LE MATEMATICHE
Vol. LXI (2006) - Fasc. I, pp. 85-107

SOME RESULTS ON SPECIAL BINOMIAL IDEALS

LUCIA MARINO

The main goal of this paper is to characterize a particular class of ideals whose structure can still be interpreted directly from their generators: binomial ideals having a finite number of binomials generators in a polynomial ring $k[\mathbf{x}] = k[x_1, ..., x_n]$. More in details, we associate to every binomial ideal a homogeneous linear system constructed on its generators. We call it *Toric System* because it supplies us Toric Ideals given as elimination ideals. We observe that the constructed Toric Ideals depend only on the rank of a matrix that is solution of the Toric System. In addition, we define a *binomial principal generator* and then, we give some results for the case of a particular class of binomial ideals.

1. Introduction.

In the first section we recall the algebraic definition of Toric Ideal, given by Sturmfels in [11] and reviews some basic definitions on terminology on the Binomial Ideals. The study of the binomial prime ideals has particular interest in applied mathematics, including integer programming (see Conti-Traverso [4]), and computational statistics (see [6]). Within computer algebra, they arise in the extension of *Gröbner bases* theory to canonical subalgebra bases suggested by Robbiano-Sweedler [10], where the role of a single $S-pair$ is played by an entire binomial ideal. In Section 2, we shall

---

Entrato in redazione il 17 Dicembre 2005.
1991 *AMS 2000 Mathematics subject classification* :16D99,16D80,16D25

review some basic definitions and terminology on the Binomial Ideals and Toric Ideals. In Section 3 it is introduced a particular homogeneous linear system, called Toric System. We prove that the set of all solutions of it will give us Toric Ideals. Moreover, we observe that these depend on the rank of a matrix that is solution of the Toric System (see Section 4). In the last section, we give some results for the case of a particular class of binomial ideals and we show how to distinguish if these are Toric ideals or not.

2. Preliminaries.

We shall review some basic definitions in this section. Most of the materials presented here are standard and can be found in [2], [5], [8], [11].

Let k be any field and let $k[\mathbf{x}] = k[x_1, ..., x_n]$ be the polynomial ring in n indeterminates.

First recall that the set of power products is denoted by

$$T^n = \{x_1^{\beta_1} ... x_n^{\beta_n} \mid \beta_i \in \mathbb{N}_0,\ i = 1.....n\}$$

Sometimes we will denote $x_1^{\beta_1} \dots x_n^{\beta_n}$ by $\mathbf{x}^\beta$, where $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$. We would like to emphasize that *power product* will always refer to a product of the x_i variables, and *term* will always refer to a coefficient times a power product. We will also always assume that different terms in a polynomial have different power products.

There are many orderings on T^n but we must be able to compare any two power products. Thus the order must be a *total order*.

Definition 2.1. By a *term ordering* on T^n we mean a total order $<$ on T^n satisfying the following two conditions :

(i) $1 < \mathbf{x}^\beta$ for all $\mathbf{x}^\beta \in T^n$, $\mathbf{x}^\beta \neq 1$;

(ii) If $\mathbf{x}^\alpha < \mathbf{x}^\beta$, then

$$\mathbf{x}^\alpha \mathbf{x}^\gamma < \mathbf{x}^\beta < \mathbf{x}^\gamma,$$

for all $\mathbf{x}^\gamma \in T^n$.

Given a term ordering $\prec$, every non-zero polynomial $f \in k[\mathbf{x}]$ has a unique *initial monomial*, denoted $in_\prec(f)$.

If I is an ideal in $k[\mathbf{x}]$, then its *initial ideal* is the monomial ideal

$$in_\prec(I) =< in_\prec(f) :\ f \in I > .$$

The power products which do not lie in $in_{\prec}(I)$ are called *standard monomials*. A finite subset $G \subset I$ is a *Gröbner bases* for I with respect to $\prec$ if $in_{\prec}(I)$ is generated by $\{in_{\prec}(g) : g \in G\}$. If no monomial in this set is redundant, then the *Gröbner bases* G is *minimal*. It is called *reduced* if, for any two distinct elements $g, g' \in G$, no term of g' is divisible by $in_{\prec}(g)$.

The reduced *Gröbner bases* is unique for an ideal and a term ordering if one requires the coefficient of $in_{\prec}(g)$ in g to be 1 for each $g \in G$. Starting with any set of generators for I, the *Buchberger algorithm* computes the reduced *Gröbner bases* G. The *division algorithm* rewrites each polynomial f modulo I uniquely as a k-linear combination of standard monomials.

Throughout this paper, k denotes a field and $S = k[x_1, ..., x_n]$ is the polynomial ring in n variables over k. In this section, we present some elementary facts about Toric Ideals.

For binomial ideals, Toric Ideals and Gröbner bases, see Buchberger [2], Cox, Little, and O'Shea [5], Eisenbud [7], Eisenbud-Sturmfels [8].

In this section it is studied a class of ideals whose structure can still be interpreted directly from their generators: binomial ideals. By *binomial* in a polynomial ring S, we mean a polynomial with at most two terms, say $ax^{\alpha} + bx^{\beta}$, where $a, b \in k$ and $\alpha, \beta \in \mathbb{N}_0^n$.

We define a *binomial ideal* to be an ideal of S that can be generated by binomials, and a *binomial scheme* (or *binomial variety*, or *binomial algebra*) to be a scheme (or variety or algebra) defined by a binomial ideal.

Proposition 2.2. *Let $''<''$ be a monomial order on S, and let $I \subset S$ be a binomial ideal.*

(a) The reduced Gröbner bases G of I with respect to $''<''$ consists of binomials.

(b) The normal form with respect to $''<''$ of any term modulo G is again a term.

Proof. If we start with a binomial generating set for I , then the new *Gröbner bases* elements produced by a step in the Buchberger algorithm are binomials.

(b) Each step of the division algorithm modulo a set of binomials takes a term to another term. □

This proposition and the uniqueness of the reduced *Gröbner bases* with respect to a fixed monomial order $'' <''$ give the following corollaries:

Corollary 2.2. *If $I \subseteq k[x_1, ..., x_n]$ is a binomial ideal, then the elimination ideal $I \cap k[x_1, ..., x_r]$ is a binomial ideal for every $r \leq n$.*

Proof. See [8]. □

In the next section we shall study a special class of binomial ideals in $k[\mathbf{x}] = k[x_1, ..., x_n]$.

Let's recall the well known definition of Toric Ideal, see Sturmfels [11]. We fix a subset $\mathbf{A} = \{\mathbf{a}_1,..., \mathbf{a}_n\}$ of $\mathbb{Z}^d$.

Each vector $\mathbf{a}_i$ is identified with a monomial $\mathbf{t}^{a_i}$ in the Laurent polynomial ring

$$k[\mathbf{t}^{\pm 1}] = k[t_1, ..., t_d, t_1^{-1}, ..., t_d^{-1}].$$

Let the semigroup homomorphism

$$\pi : \mathbb{N}^n \longrightarrow \mathbb{Z}^d$$

be defined by

$$\mathbf{u} = (u_1, ..., u_n) \longmapsto u_1\mathbf{a}_1 + ... + u_n\mathbf{a}_n = \pi(\mathbf{u}).$$

The image of π is the semigroup

$$\mathbf{N}A = \{\lambda_1\mathbf{a}_1 + ... + \lambda_n\mathbf{a}_n : \lambda_1, ..., \lambda_n \in \mathbb{N}\}.$$

The map π lifts to a homomorphism of semigroup algebras :

$$\hat{\pi} : k[\mathbf{x}] \to k[\mathbf{t}^{\pm 1}], x_i \mapsto \mathbf{t}^{a_i}.$$

The kernel of $\hat{\pi}$ is denoted by I_A and it is called the *Toric Ideal* of A. I_A is prime and it is the defining ideal of an affine variety of zeros in k^n, that is irreducible. It is the Zariski closure of the set of points satisfying the system $x_i = \mathbf{t}^{a_i}, \mathbf{a}_i \neq 0$ with $i = 1, .., n$, $t \in K^d$. Here k^* denotes $k \setminus \{0\}$. The multiplicative group $(k^*)^d$ is called the *(d- dimensional algebraic) torus*. A variety of the form $v(I_A)$ is an *affine toric variety*. This definition is different from the definition of toric variety in algebraic geometry, see Fulton [9], because we do not require that toric varieties are normal. The following lemma permits to compute an infinite generating set for the Toric Ideal I_A.

