LE MATEMATICHE
Vol. LXI (2006) - Fasc. I, pp. 69-84

NONISOTRIVIAL FAMILIES OVER CURVES WITH FIXED POINT FREE AUTOMORPHISMS

DAJANO TOSSICI - FRANCESCA VETRO

We construct for any smooth projective curve of genus $q \geq 2$ with a fixed point free automorphism a nonisotrivial family of curves. Moreover we study the space of the modular curves and one of the parameters.

1. Introduction.

It has been conjectured by Shafarevich in 1962 that for every $g \geq 2$, every curve B and every finite subset S of B the set of nonisotrivial families of curves of genus g over $B \setminus S$ is finite (see below for what we mean for curve). The proof of this statement is due to Parshin ($S = \emptyset$, [15]) and Arakelov ($S \neq \emptyset$, [1]). We are interested to the case $S = \emptyset$. We stress that there are no nonisotrivial families of curves over curves with genus at most one ([4]). The first examples of nonisotrivial families of curves over a curve has been given by Atiyah ([2]) and, independently, by Kodaira ([13]).

It is well known that to every family $X \to B$ of curves of genus g over a curve B is associated a *moduli map* $B \to \mathrm{M}_g$, where M_g is the moduli space of curves of genus g. The images of these maps are called *modular curves*. If we consider only curves B of genus q we indicate the

Entrato in redazione il 10 Novembre 2005.

space of these modular curves with $\mathrm{W}_g^q \subseteq \mathrm{M}_g$. We prove that for every $g \geq 3$ there exists $q_0(g) \geq g$ such that the generic curve of genus g is the fiber of a family over a curve of genus $q_0(g)$, i.e. $\mathrm{M}_g = \overline{\mathrm{W}_g^{q_0(g)}}$. Moreover we call $\mathrm{Z}_g^q \subseteq \mathrm{M}_q$ the space of curves of genus q which are base of some families of curves of genus g. We don't know if also the *space of parameters*, $\mathrm{Z}^q = \bigcup_g \overline{\mathrm{Z}_g^q}$, coincides with the space of moduli curves M_q.

In the last two sections of the paper we give examples of nonisotrivial families of curves over a curve and we investigate the spaces determined by these families in the space of parameters. We observe that the example of Flexor ([11]) and Kodaira ([13]) have two different fibrations one of which gives

$$\dim \mathrm{Z}^q \geq \dim \mathrm{E}^q$$

where the last one is the space of curves of genus q which are étale covering of some curve. (In reality the example of Kodaira is valid only for q odd). As a particular case of our construction we obtain nonisotrivial families over every curve with a fixed point free automorphism. So we can give a better lower bound for the dimension of the space of parameters. In fact this one is greater than the dimension of the space of curves of genus q with a free fixed point automorphism.

Our examples are a generalization of an example of Bryan and Donagi ([6]).

Notation. In the paper, if not differently specified, for curve we mean a nonsingular complete irreducible curve over $\mathbb{C}$. And for a family of curves of genus g we mean a flat surjective morphism $X \to Z$ with X and Z integral varieties and curves of genus g as fibers.

2. The space of modular curves and the space of parameters.

In this section we give some result about the space of modular curves.

Definition 2.1. A family of curves $X \to Z$ is called *isotrivial* if there exists a curve C such that for every $z \in Z$ the fiber X_z is isomorphic to C.

Remark 2.2. We recall that every family $f : X \to Z$ of curves of genus g determines a morphism $\varphi_f : Z \to \mathrm{M}_g$ defined by $z \to [X_z]$. This morphism is called *moduli map* of f. So we have that a family is isotrivial if and only if its associated moduli map is constant. In particular we have that a family is isotrivial if and only if it is true in an open of Z.

Definition 2.3. We call *modular curve* a curve $C \subseteq \mathrm{M}_g$, not necessarily nonsingular, which is image of a moduli map $\varphi : B \to \mathrm{M}_g$, with B curve.

Definition 2.4. For every q and g we define

$$\mathrm{W}_g^q = \bigcup_{\substack{\varphi \in \mathrm{Hom}^*(B, \mathrm{M}_g) \\ [B] \in \mathrm{M}_q}} \mathrm{Im}\,\varphi \subseteq \mathrm{M}_g$$

where $\mathrm{Hom}^*(B, \mathrm{M}_g)$ is the set of nonconstant moduli maps, i.e. maps induced by nonisotrivial families. And we call $\mathrm{W}_g = \bigcup_q \overline{\mathrm{W}_g^q}$ the *space of modular curves*.

Symmetrically we have the following definition.

