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SOME PROPERTIES OF GENERALIZED TWO-FOLD SYMMETRIC NON-BAZILEVIC ANALYTIC FUNCTIONS

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In this paper, we introduce a new class of generalized two-fold symmetric non-Bazilevic functions analytic in the unit disc E. We prove such results as subordination and superordination properties, convolution properties, distortion theorems, and inequality properties of this new class.

1. Introduction

Let $\mathcal{H}(E)$ be the class of functions analytic in $E = \{z : z \in E \text{ and } |z| < 1\}$ and $\mathcal{H}[a, m+1]$ be the subclass of $\mathcal{H}(E)$ consisting of functions of the form

$$f(z) = a + a_{m+1}z^{m+1} + a_{m+2}z^{m+2} + \dots,\ z \in E.$$

Also, let $\mathcal{A}(m)$ be the subclass of $\mathcal{H}(E)$ consisiting of functions of the form

$$f(z) = z + \sum_{k=m+1}^{\infty} a_k z^k, \quad m \in \mathbb{N} = \{1, 2, ..\}, \tag{1}$$

for simplicity we write $\mathcal{A}(1) = \mathcal{A}$.

If f and g are analytic in E, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w with $w(0) = 0$ and $|w(z)| < 1$

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in E such that $f(z) = g(w(z))$. Furthermore, if the function $g(z)$ is univalent in E, then the following equivalence holds (see [4,5])

$$f(z) \prec g(z) \ \ (z \in E) \Longleftrightarrow f(0) = g(0) \text{ and } f(E) \subset g(E).$$

Suppose that h and k are two analytic functions in E, let

$$\varphi(r,s,t;z) : \mathbb{C}^3 \times E \longrightarrow \mathbb{C}$$

If h and $\varphi(h(z), zh'(z), z^2h''(z); z)$ are univalent functions in E and if h satisfies the second order superordination

$$k(z) \prec \varphi(h(z), zh'(z), z^2h''(z); z), \tag{2}$$

then k is said to be a solution of the differential superordination (2). A function $q \in \mathcal{H}(E)$ is called a subordinant to (2), if $q(z) \prec h(z)$ for all the functions h satisfying (2).

A univalent subordinant $\widetilde{q}$ that satisfies $q(z) \prec \widetilde{q}(z)$ for all of the subordinants q of (2), is said to be the best subordinant.

For functions f, $g \in \mathcal{A}(m)$, where f is given by (1) and g is defined by

$$g(z) = z + \sum_{k=m+1}^{\infty} b_k z^k, \quad m \in \mathbb{N} = \{1, 2, ..\},$$

the Hadamard product (or convolution) $f * g$ of the function f and g is defined by

$$(f * g)(z) = z + \sum_{k=m+1}^{\infty} a_k b_k z^k = (g * f)(z).$$

In [8], Sakaguchi defined the class of starlike functions with respect to symmetrical points as follows:

Let $f \in \mathcal{A}$. Then, f is said to be starlike with respect to symmetrical points in E if, and only if,

$$\Re \frac{zf'(z)}{f(z) - f(-z)} > 0, \, z \in E.$$

Obviously, it forms a subclass of close-to-convex functions and hence univalent. Moreover, this class includes the class of convex functions and odd starlike functions with respect to the origin, see [8].

Definition 1.1. A function $f \in \mathcal{A}(m)$ is said to be in the class $\mathcal{N}^{\lambda,\mu}(m, A, B)$, if it satisfies the following subordination condition:

$$(1+\lambda)\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} - \lambda \frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} \prec \frac{1+Az}{1+Bz},$$

where, and throughout this paper unless otherwise mentioned, the parameters λ, μ, A and B are constrained as follows:

$$\lambda \in \mathbb{C}: \ 0 < \mu < 1: \ -1 \leq B \leq 1, \ A \neq B, \ A \in \mathbb{R} \text{ and } m \in \mathbb{N},$$

and all powers are understood as principal values.

In this paper, we prove such results as subordination and superordination properties, convolution properties, distortion theorems, and inequality properties of the class $\mathcal{N}^{\lambda,\mu}(m,A,B)$.

For interested readers, see the work done by the authors $[1,2,10,11,12,13]$.

2. Preliminary Results

Definition 2.1 ([5])**.** Let $\mathcal{Q}$ be the set of all functions f that are analytic and injective on $\overline{E}\backslash U(f)$, where

$$U(f) = \left\{ \zeta \in \partial E : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial E \backslash U(f)$.

To establish our main results we need the following Lemmas.