Lemma 2.4. *The Toric Ideal I_A is spanned as a k-vector space by the set of binomials*

$$\{\mathbf{x}^u - \mathbf{x}^v : \mathbf{u}, \mathbf{v} \in \mathbb{N}^n \mid \pi(\mathbf{u}) = \pi(\mathbf{v})\}.$$

Remark 2.5. We omit the proof.(see [11]). We next compute the dimension of the toric variety $\mathbf{V}(I_A)$.

We write **ZA** for the lattice spanned by **A** and *dim***A** for the dimension of **ZA**.

Lemma 2.6. *The Krull dimension of the residue ring* $k[x]/I_{\mathbf{A}}$ *is equal to* $dim\mathbf{A}$.

Proof. See Sturmfels [11], Lemma 4.2. □

Remark 2.7. If we consider a binomial principal Toric Ideal I_A then

$$dim(k[\mathbf{x}]/I_A) = n - 1.$$

It follows that $dim(A) = n-1$, i.e. our d is $n-1$ for the binomial principals ideals, by lemma 2.6 .

3. Toric Systems.

Let I be any binomial ideal in a polynomial ring $k[\mathbf{x}] = k[x_1, ..., x_n]$. We can suppose that I is generated by h-binomials of type

$$\{a_1\mathbf{x}^{u_1} - b_1\mathbf{x}^{v_1}, a_2\mathbf{x}^{u_2} - b_2\mathbf{x}^{v_2}, ..., a_h\mathbf{x}^{u_h} - b_h\mathbf{x}^{v_h}\}.$$

We observe that the exponents $\mathbf{u}_i$, $\mathbf{v}_i$ belong to $\mathbb{N}^n$, in particular these can be written in the following way:

$$\mathbf{u}_1 = (\mathbf{u}_{11}, \mathbf{u}_{12}, ..., \mathbf{u}_{1n}),\ \mathbf{v}_1 = (\mathbf{v}_{11}, \mathbf{v}_{12}, ..., \mathbf{v}_{1n})$$
$$...$$
$$...$$
$$\mathbf{u}_h = (\mathbf{u}_{h1}, \mathbf{u}_{h2}, ..., \mathbf{u}_{hn}),\ \mathbf{v}_h = (\mathbf{v}_{h1}, \mathbf{v}_{h2}, ..., \mathbf{v}_{hn}).$$

We give the following definition:

Definition 3.1. Let I be a binomial ideal as above. We define the *Toric System associated to* I as the homogeneous linear system $(*)$ constructed on the generators of I of type

$$(*) \begin{cases} u_{11}\mathbf{C}_1 + u_{12}\mathbf{C}_2 + ... + u_{1n}\mathbf{C}_n = v_{11}\mathbf{C}_1 + v_{12}\mathbf{C}_2 + ... + v_{1n}\mathbf{C}_n \\ u_{21}\mathbf{C}_1 + u_{22}\mathbf{C}_2 + ... + u_{2n}\mathbf{C}_n = v_{21}\mathbf{C}_1 + v_{22}\mathbf{C}_2 + ... + v_{2n}\mathbf{C}_n \\ ... \\ u_{h1}\mathbf{C}_1 + u_{h2}\mathbf{C}_2 + ... + u_{hn}\mathbf{C}_n = v_{h1}\mathbf{C}_1 + v_{h2}\mathbf{C}_2 + ... + v_{hn}\mathbf{C}_n \end{cases}$$

where C_i is a column matrix with d rows, i.e. $C_i \in \mathbb{Z}^{d,1}$, for $i = 1, .., n$, are unknown.

In order to get a solution of $(*)$ we need to find $\mathbf{A}$=$(\mathbf{C}_1, \mathbf{C}_2, ..., \mathbf{C}_n) \in \mathbb{Z}^{d,n}$, with the vectors $\mathbf{C}_i \in \mathbb{Z}^{d,1}$ for $i = 1, ..., n$.

With the above notation we can write the Toric System in the following compact form

$$\mathbf{u}_j \cdot^t \mathbf{A} = \mathbf{v}_j \cdot^t \mathbf{A}$$

for $j = 1, ..., h$.

Call S the set of all solutions of the system. We observe that $S \neq \emptyset$, since the *Toric-System* associated to a binomial ideal is a homogeneous linear system, that has always the solution $\Omega = (\mathbf{0},...,\mathbf{0})$. If the solution Ω is unique then the ideal I is a Toric ideal. Suppose that there exists at least one non-zero solution. Thus let A_I be the family of the matrices of $\mathbb{Z}^{d,n}$ having as columns-vectors a solution $(\mathbf{C}_1, \mathbf{C}_2, ..., \mathbf{C}_n) \neq (0, ..., 0)$ of the *Toric-System*. Then

$$A_I = \{\mathbf{A} \in \mathbb{Z}^{d,n}, \mathbf{A} \neq \Omega | \ \mathbf{u}_j \cdot^t \mathbf{A} = \mathbf{v}_j \cdot^t \mathbf{A} \ , \ j = 1, ..., h.\}$$

and call it *the family of the matrices associated to I.*

Definition 3.2. Let

$$\mathcal{A} = \{{}^t\mathbf{C}_1, {}^t\mathbf{C}_2, ..., {}^t\mathbf{C}_n\}$$

be the set of all column-vectors of a matrix $\mathbf{A}$ of A_I. We call it *Binomial Set of* $\mathbf{A}$.

Example 4.2 Let's consider the binomial ideal $I = (x_1^2x_2 - x_3, 2x_3^2 - 5x_4^2)$ of $k[x_1, x_2, x_3, x_4]$. The vectors $\mathbf{u}_j$ and $\mathbf{v}_j$, for $j = 1, 2$ are:

$$\mathbf{u}_1 = (2, 1, 0, 0), \ \mathbf{v}_1 = (0, 0, 1, 0), \ \mathbf{u}_2 = (0, 0, 2, 0), \ \mathbf{v}_2 = (0, 0, 0, 2).$$

The Toric Sistem is

$$\begin{cases} 2\mathbf{C}_1 + \mathbf{C}_2 = \mathbf{C}_3 \\ 2\mathbf{C}_3 = 2\mathbf{C}_4 \end{cases}$$

The generic tuple of $\mathbb{Z}^4$ of the corresponding set S of the solutions has the structure $(\mathbf{C}_1, \mathbf{C}_3 - 2\mathbf{C}_1, \mathbf{C}_3, \mathbf{C}_3)$. Thus

$$A_I = \{\mathbf{A} \in \mathbb{Z}^{d,4} | \ 2\mathbf{C}_1 + \mathbf{C}_2 - \mathbf{C}_3 = \mathbf{C}_3 - \mathbf{C}_4 = 0\}$$

Clearly if we choose $d = 3$ then one matrix of A_I is of type

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

thus the Binomial Set is

$$\mathcal{A} = \{(1, 0, 0), (1, 0, 1), (3, 0, 1), (3, 0, 1)\}.$$

Now, fixed a matrix $\mathbf{A}$ of the family A_I we construct the *Toric Ideal* of $\mathcal{A}$ as elimination ideal.

