LE MATEMATICHE
Vol. LXVIII (2013) – Fasc. II, pp. 283–295
doi: 10.4418/2013.68.2.20

# FEKETE-SZEGÖ INEQUALITIES FOR CERTAIN SUBCLASSES OF MEROMORPHIC FUNCTIONS OF COMPLEX ORDER

R. M. EL-ASHWAH - M. K. AOUF - H. M. ZAYED

In this paper, we obtain Fekete-Szegö inequalities for a certain class of meromorphic functions $f(z)$ for which $1+\frac{1}{b}\left[\frac{z(f*g)'(z)}{(\lambda-1)(f*g)(z)+\lambda z(f*g)'(z)}-1\right]$ $\prec \varphi(z)$ $(b \in \mathbb{C}^* = \mathbb{C}\backslash\{0\},\ 0 \leq \lambda < 1)$. Sharp bounds for the Fekete-Szegö functional $\left|a_1-\mu a_0^2\right|$ are obtained.

## 1. Introduction

Let $\Sigma$ denote the class of meromorphic functions of the form:

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open punctured unit disc $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U\backslash\{0\}$. Let $g(z) \in \Sigma$, be given by

$$g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} g_k z^k, \tag{1.2}$$

Entrato in redazione: 7 gennaio 2013

*AMS 2010 Subject Classification:* 30C45.
*Keywords:* Analytic, Meromorphic starlike function, Fekete-Szegö problem, Convolution.

then the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f*g)(z) = \tfrac{1}{z} + \sum_{k=0}^{\infty} a_k g_k z^k = (g*f)(z). \tag{1.3}$$

A function $f \in \Sigma$ is meromorphic starlike of order $\alpha$, denoted by $\Sigma^*(\alpha)$, if

$$-\Re\left\{\tfrac{zf'(z)}{f(z)}\right\} > \alpha \ (0 \le \alpha < 1;\ z \in U^*). \tag{1.4}$$

The class $\Sigma^*(\alpha)$ was introduced and studied by Pommerenke [22], Miller [18], Mogra et al. [19], Cho [8], Cho et al. [9] and Aouf ([4] and [5]).

Let $\varphi(z)$ be an analytic function with positive real part on $U$ satisfies $\varphi(0) = 1$ and $\varphi'(0) > 0$ which maps $U$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Let $\Sigma^*(\varphi)$ be the class of functions $f \in \Sigma$ for which

$$-\tfrac{zf'(z)}{f(z)} \prec \varphi(z) \ (z \in U^*). \tag{1.5}$$

The class $\Sigma^*(\varphi)$ was introduced and studied by Silverman et al. [26]. The class $\Sigma^*(\alpha)$ is the special case of $\Sigma^*(\varphi)$ when $\varphi(z) = \dfrac{1+(1-2\alpha)z}{1-z}$ $(0 \le \alpha < 1)$.

For $0 \le \lambda < 1$ and $b \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, we let $\Sigma^*_{\lambda,b}(g,\varphi)$ be the subclass of $\Sigma$ consisting of functions $f(z)$ of the form (1.1), the functions $g(z)$ of the form (1.2) with $g_k > 0$ and satisfying the analytic criterion:

$$1 + \tfrac{1}{b}\left[\tfrac{z(f*g)'(z)}{(\lambda-1)(f*g)(z)+\lambda z(f*g)'(z)} - 1\right] \prec \varphi(z). \tag{1.6}$$

We note that for suitable choices of $g(z), \lambda, b$ and $\varphi(z)$, we obtain the following subclasses:

(1) $\Sigma^*_{0,1}(g,\varphi) = M^s_g(\varphi)$ (see Shanmugam and Jeyaraman [25]);

(2) $\Sigma^*_{0,1}\left(\dfrac{1}{z(1-z)}, \varphi(z)\right) = \Sigma^*(\varphi)$ (see Silverman et al. [26] and Ali and Ravichandran [1, with $\alpha = 0$]);

(3) $\Sigma^*_{0,b}\left(\dfrac{1}{z(1-z)}, \dfrac{1+z}{1-z}\right) = F^*(b)$ (see Aouf [2]);

(4) $\Sigma^*_{0,b}\left(\dfrac{1}{z(1-z)}, \varphi(z)\right) = M^*_b(\varphi)$ (see Mohammed and Darus [20] and Reddy and Sharma [24, with $\gamma = 1$]);

(5) $\Sigma^*_{0,1}\left(\dfrac{1}{z(1-z)}, \dfrac{1+(1-2\alpha)z}{1-z}\right) = \Sigma^*(\alpha)$ $(0 \le \alpha < 1)$ (see Pommerenke [22]);

