LE MATEMATICHE
Vol. LXIX (2014) – Fasc. II, pp. 193–204
doi: 10.4418/2014.69.2.17

SOME SIMPSON TYPE INTEGRAL INEQUALITIES FOR *s*-GEOMETRICALLY CONVEX FUNCTIONS WITH APPLICATIONS

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In this paper, we establish some new Simpson type integral inequalities by using s-geometrically convex function which is given below as

$$f\left(x^{\lambda}y^{1-\lambda}\right) \leq [f(x)]^{\lambda^{s}}[f(y)]^{(1-\lambda)^{s}}$$

where $f : I \subseteq \mathbb{R}_{+} \to \mathbb{R}_{+}$ for some fixed $s \in (0,1]$, $x,y \in I$ and $\lambda \in [0,1]$. Also we get some applications for special means for positive numbers.

1. Introduction

The following inequality is well-known in the literature as Simpson's inequality:

Let $f : [a,b] \to \mathbb{R}$ be a four times continuously differentiable mapping on $[a,b]$ and $\left\|f^{(4)}\right\|_{\infty} = \sup\limits_{x\in[a,b]} \left|f^{(4)}(x)\right| < \infty$. Then the following inequality holds:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_{\infty}(b-a)^{4}.$$

For the recent results based on the above definition see the papers [1], [3], [4], [8], [10] and [11].

Entrato in redazione: 8 novembre 2013

AMS 2010 Subject Classification: Primary 26D10; Secondary 26D15.

Keywords: s-geometrically convex functions, Simpson inequality, integral inequalities.

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In [2], Hudzik and Maligranda considered among others the class of functions which are $s-$convex in the second sense. This class is defined in the following way: A function $f:\mathbb{R}^+\to\mathbb{R}$, where $\mathbb{R}^+=[0,\infty)$, is said to be $s-$convex in the second sense if

$$f(\alpha x+\beta y)\leq\alpha^s f(x)+\beta^s f(y)$$

for all $x,y\in[0,\infty)$, $\alpha,\beta\geq 0$ with $\alpha+\beta=1$ and for some fixed $s\in(0,1]$. The class of $s-$convex functions in the second sense is usually denoted by K_s^2. It can be easily seen that for $s=1$, $s-$convexity reduces to ordinary convexity of functions defined on $[0,\infty)$. We refer the papers [2], [4]-[7] and [12]-[15].

Definition 1.1 ([13])**.** A function $f:I\subset\mathbb{R}_+=(0,\infty)\to\mathbb{R}_+$ is said to be a geometrically convex function if

$$f\left(x^\lambda y^{1-\lambda}\right)\leq[f(x)]^\lambda[f(y)]^{1-\lambda}$$

for $x,y\in I$ and $\lambda\in[0,1]$.

In [13], Zhang et al. introduced the s-geometrically convex functions as following:

Definition 1.2. A function $f:I\subset\mathbb{R}_+\to\mathbb{R}_+$ is said to be *s-geometrically convex function* if

$$f\left(x^\lambda y^{1-\lambda}\right)\leq[f(x)]^{\lambda^s}[f(y)]^{(1-\lambda)^s}$$

for some $s\in(0,1]$, where $x,y\in I$ and $\lambda\in[0,1]$.

If $s=1$, the s-geometrically convex function becomes a geometrically convex function on $\mathbb{R}_+$.

In this paper, we establish some new Simpson type integral inequalities for s-geometrically convex functions and then obtain some applications to special means of real numbers.

2. Main Results

We use the following lemma in the literature to obtain our results.

Lemma 2.1 ([9])**.** *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be an absolutely continuous mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$, then the following equality*

holds:

$$\begin{aligned}
&\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx \\
&=\frac{b-a}{2}\int_0^1\left[\left(\frac{t}{2}-\frac{1}{3}\right)f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right. \\
&\left.+\left(\frac{1}{3}-\frac{t}{2}\right)f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt.
\end{aligned}$$