Definition 3.4. Let I be a binomial ideal and let $\mathcal{A}$ be a binomial set of the matrix $\mathbf{A}$ of the family A_I. We say that $\tilde{I}_{\mathcal{A}}$ is a *Toric Ideal of* $\mathcal{A}$ *associated to I and* $\mathcal{A}$ if it is the Toric Ideal constructed on the set $\mathcal{A}$ as elimination ideal.

4. Matrices of A_I having different rank.

We remark that in example 3.3 by using CoCoA, see [3], the *Toric Ideal* of $\mathcal{A}$ was computed as elimination ideal is $(-x_1^2x_2 + x_3, -x_3^2 + x_4)$. The computation of a set of generators of a Toric Ideal by its own definition is difficult. Toric Ideals are binomial ideals which represent the algebraic relations of sets of power products. They appear in many problems arising from different branches of mathematics. Bigatti and Robbiano develop new theories which allow us to devise a parallel algorithm and efficient elimination algorithm. (see Bigatti-Robbiano [1]).

We observe that $\rho(\mathbf{A}) \leq d$ where $\rho(\mathbf{A})$ is the rank of $\mathbf{A}$. Moreover, there exists always at least a vector $\mathbf{C}_j$ with $j = 1, ..., n$ which is a linear combination of the vectors:

$$\mathbf{C}_1, ..., \mathbf{C}_{j-1}, \mathbf{C}_{j+1}, ..., \mathbf{C}_n.$$

Then $\rho(\mathbf{A}) \leq n - 1$.

Theorem 4.1. *Let* $\mathcal{A}$ *be the Set-Binomial of* $\mathbf{A} \in A_I$*. The rank of* $\mathbf{A}$ *is* $n-1$ *if and only if the Toric Ideal of* $\mathcal{A}$ *associated to* I *is a binomial principal ideal.*

Proof. Since $\rho(\mathbf{A}) = n - 1$, by Lemma 2.6 we have $dimk[x]/\tilde{I}_{\mathcal{A}} = n - 1$ and it implies that the Toric-System has only one linear equation. So $\tilde{I}_{\mathcal{A}}$ is principal. Conversely, if $\tilde{I}_{\mathcal{A}}$ is principal then $\rho(\mathbf{A}) = n - 1$. □

We note that $\rho(\mathbf{A}) \leq n - 1$, in particular $\rho(\mathbf{A}) = n - h$ where n is the number of the coordinates $x_1, ..., x_n$, and h is the number of linearly independent equations of the Toric-System.

In next example given a binomial ideal I we choose two matrices of A_I having different rank. We will observe that if $\rho(\mathbf{A})$ is equal either to $n-3$ or to $n-2$, then the Toric Ideal of $\mathcal{A}$ associated to I is not a principal ideal.

Example 4.2. Let's consider the binomial ideal $I = (2x_1x_2 - x_3^2, 4x_3^2 - x_4)$ of $k[x_1, x_2, x_3, x_4]$. Thus $h = 2$ and the vectors-exponents are: $\mathbf{u}_1 = (1, 1, 0, 0)$, $\mathbf{v}_1 = (0, 0, 2, 0)$, $\mathbf{u}_2 = (0, 0, 2, 0)$, $\mathbf{v}_2 = (0, 0, 0, 1)$. Then, fixed d, we have

$$\mathbf{A} = (\mathbf{C}_1, ..., \mathbf{C}_4),$$

such that

$$\begin{cases} \mathbf{u}_1 \cdot^t \mathbf{A} = \mathbf{v}_1 \cdot^t \mathbf{A} \\ \mathbf{u}_2 \cdot^t \mathbf{A} = \mathbf{v}_2 \cdot^t \mathbf{A} \end{cases}$$

Hence the *Toric-System* can be written as

$$\begin{cases} \mathbf{C}_1 + \mathbf{C}_2 = 2\mathbf{C}_3 \\ 2\mathbf{C}_3 = \mathbf{C}_4 \end{cases}$$

This means that $A_I = \{\mathbf{A} \in \mathbb{Z}^{d,4} \mid \mathbf{C}_1 + \mathbf{C}_2 - 2\mathbf{C}_3 = 2\mathbf{C}_3 - \mathbf{C}_4 = 0\}$. The generic solution is $(2\mathbf{C}_3 - \mathbf{C}_2, \mathbf{C}_2, \mathbf{C}_3, 2\mathbf{C}_3)$.

Thus A_I is the set of matrices $\mathbf{A}$ with -d rows and four columns of type

$$\mathbf{A} = (2\mathbf{C}_3 - \mathbf{C}_2, \mathbf{C}_2, \mathbf{C}_3, 2\mathbf{C}_3),$$

This means that the rank of all the matrices of A_I can be at most 2. We consider two matrices of A_I having different rank: A_1 with rank equal to one and A_2 with rank equal to two.

For example, let's suppose $d = 3$ by choosing ${}^{\mathbf{t}}\mathbf{C}_2 = {}^{\mathbf{t}}\mathbf{C}_3 = (1, 1, 1)$, we have

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

thus $\rho(\mathbf{A}_1) = 1$. If we construct the *Toric Ideal of* $\mathcal{A}_1$ by using CoCoA, it will be generated by

$$\tilde{I}_{\mathcal{A}_1} = (x_2 - x_3, x_1 - x_3, -x_3^2 + x_4).$$

Let's see now a matrix $\mathbf{A}_2$ having $\rho(\mathbf{A}_2) = 2$. For example if we choose ${}^{\mathbf{t}}\mathbf{C}_2 = (1, 0, 1)$, ${}^{\mathbf{t}}\mathbf{C}_3 = (1, 0, 0)$ then

$$\mathbf{A}_2 = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$$

and we have by using CoCoA that the *Toric Ideal of* $\mathbf{A}_2$ is generated by

$$\tilde{I}_{\mathcal{A}_2} = (-x_3^2 + x_4, x_1x_2 - x_4).$$

We observe that $\tilde{I}_{\mathcal{A}_1}$ and $\tilde{I}_{\mathcal{A}_2}$ are not principal ideals, this result follows by Theorem 4.1. In fact in our example the rank of all the matrices of A_I can be at most 2, i.e. different by $n-1$ which is three. Moreover we observe that changing the rank we obtain different Toric Ideals on the same binomial ideal

$$\tilde{I}_{\mathcal{A}_1} \neq \tilde{I}_{\mathcal{A}_2}.$$

If we consider $d = 5$, choosing as ${}^{\mathbf{t}}\mathbf{C}_2 = {}^{\mathbf{t}}\mathbf{C}_3 = (1, 1, 1, 1, 1)$ then the rank of $\mathbf{A}_1$ is one and $\tilde{I}_{\mathcal{A}_1}$ is the same.

We can conclude observing that by changing d does not imply that the *Toric Ideal of* $\mathcal{A}$ will be different.

In the next example we show that *Toric Ideal of* $\mathcal{A}$ depends on the rank of the matrices of A_I, in fact we obtain different Toric Ideals if we have different ranks.

Example 4.3.

Let $I = (x_1^2x_2 - x_3, x_1x_4 - x_2^2, x_3^2 - x_4^2, x_3 + x_4)$ in $k[x_1, x_2, x_3, x_4]$.

The *Toric-System* can be written as

$$\begin{cases} 2\mathbf{C}_1 + \mathbf{C}_2 = \mathbf{C}_3 \\ \mathbf{C}_1 + \mathbf{C}_4 = 2\mathbf{C}_2 \\ 2\mathbf{C}_3 = 2\mathbf{C}_4 \\ \mathbf{C}_3 = \mathbf{C}_4 \end{cases}$$

It is a linear system having four indeterminate and three linearly independent equations, so every matrix of A_I has maximal rank either less than or equal to one.