(6) $\Sigma^*_{0,1}\left(\dfrac{1}{z(1-z)},\dfrac{1+\beta(1-2\alpha\gamma)z}{1+\beta(1-2\gamma)z}\right)=\Sigma(\alpha,\beta,\gamma)$ $(0\leq\alpha<1,\ 0<\beta\leq 1,\ \frac{1}{2}\leq\gamma\leq 1)$ (see Kulkarni and Joshi [15]);
(7) $\Sigma^*_{0,1}\left(\frac{1}{z(1-z)},\frac{1+Az}{1+Bz}\right)=K_1(A,B)$ $(0\leq B<1,\ -B<A<B)$ (see Karunakaran [14]);
(8) $\Sigma^*_{0,(1-\rho)e^{-i\alpha}\cos\alpha}\left(\frac{1}{z(1-z)},\frac{1+z}{1-z}\right)=\Sigma^{\alpha}(\rho)$ $(|\alpha|<\dfrac{\pi}{2};\ 0\leq\rho<1)$ (see Kaczmarski [13], Aouf [3] and Ravichandran et al. [23]).

Also, we note that:
(1) $\Sigma^*_{0,b}(g,\varphi)=\Sigma^*_{b}(g,\varphi)$

$$=\left\{f(z)\in\Sigma:1-\tfrac{1}{b}\left[\tfrac{z(f*g)'(z)}{(f*g)(z)}+1\right]\prec\varphi(z)\ (z\in U^*)\right\};$$

(2) $\Sigma^*_{\lambda,b}\left(\dfrac{1}{z(1-z)},\varphi(z)\right)=\Sigma^*_{\lambda,b}(\varphi)$

$$=\left\{f(z)\in\Sigma:1+\tfrac{1}{b}\left[\tfrac{zf'(z)}{(\lambda-1)f(z)+\lambda zf'(z)}-1\right]\prec\varphi(z)\ (z\in U^*)\right\};$$

(3) $\Sigma^*_{\lambda,(1-\rho)e^{-i\alpha}\cos\alpha}\left(\dfrac{1}{z(1-z)},\varphi(z)\right)=\Sigma^{*\alpha}_{\lambda,\rho}(\varphi)$

$$=\left\{f(z)\in\Sigma:\frac{e^{i\alpha}\left[\dfrac{zf'(z)}{(\lambda-1)f(z)+\lambda zf'(z)}\right]-\rho\cos\alpha-i\sin\alpha}{(1-\rho)\cos\alpha}\prec\varphi(z)\right.$$
$$\left.(|\alpha|<\frac{\pi}{2};\ 0\leq\rho<1;\ z\in U^*)\right\};$$

(4) $\Sigma^*_{0,(1-\rho)e^{-i\alpha}\cos\alpha}\left(\dfrac{1}{z(1-z)},\varphi(z)\right)=\Sigma^{*\alpha}_{\rho}(\varphi)$

$$=\left\{f(z)\in\Sigma:\frac{e^{i\alpha}\left[-\frac{zf'(z)}{f(z)}\right]-\rho\cos\alpha-i\sin\alpha}{(1-\rho)\cos\alpha}\prec\varphi(z)\right.$$
$$\left.(|\alpha|<\tfrac{\pi}{2};\ 0\leq\rho<1;\ z\in U^*)\right\};$$

(5) $\Sigma^*_{\lambda,b}\left(\dfrac{1}{z}+\sum\limits_{k=0}^{\infty}\left[\frac{\ell+\gamma(k+1)}{\ell}\right]^m z^k,\varphi(z)\right)=\Sigma^{*m}_{\lambda,b,\gamma,\ell}(\varphi)$

$$=\left\{f(z)\in\Sigma:1+\tfrac{1}{b}\left[\tfrac{z(I^m(\gamma,\ell)f(z))'}{(\lambda-1)(I^m(\gamma,\ell)f(z))+\lambda z(I^m(\gamma,\ell)f(z))'}-1\right]\prec\varphi(z)\right.$$
$$\left.(\gamma\geq 0;\ \ell>0;\ m\in\mathbb{Z}=\{0,\pm 1,\pm 2,...\})\right\};$$

where the operator

$$I^m(\gamma,\ell)(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \left[\frac{\ell+\gamma(k+1)}{\ell}\right]^m z^k, \tag{1.7}$$

was introduced and studied by El-Ashwah [10, with $p = 1$] (see also Bulboaca et al. [7, with $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $\mathbb{N} = \{1,2,...\}$], El-Ashwah [11, with $p = 1$ and $m \in \mathbb{N}_0$] and El-Ashwah et al. [12, with $p = 1$ and $m \in \mathbb{N}_0$]);