Lemma 2.2 (Miller and Mocanu [4])**.** *Let the function $h(z)$ be analytic and convex (univalent) in E with $h(0) = 1$. Suppose also that the function $\Phi(z)$ given by*

$$\Phi(z) = 1 + c_{m+1} z^{m+1} + c_{m+2} z^{m+2} + \dots$$

is analytic in E,

$$\Phi(z) + \frac{z\,\Phi'(z)}{\gamma} \prec h(z) \ \ (z \in E; \ \Re\gamma \geq 0; \ \gamma \neq 0), \tag{3}$$

then

$$\Phi(z) \prec \Psi(z) = \frac{\gamma}{(m+1) z^{\frac{\gamma}{m+1}}} \int_0^z t^{\frac{\gamma}{m+1}-1} h(t) dt \prec h(z) \ \ (z \in E),$$

and $\Psi(z)$ *is the best dominant of* (3).

Lemma 2.3 (Shanmugam et al. [9])**.** *Let $\sigma \in \mathbb{C}$, $\eta \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and let q be a convex univalent function in E with*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -\Re\frac{\sigma}{\eta}\right\}, \ z \in E.$$

If p is analytic in E and

$$\sigma p(z)+\eta z p'(z) \prec \sigma q(z)+\eta z q'(z), \tag{4}$$

then $p(z) \prec q(z)$ and q is the best dominant of (4).

Lemma 2.4 ([5]). *Let $q(z)$ be convex univalent in E and $k \in \mathbb{C}$. Further assume that $\Re k > 0$. If*

$$g(z) \in H[q(0),1] \cap \mathcal{Q},$$

and

$$g(z)+kzq'(z) \prec g(z)+kzg'(z),$$

then $q(z) \prec g(z)$ and $q(z)$ is the best subordinant.

Lemma 2.5 ([3]). *Let F be analytic and convex in E. If $f,g \in A(1)$ and $f,g \prec F$, then*

$$\lambda f+(1-\lambda)g \prec F \quad (0 \le \lambda \le 1).$$

Lemma 2.6 ([7]). *Let*

$$f(z)=1+\sum_{k=1}^{\infty} a_k z^k$$

be analytic in E and

$$g(z)=1+\sum_{k=1}^{\infty} b_k z^k$$

be analytic and convex in E. If $f(z) \prec g(z)$, then

$$|a_k| < |b_1|, \ \ k \in \mathbb{N}.$$

Unless otherwise mentioned, we assume that $0<\mu<1$, $\lambda \in \mathbb{C}$, $-1 \le B \le 1$, $A \neq B$, $A \in \mathbb{R}$ and $m \in \mathbb{N}$. We begin by presenting our first subordination property given by Theorem 3.1 below.

3. Main Results

Theorem 3.1. *Let $f(z) \in \mathcal{N}^{\lambda,\mu}(m,A,B)$ with $\Re\lambda > 0$. Then,*

$$\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} \prec \psi(z)=\frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\mu}{\lambda(m+1)}-1}du \prec \frac{1+Az}{1+Bz}, \tag{5}$$

and $\psi(z)$ is the best dominant.

Proof. Set

$$\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} = h(z), \ \ z \in E. \tag{6}$$

Then $h(z)$ is analytic in E with $h(0) = 1$.

Logarithmic differentiation of (5), simple computations and Definition 1.1 yield

$$(1+\lambda)\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} \lambda \frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} \doteq h(z) + \frac{\lambda}{\mu} zh'(z) \prec \frac{1+Az}{1+Bz}. \tag{7}$$

Applying Lemma 2.2 to (7) with $\gamma = \frac{\mu}{\lambda}$, we have

$$\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} \prec \psi(z) = \frac{\mu}{\lambda(m+1)} z^{-\frac{\mu}{\lambda(m+1)}} \int_0^z \frac{1+At}{1+Bt} t^{\frac{\mu}{\lambda(m+1)}-1} dt$$
$$\doteq \frac{\mu}{\lambda(m+1)} \int_0^z \frac{1+Azu}{1+Bzu} u^{\frac{\mu}{\lambda(m+1)}-1} du \prec \frac{1+Az}{1+Bz}, \tag{8}$$

and $\psi(z)$ is the best dominant. This completes the proof. □

Theorem 3.2. *Let $q(z)$ be univalent in E, $\lambda \in \mathbb{C}^*$. Suppose also that $q(z)$ satisfies the following inequality:*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -\Re\left(\frac{\mu}{\lambda}\right)\right\}. \tag{9}$$

If $f \in \mathcal{A}(m)$ satisfies the following subordination:

$$(1+\lambda)\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} - \lambda \frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} \prec q(z) + \frac{\lambda}{\mu} zq'(z), \tag{10}$$

then

$$\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} \prec q(z),$$

and $q(z)$ is the best dominant.