Choosing $d = 3$ and the following matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 & 5 \\ 1 & 3 & 5 & 5 \\ 1 & 3 & 5 & 5 \end{pmatrix}$$

we have $\tilde{I}_{\mathcal{A}} = (-x_1^2 + x_2, -x_1^2x_2 + x_4, -x_2^2 + x_1x_4, x_3 - x_4)$.

Here we have only matrices with ranks 0 or 1. If we take every matrix $\mathbf{A}_i$ of A_I having maximal rank (equal to one) then the *Toric Ideal of* $\mathcal{A}$ is always

$$(-x_1^2 + x_2, -x_1^2x_2 + x_4, -x_2^2 + x_1x_4, x_3 - x_4).$$

Instead if we have the matrix

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

then *Toric Ideal of* $\mathcal{A}$ is generated by $(x_1 - 1, x_2 - 1, x_3 - 1, x_4 - 1)$.

5. A particular class of binomial ideals.

Proposition 5.1. *Let I be a finitely generated binomial ideal in $k[\mathbf{x}]$ having generators with coefficients one. Let A be a matrix of A_I.*

Then for every $\mathbf{A} \in A_I$ the Toric Ideal of $\mathcal{A}$ associated to I, called $\tilde{I}_{\mathcal{A}}$, contains I.

Proof. Let h be the number of generators of I, i.e I is generated by h-binomials of type

$$\{\mathbf{x}^{u_1} - \mathbf{x}^{v_1}, \mathbf{x}^{u_2} - \mathbf{x}^{v_2}, ..., \mathbf{x}^{u_h} - \mathbf{x}^{v_h}\}.$$

By construction of $\tilde{I}_{\mathcal{A}}$, with $\mathbf{A} \in A_I$, we have that

$$\mathbf{u}_j \cdot^t \mathbf{A} = \mathbf{v}_j \cdot^t \mathbf{A}$$

for $j = 1, ..., h$ where $\mathbf{u}_j = (u_{j1}, ..., u_{jn})$ and $\mathbf{v}_j = (v_{j1}, ..., v_{jn})$ are the exponent vectors of the generators of I. Thus for every generator of I, we have

$$u_1\mathbf{C}_1 + ... + u_n\mathbf{C}_n = v_1\mathbf{C}_1 + ... + v_n\mathbf{C}_n$$

hence

$$\pi(\mathbf{u}) = \pi(\mathbf{v}) \Rightarrow \hat{\pi}(\mathbf{x}^u - \mathbf{x}^v) = 0 \Rightarrow I \subseteq \tilde{I}_A.$$

□

Remark 5.2. If the coefficients of generators of the binomial ideal I are different from one, then the Toric Ideal associated to I does not contain I.

Example 5.3. Let $I = (2x - y)$ be the binomial ideal in $k[x, y]$, then $\mathbf{u}_1 = (1, 0)$, $\mathbf{v}_1 = (0, 1)$ where

$$(1, 0)\begin{pmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{pmatrix} = (0, 1)\begin{pmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{pmatrix}$$

then $A_I = \{\mathbf{A} \mid \mathbf{C}_1 = \mathbf{C}_2\}$. Fixed $\mathbf{A}$ of A_I, if $dim\mathcal{A} = 1$ then for every integer positive d by using CoCoA the Toric Ideal of $\mathcal{A}$ associated to I is generated by $(x - y)$.

Such ideal does not contain I, in fact the coefficients of the generator are different from one.

Definition 5.4. We define a *binomial principal generator*, called *b.p.g.*, a binomial in $k[\mathbf{x}]$ of type $\mathbf{x}^u - \mathbf{x}^v$ such that for some $k \in \mathbb{N}$, $\mathbf{u}$ and $\mathbf{v}$ are of type

$$\mathbf{u} = (u_1, ..., u_k, 0, ..., 0), \mathbf{v} = (0, .., 0, v_{k+1}, ..., v_n),$$

with the following condition :

$$G.C.D.(u_1, ..., u_k, v_{k+1}, ..., v_n) = 1.$$

Let $\mathcal{A}^+$ be one matrix of A_I of type $\mathcal{A} = [\tilde{\mathbf{C}}_i]$ such that $\tilde{\mathbf{C}}_i \in \mathbb{N}^d$, for $i = 1, ..., n$.

Proposition 5.5. *For every ideal I that is generated by only one $b.p.g.$ in $k[\mathbf{x}]$, there exists at least a matrix of A_I different from the matrix Ω with d rows, n columns and non negative integer entries.*

Proof. Given a ideal I, that is generated by only one $b.p.g.$ in $k[\mathbf{x}]$,

$$I = (\mathbf{x}^u - \mathbf{x}^v),$$

then the vectors $\mathbf{u}$ and $\mathbf{v} \in \mathbb{N}^n$ can be written as $\mathbf{u} = (u_1, ..., u_k, 0, ..., 0)$ and $\mathbf{v} = (0, ..., 0, v_{k+1}, ..., v_n)$.

By section 2 since I is principal, then our *Toric-System* is the following linear equation:

$$u_1\mathbf{C}_1 + ... + u_k\mathbf{C}_k = v_{k+1}\mathbf{C}_{k+1} + ... + v_n\mathbf{C}_n.$$

Let us suppose that $u_1 \neq 0$. Then it is possible to choose the column-vectors by fixing the number of the components $n - 1$ with b_i in the $i - 1$ entry

$$^t\tilde{\mathbf{C}}_i = (0, ..., 0, b_i, 0, ..., 0),\ b_i = b_i' u_1,\ b_i' \geq 0, \forall\, i = 2, ..., n - 1$$

$$^t\tilde{\mathbf{C}}_n = (c_2, c_3, c_4, ..., c_k, 0, ..., 0, c_{n-1}),\ c_{n-1} = c_{n-1}' u_1,\ c_{n-1}' \geq 0,\ c_j = c_j' u_1,\ c_j' \geq 0$$

By our *Toric-System*, we have

$$u_1\tilde{\mathbf{C}}_1 + u_2\begin{pmatrix} b_2 \\ ... \\ 0 \\ 0 \\ 0 \\ 0 \\ ... \\ 0 \end{pmatrix} + ... + u_k\begin{pmatrix} 0 \\ ... \\ 0 \\ b_k \\ 0 \\ 0 \\ ... \\ 0 \end{pmatrix} = v_{k+1}\begin{pmatrix} 0 \\ ... \\ 0 \\ 0 \\ b_{k+1} \\ 0 \\ ... \\ 0 \end{pmatrix} + ... + v_n\begin{pmatrix} c_2 \\ c_3 \\ ... \\ c_k \\ 0 \\ ... \\ 0 \\ c_{n-1} \end{pmatrix}.$$

So the vector-column $\tilde{\mathbf{C}}_1$ is given by

$$\tilde{\mathbf{C}}_1 = \begin{pmatrix} v_n c'_2 - u_2 b'_2 \\ v_n c'_3 - u_3 b'_3 \\ \cdots \\ v_n c'_k - u_k b'_k \\ v_{k+1} b'_{k+1} \\ \cdots \\ \cdots \\ v_n c'_{n-1} \end{pmatrix}.$$

We observe that every column $\tilde{\mathbf{C}}_i$, for i=*2,...,n*, has non negative integer entries. Moreover $\tilde{\mathbf{C}}_1 \in \mathbb{N}^{n-1,1}$ if we choose $c'_2, ..., c'_k$ in such way as

$$v_n c'_j - u_j b'_j \geq 0 \,\forall\, j = 2, ..., k.$$

So if we fix $u_1, c'_2, .., c'_k$, then $\mathcal{A}^+$ is well determined.