(6) $\Sigma^*_{\lambda,b}\left(\frac{1}{z} + \sum_{k=0}^{\infty} \Gamma_{k+1}(\alpha_1)z^k, \varphi(z)\right) = \Sigma^{*\alpha_1}_{\alpha,b,q,s}(\varphi)$

$$= \left\{f(z) \in \Sigma : 1 + \frac{1}{b}\left[\frac{z(H_{q,s}(\alpha_1)f(z))'}{(\lambda-1)(H_{q,s}(\alpha_1)f(z))+\lambda z(H_{q,s}(\alpha_1)f(z))'} - 1\right] \prec \varphi(z)\right.$$
$$\left.(m \in \mathbb{N}_0;\ q \le s+1;\ q,s \in \mathbb{N}_0)\right\};$$

where the operator

$$H_{q,s}(\alpha_1)(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \Gamma_{k+1}(\alpha_1)z^k, \tag{1.8}$$

$$\Gamma_{k+1}(\alpha_1) = \frac{(\alpha_1)_{k+1} \dots (\alpha_q)_{k+1}}{(\beta_1)_{k+1} \dots (\beta_s)_{k+1}} \frac{1}{(k+1)!}, \tag{1.9}$$

for $\alpha_1, ..., \alpha_q$, $\beta_1, ..., \beta_s$ are *real* parameters and $\beta_j \notin \mathbb{Z}_0^- = \{0,-1,-2,...\}$, $j = 1,2,...,s$ was introduced and investigated by Liu and Srivastava [16, with $p = 1$] and Aouf [6, with $p = 1$].

In this paper, we obtain the Fekete-Szegö inequalities for meromorphic functions in the class $\Sigma^*_{\lambda,b}(g,\varphi)$.

## 2. Fekete-Szegö problem

To prove our results, we need the following lemmas.

**Lemma 2.1** ([17]). *If $p(z) = 1 + c_1z + c_2z^2 + ...$ is a function with positive real part in $U$ and $\mu$ is a complex number, then*

$$\left|c_2 - \mu c_1^2\right| \le 2\max\{1; |2\mu - 1|\}.$$

*The result is sharp for the functions given by*

$$p(z) = \frac{1+z^2}{1-z^2} \text{ and } p(z) = \frac{1+z}{1-z}.$$

**Lemma 2.2** ([17]). *If $p_1(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is a function with positive real part in $U$, then*

$$\left|c_2 - \nu c_1^2\right| \leq \begin{cases} -4\nu + 2 & if\ \nu \leq 0, \\ 2 & if\ 0 \leq \nu \leq 1, \\ 4\nu - 2 & if\ \nu \geq 1. \end{cases}$$

*When $\nu < 0$ or $\nu > 1$, the equality holds if and only if $p_1(z) = \dfrac{1+z}{1-z}$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $p_1(z) = \dfrac{1+z^2}{1-z^2}$ or one of its rotations. If $\nu = 0$, the equality holds if and only if*

$$p_1(z) = \left(\tfrac{1}{2} + \tfrac{1}{2}\gamma\right) \frac{1+z}{1-z} + \left(\tfrac{1}{2} - \tfrac{1}{2}\gamma\right) \frac{1-z}{1+z} \ (0 \leq \gamma \leq 1),$$

*or one of its rotations. If $\nu = 1$, the equality holds if and only if*

$$\tfrac{1}{p_1(z)} = \left(\tfrac{1}{2} + \tfrac{1}{2}\gamma\right) \frac{1+z}{1-z} + \left(\tfrac{1}{2} - \tfrac{1}{2}\gamma\right) \frac{1-z}{1+z} \ (0 \leq \gamma \leq 1),$$

*or one of its rotations. Also the above upper bound is sharp and it can be improved as follows when $0 < \nu < 1$ :*

$$\left|c_2 - \nu c_1^2\right| + \nu \left|c_1\right|^2 \leq 2 \ (0 < \nu \leq \tfrac{1}{2}),$$

*and*

$$\left|c_2 - \nu c_1^2\right| + (1-\nu) \left|c_1\right|^2 \leq 2 \ (\tfrac{1}{2} < \nu < 1).$$

Unless otherwise mentioned, we assume throughout this paper that

$0 \leq \lambda < 1,\ b \in \mathbb{C}^*$ and the function $g(z)$ is given by $(1.2)$ with $g_k > 0$ $(k \geq 0)$.