Theorem 2.2. *Let $f: I\subset\mathbb{R}_+\to\mathbb{R}_+$ be a differentiable mapping on I°, such that $f'\in L[a,b]$ where $a,b\in I$ with $a<b$. If $|f'(x)|$ is s-geometrically convex and monotonically decreasing on $[a,b]$ and $s\in(0,1]$, then*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{1}$$

$$\leq\frac{(b-a)}{2}\begin{cases}|f'(a)f'(b)|^{\frac{s}{2}}\left[h_1\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right)+h_2\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right)\right], & |f'(a)|\leq 1;\\ |f'(a)|^{1-\frac{s}{2}}|f'(b)|^{\frac{s}{2}}\left[h_1\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right)+h_2\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right)\right], & |f'(b)|\leq 1\leq|f'(a)|;\\ |f'(a)f'(b)|^{1-\frac{s}{2}}\left[h_1\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right)+h_2\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right)\right], & 1\leq|f'(b)|;\end{cases}$$

where $h_1(\alpha)=\begin{cases}\frac{5}{36}, & \alpha=1\\ \frac{6\alpha^{\frac{2}{3}}+(\alpha-2)\ln\alpha-3\alpha-3}{6\ln^2\alpha}, & \alpha\neq 1\end{cases}$

$$h_2(\alpha)=\begin{cases}\frac{5}{36}, & \alpha=1\\ \frac{6\alpha^{-\frac{2}{3}}+\left(\alpha^{-1}-2\right)\ln\alpha^{-1}-3\alpha^{-1}-3}{6\ln^2\alpha^{-1}}, & \alpha\neq 1\end{cases}$$

and $\alpha(u,v)=|f'(a)|^{-u}|f'(b)|^{v}$, $u,v>0$.

Proof. From Lemma 2.1 and by using the properties of modulus, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq\frac{b-a}{2}\int_0^1\left[\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|\right. \\
&\left.+\left|\frac{1}{3}-\frac{t}{2}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]dt.
\end{aligned}$$

Since $|f'(x)|$ is monotonically decreasing and s-geometrically convex on $[a,b]$, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq \frac{b-a}{2}\int_0^1\left[\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)\right|\right. \\
&\left.+\left|\frac{1}{3}-\frac{t}{2}\right|\left|f'\left(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}\right)\right|\right]dt \qquad (2)\\
&\leq \frac{b-a}{2}\int_0^1\left[\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'(b)\right|^{\left(\frac{1+t}{2}\right)^s}\left|f'(a)\right|^{\left(\frac{1-t}{2}\right)^s}\right. \\
&\left.+\left|\frac{1}{3}-\frac{t}{2}\right|\left|f'(a)\right|^{\left(\frac{1+t}{2}\right)^s}\left|f'(b)\right|^{\left(\frac{1-t}{2}\right)^s}\right]dt.
\end{aligned}$$

(See [14]). If $0<\mu\leq 1\leq\eta$, $0<\alpha,s\leq 1$, then

$$\mu^{\alpha^s}\leq\mu^{s\alpha} \text{ and } \eta^{\alpha^s}\leq\eta^{\alpha s+1-s}. \qquad (3)$$

i) If $0<|f'(a)|\leq 1$, by using the inequality in (3), we obtain

$$\begin{aligned}
&\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'(b)\right|^{\left(\frac{1+t}{2}\right)^s}\left|f'(a)\right|^{\left(\frac{1-t}{2}\right)^s}dt \\
&\leq\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'(b)\right|^{s\left(\frac{1+t}{2}\right)}\left|f'(a)\right|^{s\left(\frac{1-t}{2}\right)}dt \\
&=\left|f'(a)f'(b)\right|^{\frac{s}{2}}h_1\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|\left|f'(a)\right|^{\left(\frac{1+t}{2}\right)^s}\left|f'(b)\right|^{\left(\frac{1-t}{2}\right)^s}dt \\
&\leq\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|\left|f'(a)\right|^{\left(\frac{1+t}{2}\right)^s}\left|f'(b)\right|^{s\left(\frac{1-t}{2}\right)}dt \qquad (4)\\
&=\left|f'(a)f'(b)\right|^{\frac{s}{2}}h_2\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right).
\end{aligned}$$