In the case $u_1 = 0$ it is possible to proceed as above by choosing one u_i supposing that at least a $u_i \neq 0$. □

Remark 5.6. The set $\mathcal{A}^+$ is not unique because it depends on the choice of $u_1, c'_2, .., c'_k$.

We can compute now one of the sets $\mathcal{A}^+$ that we call $\mathcal{A}_1^+$. For this purpose, if $u_1 \neq 0$ we can choose $b_i = u_1$, $i = 2, ..., n-1$ and $c_{n-1} = u_1$ and $c'_j = u_j + 1$ so that $b'_i = 1$ and $c'_{n-1} = 1$ and $c_j = u_1(u_j + 1)$, $\forall\, j = 2, ..., k$.

Then our column-vectors of the sets $\mathcal{A}_1^+$

$${}^t\tilde{\mathbf{C}}_1 = (v_n(u_2+1) - u_2, ..., v_n(u_{2k}+1) - u_k, v_{k+1}, ..., v_n)$$

with u_1 in the $i-1$ entry

$$^t\tilde{\mathbf{C}}_i = (0, ..., u_1, 0, ..., 0), \ \forall\, i = 2, ..., n-1$$

$$^t\tilde{\mathbf{C}}_n = (u_1(u_2+1), ..., u_1(u_k+1), 0, ..., 0, u_1).$$

Then the set $\mathcal{A}_1^+ = [^t\tilde{\mathbf{C}}_i]$. If $u_1 = 0$, it is possible to built a similar matrix if it does exist at least a $u_s \neq 0$ (at least one of such u_s always exists) so that $b_i = u_s$, $c_{n-1} = u_s$ and $c_j = u_s(u_j+1)$.

Fixed $\mathcal{A}_1^+$ we can consider the known map $\hat{\pi} : k[\mathbf{x}] \to k[\mathbf{t}]$ defined by $x_i \mapsto \mathbf{t}^{a_i}$ where $\mathbf{a}_i$ in our case are the components of vectors $^t\tilde{\mathbf{C}}_i \in \mathcal{A}_1^+$. Thus it is possible to write

$$x_1 = t_{n-1}^{v_n} ... t_k^{v_{k+1}} t_{k-1}^{v_n(u_k+1)-u_k} ... t_1^{v_n(u_2+1)-u_2}$$

$$x_i = t_{i-1}^{u_1}, \ i = 2, ..., n-1$$

$$x_n = t_{n-1}^{u_1} t_{k-1}^{u_1(u_k+1)} ... t_1^{u_1(u_2+1)}$$

This remark suggests an equality between two ideals. We have the following result.

Call I and J the ideals in $k[\mathbf{x,t}]$ generated respectively by $t_{i-1}^{u_1} - x_i$, $\forall\, i = 2, ..., n-1$ and by α_1, α_2 for I, where

$$\alpha_1 = t_{n-1}^{u_1} t_{k-1}^{u_1(u_k+1)} ... t_1^{u_1(u_2+1)} - x_n,$$

$$\alpha_2 = t_{n-1}^{u_1 v_n} ... t_k^{u_1 v_{k+1}} t_{k-1}^{u_1(v_n(u_k+1)-u_k)} ... t_1^{u_1(v_n(u_2+1)-u_2)} - x_1^{u_1};$$

by $t_{i-1}^{u_1} - x_i$, $\forall\, i = 2, ..., n-1$ and by β_1, β_2 for J, where

$$\beta_1 = t_{n-1}^{u_1} x_k^{(u_k+1)} ... x_2^{(u_2+1)} - x_n,$$

$$\beta_2 = t_{n-1}^{v_n u_1} x_{n-1}^{v_{n-1}} ... x_{k+1}^{v_{k+1}} x_k^{(v_n(u_k+1)-u_k)} ... x_2^{(v_n(u_2+1)-u_2)} - x_1^{u_1}.$$

Proposition 5.7. $I = J$*, with notation as above.*

Proof. Both I and J are ideals in $k[\mathbf{x}, t] = A$. In order to show that $I = J$ it is therefore sufficient to show that $J/I = 0$ in A/I and in A/J we have $I/J = 0$. This is true. In fact if we have A/I then $J/I = 0$ for construction of J. The same for I. $\square$

Using the result of Proposition 5.5, i.e. that for every ideal I is generated by a only $b.p.g.$ in $k[\mathbf{x}]$ it is possible to have a set $\mathcal{A} = \{^t\mathbf{C}_1, ...,^t \mathbf{C}_n\}$ of $\mathbb{N}^d$ with some $\mathbf{C}_i \neq 0$, we obtain the following

Proposition 5.8. *A binomial ideal generated by a b.p.g. in $k[\mathbf{x}]$ is a Toric Ideal associated to I and $\mathcal{A}$ is equal to I, i.e. I is a Toric Ideal.*

Proof. Given a binomial ideal of type $I = (\mathbf{x}^u - \mathbf{x}^v)$ in $k[\mathbf{x}]$, with $\mathbf{u}$ and $\mathbf{v}$ of $\mathbb{N}^{n-1}$ so that $\mathbf{u} = (u_1, ..., u_k, 0, ..., 0)$ and $\mathbf{v} = (0, ..., 0, v_{k+1}, ..., v_n)$ satisfying the conditions

$$G.C.D.(u_1, ..., u_k, v_{k+1}, ..., v_n) = 1,$$

there exist $b_i, b_j \in \mathbb{Z}$ such that

$$1 = \sum_{i=1}^{k} b_i u_i + \sum_{j=k+1}^{n} b_j v_j.$$

We observe that in our case the *Toric-System* associated to I is the equation

$$\mathbf{u} \cdot^t \mathbf{A} = \mathbf{v} \cdot^t \mathbf{A}.$$

By hypothesis we have that $d = n - 1$, then we choose the vectors ${}^t\mathbf{C}_i \in \mathbb{Z}^{n-1}$ for $i = 1, ..., n-1$, in such way they have v_n as *i-th component* and the other components are zero, i.e

$${}^t\mathbf{C}_i = (0, ..., 0, v_n, 0, ..., 0)$$

thus the vector $\mathbf{C}_n$ in our *Toric-System* will be

$${}^t\mathbf{C}_n = (u_1, ..., u_k, -v_{k+1}, ..., -v_{n-1}).$$

Let

$$\hat{\pi} : k[\mathbf{x}] \to k[\mathbf{t}]$$

be the map defined by

$$x_i \mapsto \mathbf{t}^{a_i}.$$

If $\mathbf{a}_i =^t \mathbf{C}_i$ for $i = 1, ..., n$ then, a direct calculation shows that

$$x_j \mapsto t_j^{v_n}, \ \ j = 1, ..., n-1$$

$$x_n \mapsto t_1^{u_1} t_2^{u_2} ... t_k^{u_k} t_{k+1}^{-v_{k+1}} ... t_{n-1}^{-v_{n-1}}.$$

So, we call J the following ideal

$$J = (x_1 - t_1^{v_n}, x_2 - t_2^{v_n}, ..., x_k - t_k^{v_n}, x_{k+1} - t_{k+1}^{v_n}, ... \\ ..., x_n - t_1^{u_1} t_2^{u_2} ... t_k^{u_k} t_{k+1}^{-v_{k+1}} ... t_{n-1}^{-v_{n-1}})$$

in $k[\mathbf{x}, t]$.

Therefore if we choose as lexicographic term ordering $'' <''$ with

$$x_1 > x_2 > ... > x_n > t_1 > ... > t_{n-1}$$

then the set of generators of J is a *Gröbner bases of* J,(see B. Buchberger [2]).