Proof. Let the function $h(z)$ be defined by (6). We know that (7) holds true. Combining (7) and (10), we have

$$h(z)+\frac{\lambda}{\mu}zh'(z)\prec q(z)+\frac{\lambda}{\mu}zq'(z). \tag{11}$$

By using Lemma 2.4 and (11), we easily get the assertion of Theorem 3.2. □

Corollary 3.3. *Let $\lambda\in\mathbb{C}^*$ and $-1\le B<A\le 1$. Suppose also that*

$$\Re\left(\frac{1-Bz}{1+Bz}\right)>\max\left\{0,\ -\Re\frac{\mu}{\lambda})\right\}.$$

If $f\in\mathcal{A}(m)$ satisfies the following subordination:

$$(1+\lambda)\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}-\lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} \prec \frac{1+Az}{1+Bz}+\lambda\frac{(A-B)z}{(1+Bz)^2},$$

then

$$\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}\prec\frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

If f is subordinate to F, then F is superordinate to f. We now derive the following superordination result for the class $\mathcal{N}^{\lambda,\mu}(m,A,B)$.

Theorem 3.4. *Let q be convex univalent in E, $\lambda\in\mathbb{C}$ with $\Re\lambda>0$. Also, let*

$$\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}\in\mathcal{H}[q(0),1]\cap\mathcal{Q}$$

and

$$(1+\lambda)\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}-\lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}$$

be univalent in E. If

$$q(z)+\frac{\lambda}{\mu}zq'(z) \prec (1+\lambda)\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}-\lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu},$$

then

$$q(z)\prec\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu},$$

and q is the best subordinant.

Proof. Let the function $h(z)$ be defined by (6). Then,

$$q(z)+\frac{\lambda}{\mu}zq'(z)$$
$$\prec (1+\lambda)\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}-\lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}$$
$$\doteq h(z)+\frac{\lambda}{\mu}zh'(z).$$

An application of Lemma 2.4 yields the assertion of Theorem 3.4. □

Taking $q(z)=\frac{1+Az}{1+Bz}$ in Theorem 3.4, we obtain the following corollary.

Corollary 3.5. *Let $q(z)$ be convex univalent in E and $-1\le B<A\le 1$, $\lambda\in\mathbb{C}$ with $\Re\lambda>0$. Also, let*

$$0\neq\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}\in\mathcal{H}[q(0),1]\cap\mathcal{Q},$$

and

$$(1+\lambda)\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}-\lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}$$

be univalent in E. If

$$\frac{1+Az}{1+Bz}+\lambda\frac{(A-B)z}{(1+Bz)^2}$$
$$\prec (1+\lambda)\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}-\lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu},$$

then

$$\frac{1+Az}{1+Bz}\prec\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu},$$

and $\frac{1+Az}{1+Bz}$ is the best subordinant.

Combining the above results of subordination and superordination, we easily get the following ”sandwich-type result”.

Corollary 3.6. *Let q_1 be convex univalent and let q_2 be univalent in E, $\lambda\in\mathbb{C}$ with $\Re\lambda>0$. Let q_2 satisfy* (3.5). *If*

$$0\neq\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}\in\mathcal{H}[q(0),1]\cap\mathcal{Q},$$

and

$$(1+\lambda)\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}-\lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}$$

is univalent in E, also

$$\begin{aligned} q_1(z)+\frac{\lambda z q_1'(z)}{\mu} & \\ \prec (1+\lambda)\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} & -\lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} \\ & \prec q_2(z)+\frac{\lambda z q_2'(z)}{\mu}, \end{aligned}$$

then

$$q_1(z)\prec\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}\prec q_2(z),$$

and q_1 and q_2 are, respectively, the best subordinant and dominant.

Theorem 3.7. *If $\lambda\in\mathbb{C}$, $\mu>0$ and $f(z)\in\mathcal{N}^{0,\mu}(m,1-2\rho,-1)$ $(0\le\rho<1)$, then $f(z)\in\mathcal{N}^{\lambda,\mu}(m,1-2\rho,-1)$ for $|z|<R$, where*

$$R=\left(\left(\sqrt{\left(\frac{|\lambda|(m+1)}{\mu}\right)^2+1}\right)-\frac{|\lambda|(m+1)}{\mu}\right)^{\frac{1}{m+1}}. \tag{12}$$

The bound R is best possible.