Definition 2.5. For every q and g we set

$$\mathrm{Z}_g^q = \{[B] \in \mathrm{M}_q \text{ s.t. } \mathrm{Hom}^*(B, \mathrm{M}_g) \neq \emptyset\} \subseteq \mathrm{M}_q$$

and we call $Z^q = \bigcup_g \overline{\mathrm{Z}_g^q}$ the *space of parameters*.

We remark that these spaces are empty if $g \leq 2$ or if $q \leq 1$. In fact if $g \leq 2$, M_g is affine, as it is a point or the space of moduli of hyperelliptic curves. While for $q \leq 1$ (see [17], Theorem 4).

We prove that W_g^q and Z_g^q are constructible spaces. We use the following result.

Proposition 2.6. *Let $X \to Z$ be a family of curves of genus g. Then for every g the Z-set* $\mathrm{Hom}_Z^*(X, \mathrm{M}_g) \subseteq \mathrm{Hom}_Z(X, \mathrm{M}_g \times Z)$*, whose fiber over $z \in Z$ is the set* $\mathrm{Hom}^*(X_z, \mathrm{M}_g)$*, is constructible.*

Proof. See the proof of Proposition 3.2 in [9]. □

Lemma 2.7. *For every q and g the set $\mathrm{Z}_g^q \subseteq \mathrm{M}_q$ is constructible.*

Proof. Using the notation of the previous proposition consider as $X \to Z$ the universal family $\mathcal{C}^N \to \mathrm{M}_q^N$ of the space of curves of genus q with level N-structure, with $N \geq 3$. Then the previous result tell us that $\mathrm{Hom}^*_{\mathrm{M}_q^N}(\mathcal{C}^N, \mathrm{M}_g)$ is constructible. We recall that there is a finite morphism

$\mathrm{M}_q^N \to \mathrm{M}_q$. (For further details on M_q^N see [16], Lecture 10). So we are in the following situation

$$\begin{array}{ccc}
\mathrm{Hom}^*_{\mathrm{M}_q^N}(\mathcal{C}^N, \mathrm{M}_g) & \subseteq & \mathrm{Hom}_{\mathrm{M}_q^N}(\mathcal{C}^N, \mathrm{M}_g \times \mathrm{M}_q^N) \\
\downarrow & \swarrow & \\
\mathrm{M}_q^N & & \\
\downarrow & & \\
\mathrm{M}_q & &
\end{array}$$

The set $\mathrm{Z}_g^q \subseteq \mathrm{M}_q$ is nothing else but the image of $\mathrm{Hom}^*_{\mathrm{M}_q^N}(\mathcal{C}^N, \mathrm{M}_g)$ in M_q. So we have the lemma. □

Lemma 2.8. *For every q and g the set*

$$\mathrm{W}_g^q \subseteq \mathrm{M}_g$$

is constructible. Moreover

$$\dim \mathrm{W}_g^q \leq \dim \mathrm{Z}_g^q + 1.$$

Proof. Let

$$\begin{array}{ccc}
\mathcal{C}^N & & \mathrm{Hom}^*_{\mathrm{M}_q^N}(\mathcal{C}^N, \mathrm{M}_g) \\
& \searrow \quad \swarrow & \\
& \mathrm{M}_q^N & \\
& \downarrow & \\
& \mathrm{M}_q &
\end{array}$$

the maps described above. We consider the valuation map

$$\psi : \mathcal{C}^N \times_{\mathrm{M}_q} \mathrm{Hom}^*_{\mathrm{M}_q^N}(\mathcal{C}^N, \mathrm{M}_g) \to \mathrm{M}_g$$

defined by $\psi((x, \varphi_B)) = \varphi_B(x)$, with $x \in B$ e $\varphi_B : B \to \mathrm{M}_g$ a nonconstant moduli map. We have

$$\mathrm{Im}\psi = \mathrm{W}_g^q,$$

so W_g^q is constructible, being the image of a constructible set. Therefore we have proved the first assertion.

By the Theorem of Parshin the natural map $\mathrm{Hom}^*_{\mathrm{M}_q^N}(\mathcal{C}^N, \mathrm{M}_g) \to \mathrm{M}_q$ has finite fibers. From the other side $\mathcal{C}^N \to \mathrm{M}_q$ has fibers of dimension 1. So, by the theorem on the dimensions of fibers, we have

$$\dim \mathrm{Hom}^*_{\mathrm{M}_q^N}(\mathcal{C}^N, \mathrm{M}_g) = \dim \mathrm{Z}_g^q$$
$$\dim \mathcal{C}^N = \dim \mathrm{M}_q + 1.$$

Now applying the theorem on the dimensions of fibers to the map ψ we have

$$\dim \mathrm{Im}\psi = \dim \mathrm{W}_g^q \leq \dim \mathcal{C}^N \times_{\mathrm{M}_q} \mathrm{Hom}^*_{\mathrm{M}_q^N}(\mathcal{C}^N, \mathrm{M}_g) = \dim \mathrm{Z}_g^q + 1.$$

□

We study some properties of these sets.