In order to prove that I is a Toric Ideal it is sufficient to prove that the elimination ideal of J is the binomial principal prime ideal I. In fact, first we will show that if $\mathbf{x}^s - \mathbf{x}^z \in J$ with $\mathbf{s} = (s_1, ..., s_n)$, $\mathbf{z} = (z_1, ..., z_n) \in \mathbb{N}^n$, then $\mathbf{x}^s - \mathbf{x}^z$ is a multiple of $\mathbf{x}^u - \mathbf{x}^v$. Let $g_1 = x_1 - t_1^{v_n}$, ..., $g_n = x_n - t_1^{u_1} t_2^{u_2} ... t_k^{u_k} t_{k+1}^{-v_{k+1}} ... t_{n-1}^{-v_{n-1}}$.

Therefore we assume that $\mathbf{x}^s - \mathbf{x}^z \in J$, i.e. it reduces to zero with respect to $g_1, ..., g_n$

$$x^s - x^z \longrightarrow^{g_1,...,g_n} 0$$

where the g_i, for $i = 1, ..., n$, are the generators of J. Then beginning with g_1 we have

$$x_1^{s_1} ... x_n^{s_n} - x_1^{z_1} ... x_n^{z_n} \longrightarrow^{g_1} t_1^{v_n s_1} x_2^{s_2} ... x_n^{s_n} - t_1^{v_n z_1} x_2^{z_2} ... x_n^{z_n}$$

therefore

$$t_1^{v_n s_1} x_2^{s_2} ... x_n^{s_n} - t_1^{v_n z_1} x_2^{z_2} ... x_n^{z_n} \longrightarrow^{g_2} t_1^{v_n s_1} t_2^{v_n s_2} x_3^{s_3} ... x_n^{s_n} - t_1^{v_n z_1} t_2^{v_n z_2} x_3^{z_3} ... x_n^{z_n}$$

...

and after a finite number of steps we obtain :

$$t_1^{v_n s_1 + s_n u_1} ... t_k^{v_n s_k + s_n u_k} ... t_{n-1}^{v_n s_{n-1} - s_n v_{n-1}} - t_1^{v_n z_1 + z_n u_1} ... t_k^{v_n z_k + z_n u_k} ... t_{n-1}^{v_n z_{n-1} - z_n v_{n-1}}.$$

Since $\mathbf{x}^s - \mathbf{x}^z \rightarrow^{g_1,...,g_n} 0$ then we have

$$t_1^{v_n s_1 + s_n u_1} ... t_k^{v_n s_k + s_n u_k} ... t_{n-1}^{v_n s_{n-1} - s_n v_{n-1}} - t_1^{v_n z_1 + z_n u_1} ... t_k^{v_n z_k + z_n u_k} ... t_{n-1}^{v_n z_{n-1} - z_n v_{n-1}} = 0$$

Therefore we have the following system

$$\begin{cases} v_n s_1 + s_n u_1 = v_n z_1 + z_n u_1 \\ \cdots \\ v_n s_k + s_n u_k = v_n z_k + z_n u_k \\ v_n s_{k+1} - s_n v_{k+1} = v_n z_{k+1} - z_n v_{k+1} \\ \cdots \\ v_n s_{n-1} - s_n v_{n-1} = v_n z_{n-1} - z_n v_{n-1} \end{cases}$$

i.e.

$$(1)\begin{cases} u_1(s_n - z_n) = v_n(z_1 - s_1) \\ \dots \\ u_k(s_n - z_n) = v_n(z_k - s_k) \\ v_{k+1}(s_n - z_n) = -v_n(z_{k+1} - s_{k+1}) \\ \dots \\ v_{n-1}(s_n - z_n) = -v_n(z_{n-1} - s_{n-1}) \\ v_n(s_n - z_n) = -v_n(z_n - s_n) \end{cases}$$

As stated in the hypothesis, the 1 can be written as linear combination of u_i and v_j

$$1 = \sum_{i=1}^{k} b_i u_i + \sum_{j=k+1}^{n} b_j v_j$$

and if we multiply this equation by $(s_n - z_n)$ it becomes

$$(s_n - z_n) = (s_n - z_n)\sum_{i=1}^{k} b_i u_i + (s_n - z_n)\sum_{j=k+1}^{n} b_j v_j$$

Now, by using the system (1), this equation becomes

$$(s_n - z_n) = \sum_{i=1}^{k} b_i v_n(z_i - s_i) - \sum_{j=k+1}^{n} b_j v_n(z_j - s_j)$$

i.e.

$$(s_n - z_n) = v_n(\sum_{i=1}^{k} b_i(z_i - s_i) - \sum_{j=k+1}^{n} b_j(z_j - s_j)) \ (6.1)$$

If we call

$$k = \sum_{i=1}^{k} b_i(z_i - s_i) - \sum_{j=k+1}^{n} b_j(z_j - s_j)$$

then the equation (6.1) becomes

$$(s_n - z_n) = v_n k.$$

Thus for this result we obtain the new system:

$$\begin{cases} u_1 k v_n = v_n(z_1 - s_1) \\ \dots \\ u_k k v_n = v_n(z_k - s_k) \\ v_{k+1} k v_n = -v_n(z_{k+1} - s_{k+1}) \\ \dots \\ v_{n-1} k v_n = -v_n(z_{n-1} - s_{n-1}) \end{cases}$$

and computing the components of the vectors-exponents **s** and

$$(2) \begin{cases} z_1 = u_1 k + s_1 \\ \dots \\ z_k = u_k k + s_k \\ s_{k+1} = k v_{k+1} + z_{k+1} \\ \dots \\ s_{n-1} = k v_{n-1} + z_{n-1} \end{cases} .$$

If we consider the binomial

$$x_1^{s_1} \dots x_n^{s_n} - x_1^{z_1} \dots x_n^{z_n} =$$

$$= x_1^{s_1} \dots x_k^{s_k} x_{k+1}^{kv_{k+1}+z_{k+1}} \dots x_{n-1}^{kv_{n-1}+z_{n-1}} x_n^{s_n} - x_1^{ku_1+s_1} \dots x_k^{ku_k+s_k} x_{k+1}^{z_{k+1}} \dots x_{n-1}^{z_{n-1}} x_n^{z_n} =$$

$$= x_1^{s_1} \dots x_k^{s_k} x_{k+1}^{z_{k+1}} \dots x_{n-1}^{z_{n-1}} x_n^{z_n} (x_{k+1}^{kv_{k+1}} \dots x_n^{kv_n} - x_1^{ku_1} \dots x_k^{ku_k}) =$$

$$= x_1^{s_1} \dots x_k^{s_k} x_{k+1}^{z_{k+1}} \dots x_{n-1}^{z_{n-1}} x_n^{z_n} ((\mathbf{x}^v)^k - (\mathbf{x}^u)^k)$$

note that it is a multiple of $\mathbf{x}^v - \mathbf{x}^u$. Thus the elimination ideal of J is our

$$I = (\mathbf{x}^u - \mathbf{x}^v).$$

□

Proposition 5.8 implies the following corollary

Corollary 0.0. *If I is a binomial ideal generated by a b.p.g. of the type $(\mathbf{x}^u - \mathbf{x}^v)$ then it is a prime ideal.*

Proof. The definition of a Toric Ideal of A says that it is the kernel of $\hat{\pi}$. Thus it is a prime ideal, (see Sturmfels [11]). □

Proposition 5.10. *Let I be a binomial ideal in $k[\mathbf{x}]$ having the Krull dimension of the residue ring $k[\mathbf{x}]/I = s$. If every generator of I is a b.p.g. and precisely $I = \{\mathbf{x}^{u_1} - \mathbf{x}^{v_1}, \mathbf{x}^{u_2} - \mathbf{x}^{v_2}, \dots, \mathbf{x}^{u_h} - \mathbf{x}^{v_h}\}$ and if given a matrix $\mathbf{A}$ of A_I with $\rho(A) = s$ we have $\tilde{I}_{\mathcal{A}} = I$ then I is a Toric Ideal.*