ii) If $|f'(b)|\leq 1\leq|f'(a)|$, by using the inequality in (3), we obtain

$$\begin{aligned}
&\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'(b)\right|^{\left(\frac{1+t}{2}\right)^s}\left|f'(a)\right|^{\left(\frac{1-t}{2}\right)^s}dt \\
&\leq\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'(b)\right|^{s\left(\frac{1+t}{2}\right)}\left|f'(a)\right|^{s\left(\frac{1-t}{2}\right)+1-s}dt \\
&=\left|f'(a)\right|^{1-\frac{s}{2}}\left|f'(b)\right|^{\frac{s}{2}}h_1\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right),
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left|\frac{1}{3}-\frac{t}{2}\right| \left|f'(a)\right|^{\left(\frac{1+t}{2}\right)^s} \left|f'(b)\right|^{\left(\frac{1-t}{2}\right)^s} dt \\
&\leq \int_0^1 \left|\frac{1}{3}-\frac{t}{2}\right| \left|f'(a)\right|^{s\left(\frac{1+t}{2}\right)+1-s} \left|f'(b)\right|^{s\left(\frac{1-t}{2}\right)} dt \qquad (5)\\
&= \left|f'(a)\right|^{1-\frac{s}{2}} \left|f'(b)\right|^{\frac{s}{2}} h_2\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right).
\end{aligned}$$

iii) If $1 \leq |f'(b)|$, by using the inequality in (3), we obtain

$$\begin{aligned}
&\int_0^1 \left|\frac{t}{2}-\frac{1}{3}\right| \left|f'(b)\right|^{\left(\frac{1+t}{2}\right)^s} \left|f'(a)\right|^{\left(\frac{1-t}{2}\right)^s} dt \\
&\leq \int_0^1 \left|\frac{t}{2}-\frac{1}{3}\right| \left|f'(b)\right|^{s\left(\frac{1+t}{2}\right)+1-s} \left|f'(a)\right|^{s\left(\frac{1-t}{2}\right)+1-s} dt \\
&= \left|f'(a)f'(b)\right|^{1-\frac{s}{2}} h_1\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left|\frac{1}{3}-\frac{t}{2}\right| \left|f'(a)\right|^{\left(\frac{1+t}{2}\right)^s} \left|f'(b)\right|^{\left(\frac{1-t}{2}\right)^s} dt \\
&\leq \int_0^1 \left|\frac{1}{3}-\frac{t}{2}\right| \left|f'(a)\right|^{s\left(\frac{1+t}{2}\right)+1-s} \left|f'(b)\right|^{s\left(\frac{1-t}{2}\right)+1-s} dt \qquad (6)\\
&= \left|f'(a)f'(b)\right|^{1-\frac{s}{2}} h_2\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right)
\end{aligned}$$

If we use the inequalities from (2) to (6) we get the result inequality in (1). So, the proof is completed. □

Theorem 2.3. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable mapping on I°, such that $f' \in L[a,b]$ where $a,b \in I$ with $a < b$. If $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$ and $s \in (0,1]$, then*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq \frac{(b-a)}{2}\left(\frac{2\left(1+2^{p+1}\right)}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} \qquad (7)
\end{aligned}$$

$$\begin{cases} |f'(a)f'(b)|^{\frac{s}{2}}\left[\left[h_3\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\left[h_4\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}\right], \\ \hfill |f'(a)|\leq 1; \\ |f'(a)|^{1-\frac{s}{2}}|f'(b)|^{\frac{s}{2}}\left[\left[h_3\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\left[h_4\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}\right], \\ \hfill |f'(b)|\leq 1\leq |f'(a)| \\ |f'(a)f'(b)|^{1-\frac{s}{2}}\left[\left[h_3\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\left[h_4\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}\right], \\ \hfill 1\leq |f'(b)| \end{cases}$$

where $h_3(\alpha)=\begin{cases}1, & \alpha=1\\ \frac{\alpha-1}{\ln\alpha}, & \alpha\neq 1\end{cases}$, $h_4(\alpha)=\begin{cases}1, & \alpha=1\\ \frac{\alpha^{-1}-1}{\ln\alpha^{-1}}, & \alpha\neq 1\end{cases}$
and $\alpha(u,v)$ *is defined as in Theorem 2.2,* $\frac{1}{p}+\frac{1}{q}=1$ *and* $p>1$.

Proof. From Lemma 2.1, by using the properties of modulus and Hölder integral inequality, we have

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\leq \frac{b-a}{2}\left\{\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

Since $|f'(x)|^q$ is monotonically decreasing and s-geometrically convex on $[a,b]$, we have