Lemma 2.9. *Let $X \to B$ be a family of curves of genus g over a curve B of genus $q \geq 2$. Then for every q' such that $(q-1)|(q'-1)$ there exists a family $X' \to B'$ of curves of genus g such that $g(B') = q'$. Moreover the moduli maps associated to $B \to \mathrm{M}_g$ and $B' \to \mathrm{M}_g$ have the same image.*

Proof. In fact for every n there exists an étale covering $f : B' \to B$ of degree n. We set $n = \dfrac{q'-1}{q-1}$ and we consider

$$X' = X \times_B B'.$$

For every $b \in B'$ we have $X'_b = X_{f(b)}$ so the fibers of $X' \to B'$ have the same genus of ones of $X \to B$ and moreover their moduli maps have the same image. On the other hand, using Riemann-Hurwitz formula we obtain the assertion about the genus of B'. □

Remark 2.10. From any family of curves of genus g over a curve of genus 2, by the previous lemma, we can construct families of curves of genus g over curves of any genus.

Now we prove a result about the space of the modular curves. This has been proved, independently by us, also by Caporaso and Sernesi (unpublished).

Lemma 2.11. *Let Z be an integral variety and $\varphi : Z \to \mathrm{M}_g$ a map. Then there exists a finite dominant map $f : Z' \to Z$, with Z' integral such that $\varphi \circ f$ is a moduli map. In particular every complete curve $C \subseteq \mathrm{M}_g$, not necessarily nonsingular, is modular.*

Proof. We consider the finite morphism $F : \mathrm{M}_g^N \to \mathrm{M}_g$ with $N \geq 3$. Then we consider the pull-back of the map φ

$$\begin{array}{ccc} Z' & \xrightarrow{\psi} & \mathrm{M}_g^N \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ Z & \xrightarrow{\varphi} & \mathrm{M}_g \end{array}$$

and we obtain a finite map $f : Z' \to Z$. Taking, possibly, one of its irreducible component which dominate Z and the reduced variety associated to this one, we can suppose Z' integral (the commutative diagram above will not remain cartesian in general). Then, as M_g^N is a fine moduli space, there exists a family of curves of N-level $X \to Z'$ associated to the map $\psi : Z' \to \mathrm{M}_g^N$. If we consider this family simply as family of curves, the associated moduli map is

$$\varphi \circ f = F \circ \psi : Z' \to \mathrm{M}_g.$$

So we have the first assertion of the lemma.

By what we have proved, for a curve $C \subseteq \mathrm{M}_g$, there exists a family over a finite cover of the normalization of C such that the image of the associated moduli map is C. So C is a modular curve. □

Proposition 2.12. *For every $g \geq 3$ there exists $q_0(g)$ such that the generic curve of genus g is a fiber of a family of curves with base of genus $q_0(g)$, i.e.*

$$\mathrm{M}_g = \overline{\mathrm{W}_g^{q_0(g)}}.$$

Moreover for every such $q_0(g)$ we have $q_0(g) \geq g$.

Proof. If $g \geq 3$ the Satake compactification of M_g is a projective compactification of M_g such that the complement of M_g has codimension 2. So every point $[B]$ of M_g lies on a complete curve C, not necessarily nonsingular, entirely contained in M_g. But thanks to the previous lemma we have that C is modular. So

$$\mathrm{M}_g = \bigcup_q \mathrm{W}_g^q = \bigcup_q \overline{\mathrm{W}_g^q}. \tag{2.1}$$

By (2.1) we have that M_g is a numerable union of closed subspaces. This is clearly possible if and only if one of this subspace is all M_g. So we have proved the first assertion.

Moreover by 2.8 for every q, g

$$\dim \mathrm{W}_g^q \le \dim \mathrm{Z}_g^q + 1.$$

If $q_0(g) < g$ we would have

$$\dim \mathrm{W}_g^{q_0(g)} \le \dim \mathrm{Z}_g^{q_0(g)} + 1 \le 3q_0(g) - 2 \le 3g - 5.$$

Then $\mathrm{W}_g^{q_0(g)}$ would be a proper subset of M_g (of codimension at most 2) but it is in contradiction with the definition of $q_0(g)$. □

Now we stress some relations between the various spaces W_g^q and Z_g^q. By 2.9 we have for every g

$$\mathrm{W}_g^q \subseteq \mathrm{W}_g^{q'} \tag{2.2}$$

if $(q-1)|(q'-1)$. In particular $\mathrm{W}_g^2 \subseteq \mathrm{W}_g^q$ for every q.

