f(z) - H_{q,s}(\alpha_1+1)f(-z)}{2z}\right)^{\zeta} \cdot \left[\sigma + \zeta\left(\frac{z\left(H_{q,s}(\alpha_1+1)f(z) - H_{q,s}(\alpha_1+1)f(-z)\right)'}{H_{q,s}(\alpha_1+1)f(z) - H_{q,s}(\alpha_1+1)f(-z)} - 1\right)\right] + \varkappa \prec \sigma\frac{1+Az}{1+Bz} + \frac{(A-B)z}{(1+Bz)^2} + \varkappa, \tag{3.28}$$

then

$$\left(\frac{H_{q,s}(\alpha_1+1)f(z) - H_{q,s}(\alpha_1+1)f(-z)}{2z}\right)^{\zeta} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant of (3.28).

Putting $\rho = 0, \eta = \tau = 1$ and $q(z) = \dfrac{1+z}{1-z}$ in Theorem 3.9, we obtain the following corollary.

Corollary 3.11. *Let $f(z) \in \mathcal{A}$ such that $\frac{H_{q,s}(\alpha_1)f(z) - H_{q,s}(\alpha_1)f(-z)}{2z} \neq 0$ for all $z \in U$ and let $\zeta \in \mathbb{C}^*$. If*

$$\left(\frac{H_{q,s}(\alpha_1)f(z) - H_{q,s}(\alpha_1)f(-z)}{2z}\right)^{\zeta} \cdot \left[\sigma + \zeta\left(\frac{z\left(H_{q,s}(\alpha_1)f(z) - H_{q,s}(\alpha_1)f(-z)\right)'}{H_{q,s}(\alpha_1)f(z) - H_{q,s}(\alpha_1)f(-z)} - 1\right)\right] + \varkappa \prec \sigma\frac{1+z}{1-z} + \frac{2z}{(1-z)^2} + \varkappa, \tag{3.29}$$

then

$$\left(\frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z}\right)^{\zeta} \prec \frac{1+z}{1-z},$$

and $\dfrac{1+z}{1-z}$ *is the best dominant of (3.29).*

4. Superordination and sandwich results

Theorem 4.1. *Let* $q(z)$ *be convex in* U, *with* $q(0)=1$ *and*

$$\alpha_1\Re(\lambda)>0. \tag{4.1}$$

Let $f(z)\in\mathcal{A}$ *and suppose that* $\frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z}\in H[q(0),1]\cap\mathcal{Q}$. *If the function*

$$(1-\lambda)\left(\frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z}\right)+\lambda\left(\frac{H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)}{2z}\right),$$

is univalent in U *and*

$$q(z)+\frac{\lambda zq'(z)}{\alpha_1} \prec (1-\lambda)\left(\frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z}\right)+\lambda\left(\frac{H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)}{2z}\right), \tag{4.2}$$

then

$$q(z)\prec\frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z}$$

and q *is the best subordinant of (4.2).*

Proof. Let $k(z)$ defined by (3.3), we see that (3.4) holds. After some computations, we obtain

$$(1-\lambda)\left(\frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z}\right)+\lambda\left(\frac{H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)}{2z}\right) = k(z)+\frac{\lambda zk'(z)}{\alpha_1} \tag{4.3}$$

and now, by using Lemma 2.5 we obtain the desired result. □

Taking $q(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$ in Theorem 4.1, we obtain the following corollary.

Corollary 4.2. *Let $q(z)$ be convex in U, with $q(0)=1$ and $[\alpha_1\Re(\lambda)]>0$. Let $f(z)\in\mathcal{A}$ and suppose that $\frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z}\in H[q(0),1]\cap\mathcal{Q}$. If the function*

$$(1-\lambda)\left(\frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z}\right)+\lambda\left(\frac{H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)}{2z}\right),$$

is univalent in U and

$$\frac{1+Az}{1+Bz}+\frac{\lambda}{\alpha_1}\frac{(A-B)z}{(1+Bz)^2}\prec(1-\lambda)\left(\frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z}\right)+\lambda\left(\frac{H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)}{2z}\right), \tag{4.4}$$

then

$$\frac{1+Az}{1+Bz}\prec\frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z}$$

and $\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$ is the best subordinant of (4.4).