Proof. Set

$$\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}=(1-\rho)h(z)+\rho,\ \ z\in E, 0\le\rho<1. \tag{13}$$

Then, clearly the function $h(z)$ is analytic in E with $h(0)=1$. Proceeding as an Theorem 3.1, we have

$$\begin{aligned}\frac{1}{1-\rho}\left\{(1+\lambda)\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}-\lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}-\rho\right\} & \\ \doteq h(z)+\frac{\lambda z h'(z)}{\mu}. & \end{aligned} \tag{14}$$

Using the following well-known estimate, see [6]

$$|zh'(z)|\le\frac{2(m+1)r^{m+1}\Re(h(z))}{(1-r^{2(m+1)})}(|z|=r<1)$$

in (14), we obtain that

$$\Re\frac{1}{1-\rho}\left\{(1+\lambda)\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}-\lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}-\rho\right\}$$

$$\geq\Re(h(z))\left\{1-\frac{2|\lambda|(m+1)r^{m+1}}{\mu(1-r^{2(m+1)})}\right\}. \tag{15}$$

Right hand side of (15) is positive, provided that $r<R$, where R is given by (12).

In order to show that the bound R is best possible, we consider the function $f(z)\in A(m)$ defined by

$$\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}=(1-\rho)\frac{1+z^{m+1}}{1-z^{m+1}}+\rho, z\in E, 0\leq\rho<1.$$

We note that

$$\frac{1}{1-\rho}\left\{(1+\lambda)\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}-\lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}-\rho\right\}$$

$$\doteq\frac{1+z^{m+1}}{1-z^{m+1}}+\frac{2|\lambda|(m+1)z^{m+1}}{\mu(1-z^{m+1})^2}=0,$$

for $|z|=R$, we conclude that the bound is the best possible and this proves the theorem. □

Theorem 3.8. *Let* $0\leq\lambda_1\leq\lambda_2$ *and* $-1\leq B_1\leq B_2<A_2\leq A_1\leq 1$. *Then,*

$$\mathcal{N}^{\lambda_2,\mu}(m,A_2,B_2)\subset\mathcal{N}^{\lambda_1,\mu}(m,A_1,B_1). \tag{16}$$

Proof. Suppose that $f\in\mathcal{N}^{\lambda_2,\mu}(m,A_2,B_2)$. We know that

$$\left\{(1+\lambda_2)\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}-\lambda_2\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}\right\}$$
$$\prec\frac{1+A_2z}{1+B_2z}. \tag{17}$$

Since $-1\leq B_1\leq B_2<A_2\leq A_1\leq 1$, we easily find that

$$\left\{(1+\lambda_2)\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}-\lambda_2\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}\right\}$$
$$\prec\frac{1+A_2z}{1+B_2z}\prec\frac{1+A_1z}{1+B_1z}, \tag{18}$$

that is $f \in \mathcal{N}^{\lambda_2,\mu}(m,A_1,B_1)$. Thus, the assertion (16) holds true for $0 \leq \lambda_1 = \lambda_2$. If $\lambda_2 > \lambda_1 \geq 0$, by Theorem 3.1 and (18), we know that $f \in \mathcal{N}^{0,\mu}(m,A_2,B_2)$, that is,

$$\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} \prec \frac{1+A_1z}{1+B_1z}. \tag{19}$$

At the same time, we have

$$\left\{(1+\lambda_1)\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} - \lambda_1 \frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}\right\}$$

$$\doteq \left(1-\frac{\lambda_1}{\lambda_2}\right)\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} + \frac{\lambda_1}{\lambda_2}\left((1+\lambda_2)\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}\right)$$
$$-\lambda_2\left(\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}\right). \tag{20}$$

Moreover,

$$0 \leq \frac{\lambda_1}{\lambda_2} < 1,$$

and the function $\frac{1+A_1z}{1+B_1z}$, $-1 \leq B_1 < A_1 \leq 1$, $z \in E$, is analytic and convex in E. Combining (18)-(20) and Lemma 2.5, we find that

$$\left\{(1+\lambda_1)\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} - \lambda_1 \frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu}\right\} \prec \frac{1+A_1z}{1+B_1z},$$

that is $f \in \mathcal{N}^{\lambda_1,\mu}(m,A_1,B_1)$, which implies that the assertion (16) of Theorem 3.8 holds and this completes the proof. □

Theorem 3.9. *Let $f \in \mathcal{N}^{\lambda,\mu}(m,A,B)$ with $\Re(\lambda) > 0$ and $-1 \leq B_1 < A_1 \leq 1$. Then,*

$$\frac{\mu}{\lambda(m+1)}\int 0^1 \frac{1-Au}{1-Bu}u^{\frac{\mu}{\lambda(m+1)}-1}du$$
$$< \Re\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} < \frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\mu}{\lambda(m+1)}-1}du. \tag{21}$$