Proof. The proof is trivial, see Section 4. □

Example 5.11. Consider the

$$I = (x_3 - x_1^2, x_4 - x_3^3)$$

in $k[x_1, x_2, x_3, x_4]$. Since it has the Krull dimension of the residue ring $k[\mathbf{x}]/I$ equal to 2 this implies $s = 2$. The *Toric-System* associated to I is

$$\begin{cases} \mathbf{C}_3 = 2\mathbf{C}_1 \\ \mathbf{C}_4 = 3\mathbf{C}_3. \end{cases}$$

From this we immediately deduce that A_I is the set of matrices $\mathbf{A}$ with four rows and four columns of type

$$\mathbf{A} = (\mathbf{C}_1, \mathbf{C}_2, 2\mathbf{C}_1, 6\mathbf{C}_1),$$

This means that

$$A_I = \{\mathbf{A} \in \mathbb{Z}^{d,4} \mid \mathbf{C}_3 - 2\mathbf{C}_1 = \mathbf{C}_4 - 6\mathbf{C}_1 = 0\}.$$

If we consider d-components for the column-vectors then we can consider as independent vectors $\mathbf{C}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ .. \\ 0 \end{pmatrix}$ and $\mathbf{C}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ .. \\ 0 \end{pmatrix}$ and we have

$$\mathcal{A} = \{(1, 0, 0, ..., 0), (0, 1, 0, ..., 0), (2, 0, ..., 0), (6, 0, ..., 0)\}.$$

Then we construct as elimination ideal the *Toric Ideal* of $\mathcal{A}$, by using CoCoA the *Toric Ideal* of $\mathcal{A}$ is I. Thus I is a Toric Ideal.

Proposition 5.12. *Let I be a binomial ideal in $k[\mathbf{x}]$ having the Krull dimension of the residue ring $dimk[\mathbf{x}]/I = s$ with every generator $b.p.g.$ and precisely*

$$I = \{\mathbf{x}^{u_1} - \mathbf{x}^{v_1}, \mathbf{x}^{u_2} - \mathbf{x}^{v_2}, ..., \mathbf{x}^{u_h} - \mathbf{x}^{v_h}\}$$

and let $\mathbf{A}$ be a matrix of A_I with $\rho(A) = s$. If $\tilde{I}_{\mathcal{A}} \supset I$ then I is not a Toric Ideal.

Proof. Suppose that $\tilde{I}_{\mathcal{A}} \supset I$ and I is a Toric Ideal. Thus there exists a subset $\tilde{A}$ of $\mathbb{Z}^d$ with dimension equal to s (See lemma 4.2 Sturmfels [11]).

Moreover, it is well known that a Toric Ideal $I_{\tilde{\mathbf{A}}}$ is a ideal of type

$$I_{\tilde{\mathbf{A}}} =< \mathbf{x}^{u^+} - \mathbf{x}^{u^-} : \mathbf{u} \in Ker(\pi) > .$$

Note that every vector $\mathbf{u}$ of $\mathbb{Z}^d$ can be written uniquely as

$$\mathbf{u} = u^+ - u^-$$

where u^+ and u^- are non-negative and have disjoint support. (More precisely, the $i-th$ coordinate of u^+ equals u_i if $u_i > 0$ and it equals 0 otherwise). It follows that $Ker(\pi)$ for the sublattice of $\mathbb{Z}^d$ consists of all vectors $\mathbf{u}$ such that

$$\pi(u^+) = \pi(u^-).$$

It means that the vectors $\mathbf{a}_i$ of the subset $\tilde{A}$ of $\mathbb{Z}^d$ are a solution of the *Toric-System* associated to I. Then all subsets A of $\mathbb{Z}^d$ giving a Toric Ideal have as elements a solution of the *Toric-System* associated to I.

It suffices to show that all matrices $\mathbf{A} \in A_I$ with rank equal to s have the same *Toric Ideal of* $\mathcal{A}$ *associated to I and A*.

We choose two matrices $\mathbf{A}', \mathbf{A}'' \in A_I$ with rank equal to s. Thus the binomial sets $\mathcal{A}'$ and $\mathcal{A}''$ have

$$\rho(\mathcal{A}') = \rho(\mathcal{A}'') = s.$$

In this case there exist two Toric Ideals and precisely a *Toric Ideal of* $\mathcal{A}'$*associated to I and* $\mathbf{A}'$, and a *Toric Ideal of* $\mathcal{A}''$ *associated to I and* $\mathbf{A}''$ of type

$$\tilde{I}_{\mathcal{A}'} =< \mathbf{x}^{u^+} - \mathbf{x}^{u^-} : \mathbf{u} \in Ker(\pi) >,$$
$$\tilde{I}_{\mathcal{A}''} =< \mathbf{x}^{u^+} - \mathbf{x}^{u^-} : \mathbf{u} \in Ker(\pi) > .$$

with $\mathbf{u} \in \mathbb{Z}^d$, $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$. This implies that the two Toric Ideals are generated by binomials which do not depend on the choice of the vectors $\mathbf{a}_i$ of $\mathbf{A}'$ or $\mathbf{A}''$ of $\mathbb{Z}^d$. Indeed the fundamental fact is that the Krull dimension of the residue ring $k[\mathbf{x}]/\tilde{I}_{\mathcal{A}}$ is equals to s and that $\mathbf{u} \in Ker(\pi)$, i.e. the sets $\mathcal{A}'$ and $\mathcal{A}''$ are two solutions of the *Toric-System* associated to I.

Then

$$\tilde{I}_{\mathcal{A}'} = \tilde{I}_{\mathcal{A}''} = I.$$

This is not possible since we supposed that

$$\tilde{I}_{\mathcal{A}} \supset I.$$

This implies that I is not a Toric Ideal. □

Example 5.13. Given

$$I = (x_1^2 - x_4, x_1^2 - x_3^2)$$

in $k[x_1, x_2, x_3, x_4]$, it has the Krull dimension equal to 2. The *Toric-System* associated to I is

$$\begin{cases} \mathbf{C}_1 = \mathbf{C}_3 \\ 2\mathbf{C}_1 = \mathbf{C}_4 \end{cases}$$

Thus A_I is the set of matrices $\mathbf{A}$ with four rows and four columns of type

$$\mathbf{A} = (\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_1, 2\mathbf{C}_1),$$

This means that $A_I = \{\mathbf{A} \in \mathbb{Z}^{d,4} \mid \mathbf{C}_3 - \mathbf{C}_1 = \mathbf{C}_4 - 2\mathbf{C}_1 = 0\}$.

If we choose two linearly independent vectors

$$\mathbf{C}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ .. \\ 0 \end{pmatrix}, \ \mathbf{C}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ .. \\ 0 \end{pmatrix},$$

then the binomial set is

$$\mathcal{A} = \{(1, 0, ..., 0), (0, 1, ..., 0), (1, 0, ..., 0), (2, 0, ..., 0)\}.$$

Thus $dim \mathcal{A} = D(I) = 2$. Then we construct as elimination ideal the *Toric Ideal* of $\mathcal{A}$, by using CoCoA, obtaining

$$\tilde{I}_{\mathcal{A}} = (x_1 - x_3, x_4 - x_3^2) \supset I.$$

Thus I is not a Toric Ideal.

Proposition 5.12 can be rephrased as follows.

Corollary 5.14. *Let I be a binomial ideal in $k[\mathbf{x}]$ having the Krull dimension of the residue ring $k[\mathbf{x}]/I$ equal to s. If every generator of I is a $b.p.g.$ and precisely*

$$I = \{\mathbf{x}^{u_1} - \mathbf{x}^{v_1}, \mathbf{x}^{u_2} - \mathbf{x}^{v_2}, ..., \mathbf{x}^{u_h} - \mathbf{x}^{v_h}\}$$

and if $\mathbf{A} \in A_I$ has rank less than s, then $\tilde{I}_{\mathcal{A}} \supset I$.