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\leq \frac{(b-a)}{2}\left(\frac{2\left(1+2^{p+1}\right)}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} \\ &\times\left[\left(\int_0^1\left|f'\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|f'\left(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}}\right] \qquad (8) \\ &\leq \frac{(b-a)}{2}\left(\frac{2\left(1+2^{p+1}\right)}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} \\ &\times\left[\left(\int_0^1 |f'(b)|^{q\left(\frac{1+t}{2}\right)^s}|f'(a)|^{q\left(\frac{1-t}{2}\right)^s}dt\right)^{\frac{1}{q}}+\left(\int_0^1 |f'(a)|^{q\left(\frac{1+t}{2}\right)^s}|f'(b)|^{q\left(\frac{1-t}{2}\right)^s}dt\right)^{\frac{1}{q}}\right] \end{aligned}$$

i) If $0<|f'(a)|\le 1$, by using the inequality in (3), we obtain

$$\begin{aligned}
&\int_0^1 \left|f'(b)\right|^{q\left(\frac{1+t}{2}\right)^s}\left|f'(a)\right|^{q\left(\frac{1-t}{2}\right)^s}dt\\
&\le \int_0^1 \left|f'(b)\right|^{qs\left(\frac{1+t}{2}\right)}\left|f'(a)\right|^{qs\left(\frac{1-t}{2}\right)}dt\\
&=\left|f'(a)f'(b)\right|^{\frac{qs}{2}}\int_0^1\left[\left(\frac{|f'(b)|}{|f'(a)|}\right)^{\frac{qs}{2}}\right]^t dt\\
&=\left|f'(a)f'(b)\right|^{\frac{qs}{2}} h_3\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left|f'(a)\right|^{q\left(\frac{1+t}{2}\right)^s}\left|f'(b)\right|^{q\left(\frac{1-t}{2}\right)^s}dt\\
&\le \int_0^1 \left|f'(a)\right|^{qs\left(\frac{1+t}{2}\right)}\left|f'(b)\right|^{qs\left(\frac{1-t}{2}\right)}dt\\
&=\left|f'(a)f'(b)\right|^{\frac{qs}{2}}\int_0^1\left[\left(\frac{|f'(a)|}{|f'(b)|}\right)^{\frac{qs}{2}}\right]^t dt \qquad (9)\\
&=\left|f'(a)f'(b)\right|^{\frac{qs}{2}} h_4\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right).
\end{aligned}$$

ii) If $|f'(b)|\le 1\le |f'(a)|$, by using the inequality in (3), we obtain

$$\begin{aligned}
&\int_0^1 \left|f'(b)\right|^{q\left(\frac{1+t}{2}\right)^s}\left|f'(a)\right|^{q\left(\frac{1-t}{2}\right)^s}dt\\
&\le \int_0^1 \left|f'(b)\right|^{qs\left(\frac{1+t}{2}\right)}\left|f'(a)\right|^{q\left[s\left(\frac{1-t}{2}\right)+1-s\right]}dt\\
&=\left|f'(b)\right|^{\frac{qs}{2}}\left|f'(a)\right|^{q-\frac{qs}{2}}\int_0^1\left(\frac{|f'(b)|^{\frac{qs}{2}}}{|f'(a)|^{\frac{qs}{2}}}\right)^t dt\\
&=\left|f'(b)\right|^{\frac{qs}{2}}\left|f'(a)\right|^{q-\frac{qs}{2}} h_3\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left|f'(a)\right|^{q\left(\frac{1+t}{2}\right)^s}\left|f'(b)\right|^{q\left(\frac{1-t}{2}\right)^s}dt\\
&\le \int_0^1 \left|f'(a)\right|^{q\left[s\left(\frac{1+t}{2}\right)+1-s\right]}\left|f'(b)\right|^{qs\left(\frac{1-t}{2}\right)}dt\\
&=\left|f'(b)\right|^{\frac{qs}{2}}\left|f'(a)\right|^{q-\frac{qs}{2}}\int_0^1\left(\frac{|f'(a)|^{\frac{sq}{2}}}{|f'(b)|^{\frac{sq}{2}}}\right)^t dt \qquad (10)\\
&=\left|f'(b)\right|^{\frac{qs}{2}}\left|f'(a)\right|^{q-\frac{qs}{2}} h_4\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right).
\end{aligned}$$