Therefore by 2.8 and (2.2) follows that

$$\dim \mathrm{W}_g^q \le \dim \mathrm{Z}_g^{q'} + 1 \tag{2.3}$$

if $(q-1)|(q'-1)$.

Remark 2.13. Thanks (2.2) we can see that, for every $g \ge 3$, there are infinite $q_0(g)$ which satisfy the proposition 2.12.

3. Families over curves with fixed point free automorphisms.

In this section we construct nonisotrivial families over curves of genus ≥ 2 with a fixed point free automorphism.

Lemma 3.1. *Let f be a fixed point free automorphism of a curve B of genus at least 2. Then, for every $n \in \mathbb{N}$, there exists an étale covering (not necessarily connected) $\delta : \widetilde{B} \to B$ of degree n^{2q}, with q equal to the genus of B, such that the divisor $D = \Gamma_{\widetilde{f}} - \Gamma_\delta$ has a n-th root in $\mathrm{Pic}(\widetilde{B} \times B)$, where $\widetilde{f} = f \circ \delta$. For any morphism $g : X \to Y$ we denote with Γ_g its graph.*

Proof. Let

$$\widetilde{B} = \{(L, b) \in \mathrm{Pic}^0(B) \times B \text{ s.t. } L^{\otimes n} \simeq f(b) - b\}$$

and δ the projection over B. Obviously δ is an étale morphism of degree n^{2q}. Then we define $\widetilde{f} = f \circ \delta$. This is a morphism of degree n^{2q}.

We fix $p_0 \in B$. Let $M \in \mathrm{Pic}^0(\widetilde{B})$ such that $M^{\otimes n} \simeq \widetilde{f}^{-1}(p_0) - \delta^{-1}(p_0)$. The Poincaré bundle $\mathcal{P} \to \mathrm{Pic}^0(B) \times B$ is uniquely determined by specifying that $\mathcal{P}$ restricted to $\mathrm{Pic}^0(B) \times \{p_0\}$ is trivial.

We use the same letter $\mathcal{P}$ to denote the pullback of $\mathcal{P}$ by the following composition

$$\tilde{B} \times B \xrightarrow{i \times \mathrm{id}} \mathrm{Pic}^0(B) \times B \times B \xrightarrow{p_{13}} \mathrm{Pic}^0(B) \times B$$

where $i : \widetilde{B} \to \mathrm{Pic}^0(B) \times B$ is the natural inclusion and p_{13} is the map defined by $(L, p, q) \mapsto (L, q)$.

We set $\mathcal{L} = \mathcal{P} \otimes \pi^* M$, where $\pi : \tilde{B} \times B \to \tilde{B}$ is the projection, and we claim that $\mathcal{L}^{\otimes n} \simeq D$.

It is sufficient to prove that $\mathcal{L}^{\otimes n}$ and D coincide on the fiber over p_0 and on all the fibers over points of $\tilde{B}$.

Let $x = (L, b)$ be any point of $\tilde{B}$. By construction we have

$$\begin{aligned} \mathcal{L}^{\otimes n}{}_{|\{x\} \times B} &= \mathcal{P}^{\otimes n}{}_{|\{x\} \times B} \otimes \pi^* M_{|\{x\} \times B} \\ &\simeq L^{\otimes n} \otimes \mathcal{O}_B \\ &\simeq L^{\otimes n} \\ &\simeq f(b) - b \\ &= \widetilde{f}(x) - \delta(x) \\ &= D_{|\{x\} \times B} \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}^{\otimes n}{}_{|\tilde{B} \times \{p_0\}} &= \mathcal{P}^{\otimes n}{}_{|\tilde{B} \times \{p_0\}} \otimes \pi^* M^{\otimes n}{}_{|\tilde{B} \times \{p_0\}} \\ &\simeq \mathcal{O}_{\widetilde{B}} \otimes M^{\otimes n} \\ &\simeq M^{\otimes n} \\ &\simeq \widetilde{f}^{-1}(p_0) - \delta^{-1}(p_0) \\ &= D_{|\tilde{B} \times \{p_0\}}. \end{aligned}$$

□

Theorem 3.2. *If B is a curve of genus $q \geq 2$ with a fixed point free automorphism f then for every $n \geq 2$ there exist families over B and over an étale connected covering $\widetilde{B}$ of degree n^{2q} with base and fiber genera (q_i, g_i) equal to*

$$(q_1, g_1) = (q, n^{2q}(nq - 1) + 1),$$
$$(q_2, g_2) = (n^{2q}(q - 1) + 1, nq).$$