We observe that a Toric Ideal is not uniquely determined by the rank of matrix that generates it.

Example 5.15. Given two different ideals having both Krull dimension of the residue ring $k[\mathbf{x}]/I$ equal to two,

$$I_1 = (x_1^2 - x_3, x_3^3 - x_4)$$

$$I_2 = (x_1^2 - x_4, x_1^2 - x_3^2)$$

we consider the two *Toric-Systems* associated to two ideals

first system:

$$\begin{cases} \mathbf{C}_3{=}2\mathbf{C}_1 \\ \mathbf{C}_4{=}3\mathbf{C}_3 \end{cases}$$

second system:

$$\begin{cases} \mathbf{C}_1 = \mathbf{C}_3 \\ 2\mathbf{C}_1 = \mathbf{C}_4 \end{cases}$$

If we choose $d = 3$ and a matrix

$$\mathbf{A}' = \begin{pmatrix} 1 & 0 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

of A'_I of the first system, then

$$\mathcal{A}' = \{(1, 0, 0), (0, 1, 0), (2, 0, 0), (6, 0, 0)\}$$

and let

$$\mathbf{A}'' = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

be a matrix of A''_I of the second system, then the binomial set is

$$\mathcal{A}'' = \{(1, 0, 0), (0, 1, 0), (1, 0, 0), (2, 0, 0)\}.$$

The matrices A', A'' have the same rank where $\rho(A) = \rho(A') = D(I) = D(I') = 2$ but

$$\tilde{I}_{\mathcal{A}'} \neq \tilde{I}_{\mathcal{A}''}.$$

Proposition 5.16 *Let I be a binomial prime ideal in $k[\mathbf{x}]$ having the Krull dimension of the residue ring $k[\mathbf{x}]/I$ equal to s. If every generator of I is a $b.p.g.$ then I is a Toric Ideal.*

Proof. Applying Proposition 5.12 to our proposition, we see that our thesis is equivalent to the condition that I is equal to $\tilde{I}_{\mathcal{A}}$.

Then we suppose that $\tilde{I}_{\mathcal{A}} \supset I$, but it is impossible because these are two prime ideals with the same Krull dimension of the residue ring equal to s. Thus $\tilde{I}_{\mathcal{A}} = I$. □

Example 5.17. Let $I = (2x - y)$ a binomial ideal in $k[x, y]$, see Example 5.3, where $A_I = \{\mathbf{A} \mid \mathbf{C}_1 = \mathbf{C}_2\}$. We choose a matrix $\mathbf{A}$ of A_I, i.e fix $\mathcal{A}$. By using CoCoA, the Toric Ideal of $\mathcal{A}$ associated to I is generated by $(x - y)$ thus I is not Toric Ideal.

If

$$\begin{cases} x = \alpha_1 X \\ y = \alpha_2 Y \end{cases}$$

then our ideal is $I = (2\alpha_1 X - \alpha_2 Y)$.

Suppose that $2\alpha_1 = \alpha_2 \neq 0$ and we choose $\alpha_1 = 1$, $\alpha_2 = 2$. Thus $I = (2(X - Y))$ and now it is a Toric Ideal because it is generated by a *b.p.g.*.

Proposition 5.18. *Every principal ideal in* $k[\mathbf{x}]$ *generated by a irreducible binomial is a Toric Ideal.*

Proof. Let I be a binomial ideal of type

$$I = (a_1 \mathbf{x}^u - b_1 \mathbf{x}^v),$$

where there exists $k \in \mathbb{N}$ such that $\mathbf{u} = (u_1, ..., u_k, 0, ..., 0)$ and $\mathbf{v} = (0, .., 0, v_{k+1}, ..., v_n)$ with the following condition:

$$G.C.D.(u_1, ..., u_k, v_{k+1}, ..., v_n) = 1.$$

Moreover let

$$\begin{cases} x_1 = \alpha_1 X_1 \\ x_2 = \alpha_2 X_2 \\ ... \\ x_n = \alpha_n X_n \end{cases}$$

be our change of coordinates.

Then the binomial ideal I becomes

$$(a_1 \alpha_1^{u_1} \alpha_2^{u_2} ... \alpha_k^{u_k} X_1^{u_1} X_2^{u_2} ... X_k^{u_k} - b_1 \alpha_{k+1}^{v_{k+1}} ... \alpha_n^{v_n} X_{k+1}^{v_{k+1}} ... X_n^{v_n})$$

Thus

$$a_1 \alpha_1^{u_1} \alpha_2^{u_2} ... \alpha_k^{u_k} = b_1 \alpha_{k+1}^{v_{k+1}} ... \alpha_n^{v_n}$$

This is an equation, which has always infinite solutions. Suppose that there exists at least a solution $\bar{\alpha}$ having $\bar{\alpha}_{\mathbf{i}} \neq 0$, $\forall i = 1, ..., n$. We choose as solution $\bar{\alpha}$ and call

$$\beta = a_1 \bar{\alpha}_{\mathbf{1}}^{u_1} \bar{\alpha}_{\mathbf{2}}^{u_2} ... \bar{\alpha}_{\mathbf{k}}^{u_k} = b_1 \bar{\alpha}_{\mathbf{k}+1}^{v_{k+1}} ... \bar{\alpha}_{\mathbf{n}}^{v_n}.$$

Then I can be written as

$$I = (\beta(X_1^{u_1} X_2^{u_2} ... X_k^{u_k} - X_{k+1}^{v_{k+1}} ... X_n^{v_n}))$$

From Proposition 5.8 it follows that I is a Toric Ideal.

Acknowledgements.

The author thanks Professor G. Carrà-Ferro for helpful suggestions on the subject of this paper.

REFERENCES

[1] A. Bigatti, R. La Scala, L. Robbiano *Computing Toric Ideals*, J. Symbolic Computation, **27** (1999), 351-365.

[2] B. Buchberger, *Gröbner bases: an algorithmic method in polynomial ideal theory*, Multidimensional System Theory, Math. Appl, **16** D. Reidel, Dordrecht, 1985.

[3] A. Capani, G. Niesi, L. Robbiano, CoCoA,*a System for doing computations in commutative algebra*. The version 4.0 is available at
`http://cocoa.dima.unige.it`.

[4] P. Conti, C. Traverso, *Buchberghe r algorithm and integer programming*, Appied Algebra, Algebraic Algorithms, and Error-Protecting Codes, Lect. Notes in Comput Sci. **539** Springer-Verlag, New York (1991), 130–139, 333-341.

[5] D. Cox, J. Little, D.O' Shea *Ideals, Varieties, and Algorithms*, Undergrad. Texts Math., Springer-Verlag, New York (1992).

[6] P. Diaconis, B. Sturmfels *Algebraic algoritms for generating from conditional distribuctions*, to appear in Ann. Statist.

[7] D. Eisenbud *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Math., Springer-Verlag, New York (1995).

[8] D. Eisenbud, B. Sturmfels *Binomial Ideals*, Duke Mathematical journal, **84** (1996), 63-83.

[9] W. Fulton *Introduction to Toric Varieties*, Ann. of Math. Stud. **131** Princeton Univ. Press, Princeton (1993).

[10] L. Robbiano, M. Sweedler, *Subalgebra bases*, in Commutative Algebra, Lecture Notes in Math. **1430**, Springer-Verlag, Berlin (1990), 61-87.

[11] B. Sturmfels, *Gröbner bases and Convex Polytopes*, A M S, Providence RI, (1995).

Dipartimento di Matematica e Informatica
Università of Catania
Italy.
e-mail: lmarino@dmi.unict.it