iii) If $1 \le |f'(b)|$, by using the inequality in (3), we obtain

$$\begin{aligned}
&\int_0^1 \left|f'(b)\right|^{q\left(\frac{1+t}{2}\right)^s} \left|f'(a)\right|^{q\left(\frac{1-t}{2}\right)^s} dt \\
&\le \int_0^1 \left|f'(b)\right|^{q\left[s\left(\frac{1+t}{2}\right)+1-s\right]} \left|f'(a)\right|^{q\left[s\left(\frac{1-t}{2}\right)+1-s\right]} dt \\
&= \left|f'(b)\right|^{q-\frac{qs}{2}} \left|f'(a)\right|^{q-\frac{qs}{2}} \int_0^1 \left(\left(\frac{|f'(b)|}{|f'(a)|}\right)^{\frac{sq}{2}}\right)^t dt \\
&= \left|f'(a) f'(b)\right|^{q-\frac{qs}{2}} h_3\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left|f'(a)\right|^{q\left(\frac{1+t}{2}\right)^s} \left|f'(b)\right|^{q\left(\frac{1-t}{2}\right)^s} dt \\
&\le \int_0^1 \left|f'(a)\right|^{q\left[s\left(\frac{1+t}{2}\right)+1-s\right]} \left|f'(b)\right|^{q\left[s\left(\frac{1-t}{2}\right)+1-s\right]} dt \\
&= \left|f'(b)\right|^{q-\frac{qs}{2}} \left|f'(a)\right|^{q-\frac{qs}{2}} \int_0^1 \left(\left(\frac{|f'(a)|}{|f'(b)|}\right)^{\frac{sq}{2}}\right)^t dt \qquad (11)\\
&= \left|f'(a) f'(b)\right|^{q-\frac{qs}{2}} h_4\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right).
\end{aligned}$$

If we use the inequalities from (8) to (11) we get the result inequality in (7). So, the proof is completed. □

Theorem 2.4. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable mapping on I°, such that $f' \in L[a,b]$ where $a,b \in I$ with $a<b$. If $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$ and $s \in (0,1]$, then*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\le \frac{(b-a)}{2}\left(\frac{5}{36}\right)^{1-\frac{1}{q}} \\
&\times \begin{cases}
|f'(a)f'(b)|^{\frac{s}{2}}\left[\left[h_1\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\left[h_2\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}\right], \\
|f'(a)| \le 1 \\
|f'(a)|^{1-\frac{s}{2}}|f'(b)|^{\frac{s}{2}}\left[\left[h_1\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\left[h_2\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}\right], \\
|f'(b)| \le 1 \le |f'(a)| \\
|f'(a)f'(b)|^{1-\frac{s}{2}}\left[\left[h_1\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\left[h_2\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}\right], \\
1 \le |f'(b)|
\end{cases}
\end{aligned}$$

where $h_1(\alpha)$, $h_2(\alpha)$ and $\alpha(u,v)$ are defined as in Theorem 2.2.

Proof. From Lemma 2.1, by using the properties of modulus and Hölder integral inequality, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq \frac{b-a}{2}\int_0^1\left[\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|\right. \\
&\left.+\left|\frac{1}{3}-\frac{t}{2}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]dt \\
&\leq \frac{b-a}{2}\left\{\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Since $|f'(x)|^q$ is monotonically decreasing and s-geometrically convex on $[a,b]$, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq \frac{(b-a)}{2}\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\left[\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|\left|f'\left(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&\leq \frac{(b-a)}{2}\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\left[\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'(b)\right|^{q\left(\frac{1+t}{2}\right)^s}\left|f'(a)\right|^{q\left(\frac{1-t}{2}\right)^s}dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|\left|f'(a)\right|^{q\left(\frac{1+t}{2}\right)^s}\left|f'(b)\right|^{q\left(\frac{1-t}{2}\right)^s}dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

The proof can be continued as in Theorem 2.2. We omit the details. □

3. Applications to Special Means

We now consider the means for non-negative real numbers $\alpha<\beta$ as follows:

1. *Arithmetic mean* :

$$A(\alpha,\beta)=\frac{\alpha+\beta}{2},\ \ \alpha,\beta\in\mathbb{R}^+.$$

2. *Geometric mean*:

$$G(\alpha,\beta)=\sqrt{\alpha\beta},\quad \alpha,\beta\in\mathbb{R}^{+}.$$

3. *Generalized log − mean*:

$$L_{s}(\alpha,\beta)=\left[\frac{\beta^{s+1}-\alpha^{s+1}}{(s+1)(\beta-\alpha)}\right]^{\frac{1}{s}},\ s\in\mathbb{R}\backslash\{-1,0\},\ \alpha,\beta\in\mathbb{R}\text{ with }\alpha\neq\beta.$$

Now using the our results, we give some applications to special means of real numbers.