**Theorem 2.3.** *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \ldots$. If $f(z)$ given by (1.1) belongs to the class $\Sigma^*_{\lambda,b}(g,\varphi)$ and $\mu$ is a complex number. Then*

$$(i)\ \left|a_1 - \mu a_0^2\right| \leq \tfrac{|B_1 b|}{2(1-\lambda)g_1} \max\left\{1, \left|\tfrac{B_2}{B_1} - \left[1 - 2\mu \frac{g_1}{(1-\lambda)g_0^2}\right] B_1 b\right|\right\},\ B_1 \neq 0, \tag{2.1}$$

$$(ii)\ |a_1| \leq \tfrac{|B_2 b|}{2(1-\lambda)g_1},\ B_1 = 0. \tag{2.2}$$

*The result is sharp.*

*Proof.* If $f(z) \in \Sigma^*_{\lambda,b}(g,\varphi)$, then there is a Schwarz function $w(z)$ in $U$ with $w(0)=0$ and $|w(z)|<1$ in $U$ and such that

$$1+\tfrac{1}{b}\left[\tfrac{z(f*g)'(z)}{(\lambda-1)(f*g)(z)+\lambda z(f*g)'(z)}-1\right]=\varphi(w(z)). \tag{2.3}$$

Define the function $p_1(z)$ by

$$p_1(z)=\tfrac{1+w(z)}{1-w(z)}=1+c_1z+c_2z^2+\dots. \tag{2.4}$$

Since $w(z)$ is a Schwarz function, we see that $Re\ p_1(z)>0$ and $p_1(0)=1$. Define

$$p(z)=1+\tfrac{1}{b}\left[\tfrac{z(f*g)'(z)}{(\lambda-1)(f*g)(z)+\lambda z(f*g)'(z)}-1\right]=1+b_1z+b_2z^2+\dots. \tag{2.5}$$

In view of (2.3), (2.4) and (2.5), we have

$$p(z)=\varphi\left(\tfrac{p_1(z)-1}{p_1(z)+1}\right). \tag{2.6}$$

Since

$$\tfrac{p_1(z)-1}{p_1(z)+1}=\tfrac{1}{2}\left[c_1z+\left(c_2-\tfrac{c_1^2}{2}\right)z^2+\left(c_3+\tfrac{c_1^3}{4}-c_1c_2\right)z^3+\dots\right].$$

Therefore, we have

$$\varphi\left(\tfrac{p_1(z)-1}{p_1(z)+1}\right)=1+\tfrac{1}{2}B_1c_1z+\left[\tfrac{1}{2}B_1\left(c_2-\tfrac{c_1^2}{2}\right)+\tfrac{1}{4}B_2c_1^2\right]z^2+\dots,$$

and from this equation and (2.6), we obtain

$$b_1=\tfrac{1}{2}B_1c_1,$$

and

$$b_2=\tfrac{1}{2}B_1\left(c_2-\tfrac{c_1^2}{2}\right)+\tfrac{1}{4}B_2c_1^2.$$

Then, from (2.5) and (1.1), we see that

$$bb_1=-(1-\lambda)a_0g_0, \tag{2.7}$$

and

$$bb_2=(1-\lambda)^2a_0^2g_0^2-2(1-\lambda)a_1g_1, \tag{2.8}$$

or, equivalently, we have

$$a_0=\tfrac{-B_1bc_1}{2(1-\lambda)g_0},$$

and

$$a_1=\tfrac{-B_1bc_2}{4(1-\lambda)g_1}+\tfrac{bc_1^2}{8(1-\lambda)g_1}[B_1-B_2+B_1^2b].$$

Therefore

$$a_1 - \mu a_0^2 = \tfrac{-B_1 b}{4(1-\lambda)g_1}\left\{c_2 - \nu c_1^2\right\}, \tag{2.9}$$

where

$$\nu = \tfrac{1}{2}\left[1 - \tfrac{B_2}{B_1} + B_1 b - 2\mu \tfrac{B_1 b g_1}{(1-\lambda)g_0^2}\right]. \tag{2.10}$$

Now, the result (2.1) follows by an application of Lemma 2.1. Also, if $B_1 = 0$, then

$$a_0 = 0 \text{ and } a_1 = \tfrac{-B_2 b c_1^2}{8(1-\lambda)g_1}.$$

Since $p(z)$ has positive real part, $|c_1| \leq 2$ (see Nehari [21]), so that

$$|a_1| \leq \tfrac{|B_2 b|}{2(1-\lambda)g_1},$$

this proving (2.2). The result is sharp for the functions

$$1 + \tfrac{1}{b}\left[\tfrac{z(f*g)'(z)}{(\lambda-1)(f*g)(z)+\lambda z(f*g)'(z)} - 1\right] = \varphi(z^2), \tag{2.11}$$

and

$$1 + \tfrac{1}{b}\left[\tfrac{z(f*g)'(z)}{(\lambda-1)(f*g)(z)+\lambda z(f*g)'(z)} - 1\right] = \varphi(z). \tag{2.12}$$