The proof of the following theorem is similar to the proof of Theorem 3.4 and then applying Lemma 2.4, so we state the theorem without proof.

Theorem 4.3. *Let $q(z)$ be convex in U, with $q(0)=1$, $\eta,\zeta\in\mathbb{C}^*$, $\rho,\tau\in\mathbb{C}$, with $\rho+\tau\neq 0$. Let $f(z)\in\mathcal{A}$ and satisfy the next conditions:*

$$\frac{\rho\big(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\big)+\tau\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)}{2(\rho+\tau)z}\neq 0\ (z\in U)$$

and

$$\left(\frac{\rho\big(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\big)+\tau\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)}{2(\rho+\tau)z}\right)^{\zeta}\in H[q(0),1]\cap\mathcal{Q}.$$

If the function

$$1+\zeta\eta\left\{\frac{\rho z\big(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\big)'+\tau z\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)'}{\rho\big(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\big)+\tau\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)}-1\right\}$$

is univalent in U and

$$1+\eta\frac{zq'(z)}{q(z)}\prec 1+\zeta\eta\times\left\{\frac{\rho z\big(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\big)'+\tau z\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)'}{\rho\big(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\big)+\tau\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)}-1\right\},$$

then

$$q(z) \prec \left(\frac{\rho\left(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\right)+\tau\left(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\right)}{2(\rho+\tau)z} \right)^{\zeta} \tag{4.5}$$

and q is the best subordinant of (4.5) .

By applying Lemma 2.4, we obtain the following theorem.

Theorem 4.4. *Let $q(z)$ be convex in U, with $q(0)=1$, $\eta,\zeta \in \mathbb{C}^*$, $\rho,\tau,\sigma,\varkappa \in \mathbb{C}$, with $\rho+\tau \neq 0$ and $\Re\left(\frac{\sigma}{\eta}q^{'}(z)\right) > 0$. Let $f(z) \in \mathcal{A}$ and satisfy the next conditions:*

$$\frac{\rho\left(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\right)+\tau\left(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\right)}{2(\rho+\tau)z} \neq 0 \ (z \in U)$$

and

$$\left(\frac{\rho\left(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\right)+\tau\left(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\right)}{2(\rho+\tau)z} \right)^{\zeta} \in H[q(0),1] \cap \mathcal{Q}.$$

If the function $\mathcal{F}$ given by (3.23) is univalent in U and

$$\sigma q(z) + \eta z q^{'}(z) + \varkappa \prec \mathcal{F}(z), \tag{4.6}$$

then

$$q(z) \prec \left(\frac{\rho\left(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\right)+\tau\left(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\right)}{2(\rho+\tau)z} \right)^{\zeta}$$

and q is the best subordinant of (4.6) .

Combining Theorem 3.1 and Theorem 4.1, we obtain the following sandwich theorem.

Theorem 4.5. *Let q_1 and q_2 be two convex functions in U, with $q_1(0) = q_2(0) = 1$ and $[\alpha_1 \Re(\lambda)] > 0$. Let $f(z) \in \mathcal{A}$ and suppose that*

$$\frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z} \in H[q(0),1] \cap \mathcal{Q}.$$

If the function

$$(1-\lambda)\left(\frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z}\right) + \lambda\left(\frac{H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)}{2z}\right)$$

is univalent in U and

$$q_1(z)+\frac{\lambda z q_1'(z)}{\alpha_1} \prec (1-\lambda)\left(\frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z}\right)$$
$$+\lambda\left(\frac{H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)}{2z}\right) \prec q_2(z)+\frac{\lambda z q_2'(z)}{\alpha_2}, \qquad (4.7)$$

then

$$q_1(z) \prec \frac{H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)}{2z} \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and dominant of (4.7).

Combining Theorem 3.4 and Theorem 4.3, we obtain the following sandwich theorem.