Proposition 3.1. *Let $a,b\in I^{\circ}$, $a<b$ and $0<s<1$. Then, we have*

$$\left|\frac{1}{6}\left[\frac{2A\left(a^{s},b^{s}\right)+4A^{s}\left(a,b\right)}{s}\right]-\frac{L_{s}^{s}\left(a,b\right)}{s}\right|$$
$$\leq\frac{b-a}{2}\times\begin{cases}G\left(a^{s(s-1)},b^{s(s-1)}\right)\left[h_{1}\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right)+h_{2}\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right)\right],\\ a^{s-1}\leq1;\\ G\left(a^{(2-s)(s-1)},b^{s(s-1)}\right)\left[h_{1}\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right)+h_{2}\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right)\right],\\ b^{s-1}\leq1\leq a^{s-1};\\ G\left(a^{(s-1)(2-s)},b^{(s-1)(2-s)}\right)\left[h_{1}\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right)+h_{2}\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right)\right],\\ 1\leq b^{s-1}\end{cases}$$

where $h_{1}\left(\alpha\right),h_{2}\left(\alpha\right)$ are defined in Theorem 2.2 and

$$\alpha\left(u,v\right)=\left(a^{s-1}\right)^{-u}\left(b^{s-1}\right)^{v},\ u,v>0.$$

Proof. The assertion follows from Theorem 2.2 applied to the function which is defined as $f:(0,1]\rightarrow\mathbb{R}_{+}$, $0<s<1$, $f\left(x\right)=\frac{x^{s}}{s}$. □

Proposition 3.2. *Let $a,b\in I^{\circ}$, $a<b$ and $0<s<1$. Then, we have*

$$\left|\frac{1}{6}\left[\frac{2A\left(a^{s},b^{s}\right)+4A^{s}\left(a,b\right)}{s}\right]-\frac{L_{s}^{s}\left(a,b\right)}{s}\right|\leq\frac{(b-a)}{2}\left(\frac{2\left(1+2^{p+1}\right)}{6^{p+1}\left(p+1\right)}\right)^{\frac{1}{p}}$$
$$\times\begin{cases}G\left(a^{s(s-1)},b^{s(s-1)}\right)\left[\left[h_{3}\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\left[h_{4}\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}\right],\\ a^{s-1}\leq1;\\ G\left(a^{(2-s)(s-1)},b^{s(s-1)}\right)\left[\left[h_{3}\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\left[h_{4}\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}\right],\\ b^{s-1}\leq1\leq a^{s-1};\\ G\left(a^{(s-1)(2-s)},b^{(s-1)(2-s)}\right)\left[\left[h_{3}\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\left[h_{4}\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}\right],\\ 1\leq b^{s-1}\end{cases}$$

where $h_3(\alpha), h_4(\alpha)$ *are defined as in Theorem 2.3 and* $\alpha(u,v)$ *is defined as in Proposition 3.1,* $\frac{1}{p}+\frac{1}{q}=1$ *and* $p>1$.

Proof. The assertion follows from Theorem 2.3 applied to the function which is defined $f:(0,1]\to\mathbb{R}_+$, $0<s<1$, $f(x)=\frac{x^s}{s}$. □

Proposition 3.3. *Let* $a,b\in I^\circ$, $a<b$ *and* $0<s<1$. *Then, we have*

$$\left|\frac{1}{6}\left[\frac{2A(a^s,b^s)+4A^s(a,b)}{s}\right]-\frac{L_s^s(a,b)}{s}\right|$$
$$\leq\frac{(b-a)}{2}\left(\frac{5}{36}\right)^{1-\frac{1}{q}}$$
$$\times\begin{cases} G\left(a^{s(s-1)},b^{s(s-1)}\right)\left[\left[h_1\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\left[h_2\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}\right], \\ a^{s-1}\leq 1; \\ G\left(a^{(s-1)(2-s)},b^{s(s-1)}\right)\left[\left[h_1\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\left[h_2\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}\right], \\ b^{s-1}\leq 1\leq a^{s-1}; \\ G\left(a^{(s-1)(2-s)},b^{(s-1)(2-s)}\right)\left[\left[h_1\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\left[h_2\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}\right], \\ 1\leq b^{s-1} \end{cases}$$

where $h_1(\alpha)$, $h_2(\alpha)$ *and* $\alpha(u,v)$ *are defined as in Proposition 3.1.*

Proof. The assertion follows from Theorem 2.4 applied to the function which is defined $f:(0,1]\to\mathbb{R}_+$, $0<s<1$, $f(x)=\frac{x^s}{s}$. □

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