This completes the proof of Theorem 2.3. $\square$

**Remark 2.4.** (1) Putting $g(z) = \dfrac{1}{z} + \sum\limits_{k=0}^{\infty} z^k$, $\lambda = 0$ and $b = 1$ in Theorem 2.3, we obtain the result obtained by Silverman et al. [26, Theorem 2.1];
(2) Putting $g(z) = \dfrac{1}{z} + \sum\limits_{k=0}^{\infty} \delta(n,k) z^k$, where $\delta(n,k) = \begin{pmatrix} n+k+1 \\ k+1 \end{pmatrix}$ $(n > -1)$, $\lambda = 0$ and $b = 1$ in Theorem 2.3, we obtain the result obtained by Silverman et al. [26, Theorem 3.3];
(3) Putting $g(z) = \dfrac{1}{z} + \sum\limits_{k=0}^{\infty} z^k$ and $\lambda = 0$ in Theorem 2.3, we obtain the result obtained by Mohammed and Darus [20, Theorem 1.1];
(4) Putting $g(z) = \dfrac{1}{z} + \sum\limits_{k=0}^{\infty} z^k$ and $B_1 \geq 0$ in Theorem 2.3, we obtain the result obtained by Reddy and Sharma [24, Theorem 2.1 with $\gamma = 1$];
(5) Putting $g(z) = \dfrac{1}{z} + \sum\limits_{k=0}^{\infty} z^k$, $b = (1-\rho)e^{-i\alpha}\cos\alpha$ $(|\alpha| < \dfrac{\pi}{2},\ 0 \leq \rho < 1)$, $\lambda = 0$ and $\varphi(z) = \dfrac{1+z}{1-z}$ in Theorem 2.3, we obtain the result obtained by Mohammed and Darus [20, Example 1.1].

Putting $\lambda = 0$ and $b = 1$ in Theorem 2.3, we obtain the following corollary.

**Corollary 2.5.** *If $f(z)$ given by (1.1) belongs to the class $M_g^s(\varphi)$ and $\mu$ is a complex number, then*

$$(i)\ \left|a_1-\mu a_0^2\right| \le \tfrac{|B_1|}{2g_1}\max\left\{1, \left|\tfrac{B_2}{B_1}-\left[1-\frac{2\mu g_1}{g_0^2}\right]B_1\right|\right\},\ B_1\neq 0,$$

$$(ii)\ |a_1| \le \tfrac{|B_2|}{2g_1},\ B_1=0.$$

Putting $g(z)=\dfrac{1}{z}+\sum\limits_{k=0}^{\infty} z^k$, $\lambda=0$ and $\varphi(z)=\dfrac{1+z}{1-z}$ in Theorem 2.3, we obtain the following corollary.

**Corollary 2.6.** *If $f(z)$ given by (1.1) belongs to the class $F^*(b)$ and $\mu$ is a complex number, then*

$$\left|a_1-\mu a_0^2\right| \le |b|\max\{1, |1-2(1-2\mu)b|\}.$$

*The result is sharp.*

Putting $g(z)=\dfrac{1}{z}+\sum\limits_{k=0}^{\infty} z^k$, $\lambda=0$, $b=1$ and $\varphi(z)=\dfrac{1+(1-2\alpha)z}{1-z}$ $(0\le\alpha<1)$ in Theorem 2.3, we obtain the following corollary.

**Corollary 2.7.** *If $f(z)$ given by (1.1) belongs to the class $\Sigma^*(\alpha)$ and $\mu$ is a complex number, then*

$$\left|a_1-\mu a_0^2\right| \le (1-\alpha)\max\{1, |1-2(1-\alpha)(1-2\mu)|\}.$$

*The result is sharp.*

Putting $\lambda=0$ in Theorem 2.3, we obtain the following corollary.

**Corollary 2.8.** *Let $\varphi(z)=1+B_1z+B_2z^2+\ldots$. If $f(z)$ given by (1.1) belongs to the class $\Sigma_b^*(g,\varphi)$ and $\mu$ is a complex number, then*

$$(i)\ \left|a_1-\mu a_0^2\right| \le \tfrac{|B_1 b|}{2g_1}\max\left\{1, \left|\tfrac{B_2}{B_1}-\left[1-2\mu\frac{g_1}{g_0^2}\right]B_1 b\right|\right\},\ B_1\neq 0,$$

$$(ii)\ |a_1| \le \tfrac{|B_2 b|}{2g_1},\ B_1=0.$$

*The result is sharp.*

Putting $g(z)=\dfrac{1}{z}+\sum\limits_{k=0}^{\infty} z^k$ in Theorem 2.3, we obtain the following corollary.