Proof. We use the notation of above lemma. We suppose for the moment that $\tilde{B}$, constructed in the proof of lemma, is connected. Then we have $\mathcal{L}$ such that $\mathcal{L}^{\otimes n} \simeq D = \Gamma_{\tilde{f}} - \Gamma_\delta$. So we can define

$$X_n = \{(v_1 : v_2) \in \mathbb{P}(\mathcal{L} \oplus \mathcal{O}) \text{ s.t. } (v_1^n : v_2^n) = (s_1 : s_2)\}$$

where s_1 is a section of $\mathcal{O}(\Gamma_{\tilde{f}})$ and s_2 a section of $\mathcal{O}(\Gamma_\delta)$ that vanish, respectively along $\Gamma_{\tilde{f}}$ and Γ_δ so that $(s_1 : s_2)$ is in $\mathbb{P}(\mathcal{O}(\Gamma_{\tilde{f}}) \oplus \mathcal{O}(\Gamma_\delta))$ which is the same of $\mathbb{P}(\mathcal{O}(\Gamma_{\tilde{f}} - \Gamma_\delta) \oplus \mathcal{O})$. Moreover $X_n \to \tilde{B} \times B$ is a cyclic covering ramified over D.

Using Hurwitz formula we obtain

$$g(\tilde{B}) = n^{2q}(q-1) + 1.$$

If we consider the fibration $X_n \to B$ (so $q_1 = q$) we have for every $b \in B$ a cyclic covering $(X_n)_b \to \tilde{B}$ of degree n ramified over $D_{|\tilde{B}\times\{b\}} = \tilde{f}^{-1}(b) - \delta^{-1}(b)$. Using again Hurwitz formula we have

$$g_1 = g((X_n)_b) = n^{2q+1}(q-1) + n^{2q}(n-1) + 1 = n^{2q}(nq-1) + 1.$$

Now the coverings $(X_n)_b \to \tilde{B}$ vary in an infinite set (as the ramification locus varies in an infinite set). Therefore the de Franchis-Severi theorem (see 3.6) ensures that the fibration is nonisotrivial.

Now we consider $X_n \to \tilde{B}$ (so $q_2 = n^{2q}(q-1)+1$). For every $x \in \tilde{B}$ we have the cyclic covering $(X_n)_x \to B$ ramified over $D_{|\{x\}\times B} = \tilde{f}(x) - \delta(x)$. So

$$g_2 = n(q-1) + n = nq.$$

Reasoning as above we have that the family is nonisotrivial.

Now we prove that $\tilde{B}$ can be taken connected. We suppose that $\tilde{B}$ is disconnected with N components. Since $\delta : \tilde{B} \to B$ is a normal covering, N must divide n^{2q}, the degree of $\tilde{f} : \tilde{B} \to B$ and $\delta : \tilde{B} \to B$. Fix a connected component $\tilde{B}'$ of $\tilde{B}$ and let X_n' be the corresponding component of X_n. Note that $X_n' \to \tilde{B}' \times B$ is the corresponding cover determined by $\mathcal{L}' := \mathcal{L}_{|\tilde{B}'\times B}$. The degree of $\tilde{f}_{|\tilde{B}'}$ and $\delta_{|\tilde{B}'}$ is $\dfrac{n^{2q}}{N}$. Now consider any connected, unramified, degree N covering $p : B'' \to \tilde{B}'$ and let $f'' = \tilde{f}_{|\tilde{B}'} \circ p : B'' \to B$ and $\delta'' = \delta_{|\tilde{B}'} \circ p$. The degree of f'' and δ'' is n^{2q}. We observe that $p^*(\mathcal{L}')^{\otimes n} \simeq \mathcal{O}(\Gamma_{f''} - \Gamma_{\delta''})$ so that $p^*\mathcal{L}'$ defines an n-cyclic branched cover of $X_n'' = B'' \times B$ ramified along $\Gamma_{f''} - \Gamma_{\delta''}$. Obviously the computations of the base and fiber genera of the fibrations $X_n'' \to B''$ and

$X''_n \to B$ proceed identically with the corresponding computations for X_n done previously. □

Corollario 3.3. *For every $n, q \geq 2$ and for every $e \geq 1$, there exist nonisotrivial smooth fibrations over B and $\widetilde{B}$ with base and fiber genera (q_i, g_i) equal to*

$$(q_1, g_1) = (e(q-1)+1, n^{2q}(nq-1)+1)$$

$$(q_2, g_2) = (n^{2q}e(q-1)+1, nq).$$

Proof. We apply the previous theorem and 2.9. □

Remark 3.4. The fibers of the families we have constructed are all curves with automorphisms because they are of type X'/G for some curve X' and some cyclic group G. Because G is a subgroup of the group of automorphisms of the curve this one is not empty.

Remark 3.5. A curve C appears as fiber of at most a finite number of families constructed above. In fact if it is a fiber of a family $X \to B$ or $X \to \widetilde{B}$ as above then there is a morphism $C \to B$. But the de Franchis-Severi theorem, ensure us that there are only a finite number of such B, and therefore of $\widetilde{B}$. Moreover the Parshin theorem tell us that for every B there are only a finite number of families over B and $\widetilde{B}$.