The extremal function of (21) is defined by

$$F_{\lambda,\mu,m,A,B}(z) = 2z\left(\frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\mu}{\lambda(m+1)}-1}du\right)^{-\frac{1}{\mu}}. \tag{22}$$

Proof. Let $f \in \mathcal{N}^{\lambda,\mu}(m,A,B)$ with $\Re\lambda > 0$. From Theorem 3.1, we know that (5) holds, which implies that

$$\begin{aligned}
\Re\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} &< \sup_{z\in E}\Re\left\{\frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\mu}{\lambda(m+1)}-1}du\right\} \\
&\leq \left\{\frac{\mu}{\lambda(m+1)}\int_0^1 \sup_{z\in E}\Re\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu}{\lambda(m+1)}-1}du\right\} \\
&< \frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\mu}{\lambda(m+1)}-1}du, \quad (23)
\end{aligned}$$

and

$$\begin{aligned}
\Re\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} &> \inf_{z\in E}\Re\left\{\frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\mu}{\lambda(m+1)}-1}du\right\} \\
&\geq \left\{\frac{\mu}{\lambda(m+1)}\int_0^1 \inf_{z\in E}\Re\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu}{\lambda(m+1)}-1}du\right\} \\
&> \frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\mu}{\lambda(m+1)}-1}du. \quad (24)
\end{aligned}$$

Combining (23) and (24), we obtain (21). Noting that the function $F_{\lambda,\mu,m,A,B}(z)$ defined by (22) belongs to the class $\mathcal{N}^{\lambda,\mu}(m,A,B)$, we get that inequality (21) is sharp. This completes the proof. □

In view of Theorem 3.9, we have the following distortion theorems for the class $\mathcal{N}^{\lambda,\mu}(m,A,B)$.

Corollary 3.10. *Let $f(z) \in \mathcal{N}^{\lambda,\mu}(m,A,B)$ with $\lambda > 0$ and $-1 \leq B < A \leq 1$. Then, for $|z| = r < 1$, we have*

$$\begin{aligned}
&2r\left(\frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1-Aur}{1-Bur}u^{\frac{\mu}{\lambda(m+1)}-1}du\right)^{-\frac{1}{\mu}} \\
&\quad < |f(z)-f(-z)| < 2r\left(\frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1+Aur}{1+Bur}u^{\frac{\mu}{\lambda(m+1)}-1}du\right)^{-\frac{1}{\mu}}. \quad (25)
\end{aligned}$$

The extremal function of (25) is defined by (22).

By noting that

$$\Re(v)^{\frac{1}{2}} \leq \Re(v^{\frac{1}{2}}) \leq |v|^{\frac{1}{2}}, \ \ v \in \mathbb{C};\ \Re(v) \geq 0,$$

from Theorem 3.9, we can easily derive the following result.

Corollary 3.11. *Let $f(z) \in \mathcal{N}^{\lambda,\mu}(m,A,B)$ with $\Re\lambda > 0$ and $-1 \le B < A \le 1$. Then*

$$\left(\frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{\mu}{\lambda(m+1)}-1}du\right)^{\frac{1}{2}} < \Re\left(\left(\frac{2z}{f(z)-f(-z)}\right)\right)^{\frac{\mu}{2}} < \left(\frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\mu}{\lambda(m+1)}-1}du\right)^{\frac{1}{2}}.$$

Theorem 3.12. *Let*

$$f(z) = z + \sum_{k=m+1}^{\infty} a_k z^k \in \mathcal{N}^{\lambda,\mu}(m,A,B),\ m \in \mathbb{N}. \tag{26}$$

Then

$$|a_{m+1}| \le \left|\frac{2(A-B)}{\lambda(m+1)+2\mu}\right|. \tag{27}$$

The inequality (27) is sharp, with the extremal function defined by (22).

Proof. Combining Definition 1.1 and (26), we have

$$\begin{aligned}(1+\lambda)\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} - \lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{2z}{f(z)-f(-z)}\right)^{\mu} \\ = 1 - [1+\frac{\lambda(m+1)}{2\mu}]\mu a_{m+1}\, z^{m+1} + \cdots \prec \frac{1+Az}{1+Bz} = 1-(A-B)z+\ldots\end{aligned} \tag{28}$$

An application of Lemma 2.6 to (28) yields

$$\left|[1-\frac{\lambda(m+1)}{2\mu}]\mu a_{m+1}\right| \le |A-B|\,. \tag{29}$$

Thus, from (29) we easily arrive to (27) asserted by Theorem 3.12. □

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