**Corollary 2.9.** *Let $\varphi(z)=1+B_1z+B_2z^2+\ldots$. If $f(z)$ given by (1.1) belongs to the class $\Sigma^*_{\lambda,b}(\varphi)$ and $\mu$ is a complex number, then*

$$(i)\ \left|a_1-\mu a_0^2\right|\leq \tfrac{|B_1b|}{2(1-\lambda)}\max\left\{1,\left|\tfrac{B_2}{B_1}-\left[1-\frac{2\mu}{(1-\lambda)}\right]B_1b\right|\right\},\ B_1\neq 0,$$

$$(ii)\ |a_1|\leq \tfrac{|B_2b|}{2(1-\lambda)},\ B_1=0.$$

*The result is sharp.*

**Remark 2.10.** (1) Putting $\lambda=0$ and $b=(1-\rho)e^{-i\alpha}\cos\alpha$ ($|\alpha|<\frac{\pi}{2}$, $0\leq\rho<1$) in Corollary 2.9, we obtain a new result for the class $\Sigma^{*\alpha}_{\rho}(\varphi)$;
(2) Putting $b=(1-\rho)e^{-i\alpha}\cos\alpha$ ($|\alpha|<\frac{\pi}{2}$, $0\leq\rho<1$) in Corollary 2.9, we obtain a new result for the class $\Sigma^{*\alpha}_{\lambda,\rho}(\varphi)$.

By using Lemma 2.2, we can obtain the following theorem.

**Theorem 2.11.** *Let $\varphi(z)=1+B_1z+B_2z^2+\ldots$ $(B_i>0,\ i\in\mathbb{N},\ b>0)$. If $f(z)$ given by (1.1) belongs to the class $\Sigma^*_{\lambda,b}(g,\varphi)$ and $\mu$ is a real number, then*

$$\left|a_1-\mu a_0^2\right|\leq\begin{cases}\dfrac{b}{2(1-\lambda)g_1}\left\{-B_2+\left[1-2\mu\dfrac{g_1}{(1-\lambda)g_0^2}\right]B_1^2b\right\} & if\ \mu\leq\sigma_1,\\ \dfrac{B_1b}{2(1-\lambda)g_1} & if\ \sigma_1\leq\mu\leq\sigma_2,\\ \dfrac{b}{2(1-\lambda)g_1}\left\{B_2-\left[1-2\mu\dfrac{g_1}{(1-\lambda)g_0^2}\right]B_1^2b\right\} & if\ \mu\geq\sigma_2,\end{cases}\tag{2.13}$$

*where*

$$\sigma_1=\frac{\left\{-(B_2+B_1)+B_1^2b\right\}(1-\lambda)g_0^2}{2g_1B_1^2b}.$$

*and*

$$\sigma_2=\frac{\left\{-(B_2-B_1)+B_1^2b\right\}(1-\lambda)g_0^2}{2g_1B_1^2b}.$$

*The result is sharp.*

*Proof.* First, let $\mu\leq\sigma_1$. Then

$$\begin{aligned}\left|a_1-\mu a_0^2\right| &\leq \frac{B_1b}{2(1-\lambda)g_1}\left\{-\tfrac{B_2}{B_1}+\left[1-2\mu\frac{g_1}{(1-\lambda)g_0^2}\right]B_1b\right\}\\ &\leq \frac{b}{2(1-\lambda)g_1}\left\{-B_2+\left[1-2\mu\frac{g_1}{(1-\lambda)g_0^2}\right]B_1^2b\right\}.\end{aligned}$$

Let, now $\sigma_1 \leq \mu \leq \sigma_2$. Then, using the above calculations, we obtain

$$\left|a_1 - \mu a_0^2\right| \leq \frac{B_1 b}{2(1-\lambda)g_1}.$$

Finally, if $\mu \geq \sigma_2$, then

$$\begin{aligned}\left|a_1 - \mu a_0^2\right| &\leq \frac{B_1 b}{2(1-\lambda)g_1}\left\{\tfrac{B_2}{B_1} - \left[1 - 2\mu\frac{g_1}{(1-\lambda)g_0^2}\right]B_1 b\right\} \\ &\leq \frac{b}{2(1-\lambda)g_1}\left\{B_2 - \left[1 - 2\mu\frac{g_1}{(1-\lambda)g_0^2}\right]B_1^2 b\right\}.\end{aligned}$$