We use the following classical result.

Theorem 3.6. ([12], de Franchis-Severi Theorem). *The set of dominant maps from a fixed curve to a (varying) curve of genus at least two is finite.*

4. Bounds for the dimensions of Z^q and Z^q_g.

In this section we want to give some lower bounds for the dimension of the spaces Z^q_g and $\mathrm{Z}^q = \bigcup_g \mathrm{Z}^q_g$ using our families. We call $\dim \mathrm{Z}^q$ the maximum dimension between those of the irreducible components of Z^q_g, varying g. We call $g_i(q)$ and $q_i(q)$, $i = 1, 2$, the fibers genus and base genus of the families of the theorem. By our construction it is clear that

$$\dim \mathrm{Z}^{q_i(q)}_{g_i(q)} \geq \dim \mathrm{F}^q \tag{4.1}$$

where, for every q, $\mathrm{F}^q = \{$curve with a fixed point free automorphism$\} \subset \mathrm{M}_q$.

As consequence, if we consider the families with $(q_1(q), g_1(q))$, we obtain

Proposition 4.1. *For every $q \geq 2$*

$$\dim \mathrm{Z}^q \geq \dim \mathrm{F}^q.$$

Remark 4.2. Using 3.3 and (2.3), the relation (4.1) and the proposition 4.9 below are valid also if we consider $q_i(q, e)$ and $g_i(q, e)$ of 3.3. But in order to avoid making notation dull reading we consider only the genera of the theorem.

Proposition 4.3. *Let $q = ph + 1 \geq 3$ with $p = \min\{r > 1$ s.t. $r|(q-1)\}$. Let*

$$\mathrm{F}^q_{et} = \{ \textit{ curve with an automorphism } \tau \textit{ s.t. } \pi : C \to C/ < \tau > \textit{ is étale}\} \subset \mathrm{M}_q$$

then

$$\dim \mathrm{F}^q_{et} = 3h.$$

Proof. If a curve C of genus q has a fixed point free automorpism τ of order r and $\pi : C \to C/ < \tau >$ is étale then $r \geq p$. In fact by Riemann-Hurwitz theorem we have

$$q - 1 = r(g(C/ < \tau >) - 1).$$

Then $r|(q-1)$. So, by definition, $r \geq p$.

Moreover we obtain

$$g(C/ < \tau >) = \frac{q-1}{r} + 1 \geq 2$$

and for every curve C' with this genus every étale covering of order r has genus q.

So if we let

$$\mathrm{F}^q_r = \{ \text{ curve in } \mathrm{F}^q_{et} \text{ s.t. } \pi \text{ has order } r\}$$

we have, using the fact that every curve of general type has only a finite number of étale coverings,

$$\dim \mathrm{F}^q_r = 3(\frac{q-1}{r} + 1) - 3 \leq \dim \mathrm{F}^q_p = 3h.$$

So

$$\dim \mathrm{F}^q_{et} = \dim \mathrm{F}^q_p = 3h.$$

□

Remark 4.4. We observe that clearly $\mathrm{F}^q_{et} \subseteq \mathrm{F}^q$.

Remark 4.5. We can see that for every $q \geq 3$

$$\dim \mathrm{F}^q_{et} \geq 3.$$

Moreover if q is odd then we have

$$\dim \mathrm{F}^q_{et} = 3\left(\frac{q-1}{2}\right) = \frac{\dim \mathrm{M}_q}{2}$$

and if q is even and $q \geq 4$ then we have

$$3 \leq \dim \mathrm{F}^q_{et} \leq q - 1$$

because $h = \dfrac{q-1}{p} \leq \dfrac{q-1}{3}$.

Clearly in general $\mathrm{F}^q_{et} \subsetneq \mathrm{F}^q$ because it may happen that a power of a fixed point free automorphism τ has a fixed point. Now we pay attention to the case of hyperelliptic curves and we determine the dimension of the space of hyperelliptic curves with a fixed point free automorphism.

Proposition 4.6. *Let* $q = rk - 1$ *be with* $r = \min\{m > 2 \text{ s.t. } m | q + 1\}$ *and*

$${}^f\mathrm{H}_q = \{\textit{hyperelliptic curve of genus } q \textit{ with a fixed point free automorphism}\} \subseteq \mathrm{H}_q$$

where H_q *is the space of hyperelliptic curves. Then*

$$\dim {}^f\mathrm{H}_q = 2k - 1.$$

Proof. For a hyperelliptic curve C, with affine form associated $y^2 = h(x)$, its equation in an affine neighborhood of the points at infinity is $w^2 = k(x')$, where $x' = 1/x$, $w = x'^{g+1}y$ and $k(x') = x'^{2g+2}h(1/x')$. Moreover an automorphism $\varphi : C \to C$ descends to any automorphism $\varphi' : \mathbb{P}^1 \to \mathbb{P}^1$ such that the polynomial h is invariant for the action induced by φ'. Moreover, if σ is the hyperelliptic involution, there is only another automorphism of C which descends to φ': it is $\sigma \circ \varphi$.