Theorem 4.6. *Let $q(z)$ be convex in U, with $q(0)=1$, $\eta,\zeta \in \mathbb{C}^*$, $\rho,\tau \in \mathbb{C}$, with $\rho+\tau \neq 0$. Let $f(z) \in \mathcal{A}$ and satisfy*

$$\frac{\rho\big(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\big)+\tau\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)}{2(\rho+\tau)z} \neq 0 \quad (z \in U)$$

and

$$\left(\frac{\rho\big(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\big)+\tau\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)}{2(\rho+\tau)z}\right)^{\zeta} \in H[q(0),1]\cap \mathcal{Q}.$$

If the function

$$1+\zeta\eta\left\{\frac{\rho z\big(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\big)'+\tau z\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)'}{\rho\big(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\big)+\tau\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)}-1\right\}$$

is univalent in U and

$$1+\eta\frac{zq_1'(z)}{q_1(z)}$$
$$\prec 1+\zeta\eta\left\{\frac{\rho z\big(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\big)'+\tau z\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)'}{\rho\big(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\big)+\tau\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)}-1\right\}$$
$$\prec 1+\eta\frac{zq_2'(z)}{q_2(z)}, \qquad (4.8)$$

then

$$q_1(z) \prec \left(\frac{\rho\big(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\big)+\tau\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)}{2(\rho+\tau)z}\right)^{\zeta} \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and dominant of (4.8).

Combining Theorem 3.9 and Theorem 4.4, we obtain the following sandwich theorem.

Theorem 4.7. *Let q_1 and q_2 be two convex functions in U, with $q_1(0) = q_2(0) = 1$, let $\eta, \zeta \in \mathbb{C}^*$, $\rho, \tau, \sigma, \varkappa \in \mathbb{C}$, with $\rho + \tau \neq 0$ and $\Re\left(\frac{\sigma}{\eta} q^{'}(z)\right) > 0$. Let $f(z) \in \mathcal{A}$ satisfies*

$$\frac{\rho\big(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\big)+\tau\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)}{2(\rho+\tau)z} \neq 0 \quad (z \in U)$$

and

$$\left(\frac{\rho\big(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\big)+\tau\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)}{2(\rho+\tau)z}\right)^{\zeta} \in H[q(0),1] \cap \mathcal{Q}.$$

If the function $\mathcal{F}$ given by (3.23) is univalent in U and

$$\sigma q_1(z) + \eta z q_1^{'}(z) + \varkappa \prec \mathcal{F}(z) \prec \sigma q_2(z) + \eta z q_2^{'}(z) + \varkappa, \tag{4.9}$$

then

$$q_1(z) \prec \left(\frac{\rho\big(H_{q,s}(\alpha_1+1)f(z)-H_{q,s}(\alpha_1+1)f(-z)\big)+\tau\big(H_{q,s}(\alpha_1)f(z)-H_{q,s}(\alpha_1)f(-z)\big)}{2(\rho+\tau)z}\right)^{\zeta} \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and dominant of (4.9).

Remark 4.8. (i) Taking $q = 2,\ s = 1,\ \alpha_1 = \alpha_2 = \beta_1 = 1$, in Theorems 3.1, 3.4, 3.9, 4.1, 4.4, 4.5, 4.7, we obtain the results obtained by Muhammad [14, Theorems 1, 2, 3, 4, 5, 6, 7];
(ii) Taking $q = 2,\ s = 1,\ \alpha_1 = n+1\ (n > -1)$, $\alpha_2 = 1$ and $\beta_1 = 2$, in Theorems 3.1, 3.4, 3.9, 4.1, 4.4, 4.5, 4.7, we obtain the results obtained by Muhammad [15, Theorems 3.1, 3.4, 3.9, 4.1, 4.3, 4.4, 4.5].

Remark 4.9. By Specializing q, s and α_1 in the above results, we obtain the corresponding results for the operators $\mathcal{F}(\alpha_1, \alpha_2; \beta_1)$, $\mathcal{L}(a;c)$, $\mathcal{D}^n f(z)$ and $I^{\lambda}(a,c)$, which are defined in introduction.

Acknowledgments

The authors would like to record their sincere thanks to the referee(s) for their valuable comments and insightful suggestions.

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