To show that the bounds are sharp, we define the functions $K_{\varphi n}(n \geq 2)$ by

$$1+\tfrac{1}{b}\left[\tfrac{z(K_{\varphi n}*g)'(z)}{(\lambda-1)(K_{\varphi n}*g)(z)+\lambda z(K_{\varphi n}*g)'(z)} - 1\right] = \varphi(z^{n-1}), K_{\varphi n}(0) = 0 = K'_{\varphi n}(0) - 1,$$

and the functions $F_\gamma$ and $G_\gamma$ $(0 \leq \gamma \leq 1)$ by

$$1+\tfrac{1}{b}\left[\tfrac{z(F_\gamma*g)'(z)}{(\lambda-1)(F_\gamma*g)(z)+\lambda z(F_\gamma*g)'(z)} - 1\right] = \varphi\left(\tfrac{z(z+\gamma)}{1+\gamma z}\right), F_\gamma(0) = 0 = F'_\gamma(0) - 1,$$

and

$$1+\tfrac{1}{b}\left[\tfrac{z(G_\gamma*g)'(z)}{(\lambda-1)(G_\gamma*g)(z)+\lambda z(G_\gamma*g)'(z)} - 1\right] = \varphi\left(-\tfrac{z(z+\gamma)}{1+\gamma z}\right), G_\gamma(0) = 0 = G'_\gamma(0) - 1.$$

Clearly the functions $K_{\varphi n},\ F_\gamma$ and $G_\gamma \in \Sigma^*_{\lambda,b}(g,\varphi)$. Also we write $K_\varphi = K_{\varphi 2}$.

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if $f$ is $K_\varphi$ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if $f$ is $K_{\varphi 3}$ or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if $f$ is $F_\gamma$ or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if $f$ is $G_\gamma$ or one of its rotations. This completes the proof of Theorem 2.11. □

**Remark 2.12.** (1) Putting $g(z) = \dfrac{1}{z} + \sum\limits_{k=0}^{\infty} z^k,\ \lambda = 0$ and $b = 1$ in Theorem 2.11, we obtain the result obtained by Ali and Ravichandran [1, Theorem 5.1]; (2) For different choices of $g(z),\ \lambda,\ b$ and $\varphi(z)$ in Theorem 2.11, we will obtain new results for different classes mentioned in the introduction.

Using arguments similar to those in the proof of Theorem 2.11, we obtain the following theorem.

**Theorem 2.13.** *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ $(B_i > 0,\ i \in \mathbb{N},\ b > 0)$ *and*

$$\sigma_3 = \frac{\left(-B_2 + B_1^2 b\right)(1-\lambda)g_0^2}{2g_1 B_1^2 b}. \tag{2.14}$$

*If* $f(z)$ *given by (1.1) belongs to the class* $\Sigma^*_{\lambda,b}(g,\varphi)$ *and* $\mu$ *is a real number, then we have*
*(i) If* $\sigma_1 \leq \mu \leq \sigma_3$, *then*

$$\left|a_1 - \mu a_0^2\right| + \tfrac{(1-\lambda)g_0^2}{2g_1 B_1^2 b}\left\{(B_1 + B_2) + \left[2\mu \tfrac{g_1}{(1-\lambda)g_0^2} - 1\right] B_1^2 b\right\}|a_0|^2 \leq \tfrac{B_1 b}{2(1-\lambda)g_1}. \tag{2.15}$$

*(ii) If* $\sigma_3 \leq \mu \leq \sigma_2$, *then*

$$\left|a_1 - \mu a_0^2\right| + \tfrac{(1-\lambda)g_0^2}{2g_1 B_1^2 b}\left\{(B_1 - B_2) + \left[1 - 2\mu \tfrac{g_1}{(1-\lambda)g_0^2}\right] B_1^2 b\right\}|a_0|^2 \leq \tfrac{B_1 b}{2(1-\lambda)g_1}, \tag{2.16}$$

*where* $\sigma_1$ *and* $\sigma_2$ *are given in Theorem 2.11.*

**Remark 2.14.** By specializing the function $g(z)$ in Theorems 2.3, 2.11 and 2.13, we will obtain new results for different classes mentioned in the introduction.

**Acknowledgments**

The authors thank the referees for their valuable suggestions which led to the improvement of this paper.

REFERENCES

[1] R. M. Ali - V. Ravichandran, *Classes of meromorphic* $\alpha$*-convex functions*, Taiwanese J. Math. 14 (4) (2010), 1479–1490.

[2] M. K. Aouf, *Coefficient results for some classes of meromorphic functions*, J. Nat. Sci. Math. 27 (2) (1987), 81–97.

[3] M. K. Aouf, *On the coefficients of some meromorphic classes of order* $\alpha$ *and type* $\beta$, Rend. Mat. 8 (1988), 211-221.

[4] M. K. Aouf, *A certain subclass of meromorphically starlike functions with positive coefficients*, Rend. Mat. 9 (1989), 225–235.