Suppose that a curve C has an automorphism φ which induces an automorphism φ' of $\mathbb{P}^1$ of order m. We can suppose that φ' is a rotation of order m with fixed point $0 \in \mathbb{P}^1$. Then C is a hyperelliptic curve with affine form given by of the following

$$\begin{cases} y^2 = (x^m - 1)(x^m - a_1)\dots(x^m - a_{\frac{2q+2}{m}-1}), & 2q+2 \equiv 0 \mod m \\ y^2 = x(x^m - 1)(x^m - a_1)\dots(x^m - a_{\frac{2q+1}{m}-1}), & 2q+2 \equiv 1 \mod m \\ y^2 = x(x^m - 1)(x^m - a_1)\dots(x^m - a_{\frac{2q}{m}-1}), & 2q+2 \equiv 2 \mod m. \end{cases}$$

We can see that the only case in which we can have a fixed point free automorphism is when $2q+2 \equiv 0 \mod m$, otherwise $(0, 0)$ is a fixed point.

In this case the automorphisms are

$$\begin{cases} \varphi(x, y) = (e^{\frac{2\pi i}{m}} x, y) \\ \varphi(x', w) = (e^{-\frac{2\pi i}{m}} x', (-1)^{\frac{2q+2}{m}} w) \end{cases}$$

and

$$\begin{cases} \sigma \circ \varphi(x, y) = (e^{\frac{2\pi i}{m}} x, -y) \\ \sigma \circ \varphi(x', w) = (e^{-\frac{2\pi i}{m}} x', (-1)^{\frac{2q+2+m}{m}} w). \end{cases}$$

So φ has a fixed point for every m. While, if $m > 2$, $\sigma \circ \varphi$ has no fixed points if and only if $q + 1 \equiv 0 \mod m$ and its order is $m' = 2^{\frac{(-1)^{(m+1)}+1}{2}} m$. We observe that if $m = 2$ there is no fixed point free automorphism which induces an isomorphism of $\mathbb{P}^1$ of order 2. So if we define

$$^f\mathrm{H}_q^l = \{\text{hyperelliptic curve of genus } q \text{ with a fixed point free automorphism of order } l\} \subseteq \mathrm{H}_q$$

then

$$\dim(^f\mathrm{H}^q_{m'}) = \frac{2q+2}{m} - 1$$

for every $m|(q+1)$. Clearly

$$\dim(^f\mathrm{H}_q^{m'}) \leq \dim(^f\mathrm{H}_q^{r'})$$

so we obtain

$$\dim(^f\mathrm{H}_q) = \dim(^f\mathrm{H}_q^{r'}) = \frac{2q+2}{r} - 1 = 2k - 1$$

where $r' = 2^{\frac{(-1)^{(r+1)}+1}{2}} r$. □

Corollario 4.7. *Let C be a curve of genus $q = ph + 1 = rk - 1 \geq 3$ with r and p as above. Then if*

$$\mathrm{F}^q = \{\textit{curve with a fixed point free automorphism}\} \subset \mathrm{M}_q$$

$$\dim F^q \geq \max\{3h, 2k-1\} = \max\left\{3\left(\frac{q-1}{p}\right), \frac{2q+2}{r} - 1\right\}$$

while $\dim F^2 = 1$.

Proof. The proof follows from previous propositions. We only observe that every curve of genus 2 is hyperelliptic. □

Remark 4.8. If q is odd then $p = 2$ and $r \geq 3$ so, using 4.5,

$$\frac{2q+2}{r} - 1 \leq \frac{2q-1}{3} \leq 3\left(\frac{q-1}{2}\right).$$

Then

$$\dim F^q \geq 3\left(\frac{q-1}{2}\right)$$

and if q is even and $q = 3k - 1$ then

$$\dim F^q \geq \frac{2q-1}{3}.$$

In fact $p \geq 5$ so

$$3\left(\frac{q-1}{p}\right) < \frac{2q-1}{3}$$

as it is easy to see.