[5] M. K. Aouf, *On a certain class of meromorphically univalent functions with positive coefficients*, Rend. Mat. 11 (1991), 209–219.

[6] M. K. Aouf, *Certain subclasses of meromorphically multivalent functions associated with generalized hypergeometric function*, Comput. Math. Appl. 55 (2008), 494–509

[7] T. Bulboaca - M. K. Aouf - R. M. El-Ashwah, *Convolution properties for subclasses of meromorphic univalent functions of complex order*, Filomat 26 (1) (2012), 153–163.

[8] N. E. Cho, *On certain class of meromorphic functions with positive coefficients*, J. Inst. Math. Comput. Sci. 3 (2) (1990), 119–125.

[9] N. E. Cho - S. H. Lee - S. Owa, *A class of meromorphic univalent functions with positive coefficients*, Kobe J. Math. 4 (1987), 43–50.

[10] R. M. El-Ashwah, *Argument properties for p-valent meromorphic functions defined by differintegral operator*, (Submitted).

[11] R. M. El-Ashwah, *Properties of certain class of p-valent meromorphic functions associated with new integral operator*, Acta Univ. Apulensis 29 (2012), 255–264.

[12] R. M. El-Ashwah - M. K. Aouf - T. Bulboaca, *Differential subordinations for classes of meromorphic p-valent functions defined by multiplier transformations*, Bull. Aust. Math. Soc. 83 (2011), 353–368.

[13] J. Kaczmarski, *On the coefficients of some classes of starlike functions*, Bull. Acad. Polon. Sci. Ser. Math. Astronom. Phy. 17 (1969), 495–501.

[14] V. Karunakaran, *On a class of meromorphic starlike functions in the unit disc*, Math. Chronicle 4 (2-3) (1976), 112–121.

[15] S. R. Kulkarni - S. S. Joshi, *On a subclass of meromorphic univalent functions with positive coefficients*, J. Indian Math. Acad. 24 (1) (2002), 197–205.

[16] J. L. Liu - H. M. Srivastava, *Classes of meromorphically multivalent functions associated with the generalized hypergeometric function*, Math. Comput. Modelling 39 (2004), 21–34.

[17] W. Ma - D. Minda, *A unified treatment of some special classes of univalent functions*, in Proceedings of the conference on complex analysis, Z. Li, F. Ren, L. Lang and S. Zhang (Eds.), Int. Press (1994), 157–169.

[18] J. E. Miller, *Convex meromorphic mapping and related functions*, Proc. Amer. Math. Soc. 25 (1970), 220–228.

[19] M. L. Mogra - T. Reddy - O. P. Juneja, *Meromorphic univalent functions with positive coefficients*, Bull. Aust. Math. Soc. 32 (1985), 161–176.

[20] A. Mohammed - M. Darus, *On the class of starlike meromorphic functions of complex order*, Rend. Mat. 31 (2011), 53–61.

[21] Z. Nehari, *Conformal Mapping*, McGraw-Hill, New York, 1952.

[22] Ch. Pommerenke, *On meromorphic starlike functions*, Pacific J. Math. 13 (1963), 221–235.

[23] V. Ravichandran - S. S. Kumar - K. G. Subramanian, *Convolution conditions for spirallikeness and convex spirallikeness of certain meromorphic p-valent functions*, J. Inequal. Pure Appl. Math. 5 (1) Art. 11 (2004), 7 pp.

[24] T. R. Reddy - R. B. Sharma, *Fekete-Szegö Inequality for certain subclasses of meromorphic functions*, Internat. Math. Forum 5 (68) (2010), 3399–3412.

[25] T. N. Shanmugam - M. P. Jeyaraman, *Certain subclasses of meromorphic functions using convolution and subordination*, General Math. 20 (1) (2012), 49–56.

[26] H. Silverman - K. Suchithra - B. A. Stephen - A. Gangadharan, *Cofficient bounds for certain classes of meromorphic functions*, J. Inequal. Pure Appl. Math. (2008), 1–9.

*RABHA M. EL-ASHWAH*
*Department of Mathematics*
*Faculty of Science*
*Damietta University*
*New Damietta 34517, Egypt*
*e-mail:* r_elashwah@yahoo.com

*MOHAMED K. AOUF*
*Department of Mathematics*
*Faculty of Science,*
*Mansoura University*
*Mansoura 35516, Egypt*
*e-mail:* mkaouf127@yahoo.com

*HANAA M. ZAYED*
*Department of Mathematics*
*Faculty of Science*
*Menofia University*
*Shebin Elkom 32511, Egypt*
*e-mail:* hanaazayed42@yahoo.com