So by (4.1), 4.1 and the above considerations about the dimension of F_q we obtain

Proposition 4.9. *Let* $q = ph + 1 = rk - 1 \geq 3$ *be with* p, r *as above. Then, for every* n,

$$\dim Z^q_{n^{2q}(nq-1)+1} \geq \max\left\{3\left(\frac{q-1}{p}\right), \frac{2q+2}{r} - 1\right\}$$

$$\dim Z^{n^{2q}(q-1)+1}_{nq} \geq \max\left\{3\left(\frac{q-1}{p}\right), \frac{2q+2}{r} - 1\right\}$$

and

$$\dim Z^2_{n^4(2n-1)+1} \geq 1.$$

Finally we have

$$\dim Z^q \geq \max\left\{3\left(\frac{q-1}{p}\right), \frac{2q+2}{r} - 1\right\}$$

and

$$\dim Z^2 \geq 1.$$

Acknowledgment.

Part of this work has been done during the summer school "Pragmatic 2004". We thank the Professores Lucia Caporaso, Olivier Debarre, Cinzia Casagrande, Gianluca Pacienza and Michael Paun for the interesting lectures which have been hold during the school and for the stimulating atmosphere which they have created. Moreover we thank the Professor Lucia Caporaso for having suggested the problem and for the many useful conversations. The first author thanks also Marco Garuti and Qing Liu for having suggested him the proof of the fact that every curve in M_g is modular, which we have written in a slightly generalized form in 2.11. Their suggestion has considerably simplified our proof of the first part of 2.12.

The authors are also very grateful to the Professor Alfio Ragusa and to University of Catania for giving the possibility to attend this school.

REFERENCES

[1] S. Ju. Arakelov, *Families of algebraic curves with fixed degeneracies,* Izv. Akad. Nauk SSSR Ser. Mat., 35 (1971), no. 6, pp. 1269-1293.

[2] M.F. Atiyah, *The signature of fibre-bundles,* Global Analysis (1969), 73-84, Papers in honor of K. Kodaira.

[3] W. Barth, C. Peters, A. Van de Ven, *Compact complex surfaces, Ergebnisse der Mathematik und ihrer Grenzgebiete,* 3. Folge . A Series of Modern Surveys in Mathematics, vol. 4, Springer Verlag, 1984.

[4] A. Beauville, *Les nombres di fibres singulieres d'une courbe stable su P^1,* Séminaire sur les pinceaux de courbes de genre aux moins deux, Astérisque, vol. 86, 1981, Exposè n. 6 , pp. 97-108.

[5] A. Beauville, *Complex algebraic surfaces,* London Mathematical Society Student Texts, vol. 34, Cambridge University Press, 1996, Translated from the 1978 French original by R. Barlow.

[6] J. Bryan, R. Donagi, *Surface bundles over surfaces of small genus,* Geom. Topol., 6 (2002), pp. 59-67 (electronic).

[7] J. Bryan, R. Donagi, A. I. Stipsicz, *Surface bundles: some interesting examples,* Turkish J. Math., 25 (2001), pp. 61-68.

[8] L. Caporaso, *Introduction to moduli curves,* course notes for Pragmatic 2005.

[9] L. Caporaso, *On certain uniformity properties of curves over function fields,* Compositio Math., 130 (2002), pp. 1-19.

[10] P. Deligne, D. Mumford, *The irreducibility of the space of curves of given genus,* Publ. Math. I.H.E.S, 36 (1969), pp. 75-120.

[11] M. Flexor, *Noveaux contrexemples aux enounces d'annulation "a la Kodaira" en caracteristic $p > 0$,* Séminaire sur les pinceaux de courbes de genre aux moins deux, Astérisque, vol. 86, 1981, Exposé n. 4, pp. 79-89.

[12] M. de Franchis, *Un teorema sulle involuzioni irrazionali,* Rend. Circ. Mat. Palermo, 36 (1913), p. 368.

[13] K. Kodaira, *A certain type of irregular algebraic surfaces,* J. Anal. Math., 19 (1967), pp. 207-215.

[14] E. Palmieri, *Gruppi di automorfismi di curve iperellittiche e di rivestimenti ciclici della retta proiettiva, Masters thesis,* University of Tor Vergata, Rome, 2003.

[15] A. N. Parshin, *Algebraic curves over function fields,* Izv. Akad. Nauk. SSSR Ser. Mat. 32 (1968), no. 5, pp. 1145-1170.

[16] H. Popp, *Moduli theory and classification theory of algebraic varieties,* Lecture notes in Mathematics, vol. 620, Springer Verlag, 1977.

[17] L. Szpiro, *Propriétés numériques du faisceau dualisant relatif,* Séminaire sur les pinceaux de courbes de genre aux moins deux, Astérisque, 1981, pp. 44-78.

D. Tossici
Department of Mathematics,
University of Roma Tre,
Italy
e-mail: dajano@mat.uniroma3.it

F. Vetro
Department of Mathematics,
University of Palermo,
Italy
e-mail: fvetro@math.unipa.it