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SOME RESULTS FOR HADAMARD-TYPE INEQUALITIES IN QUANTUM CALCULUS

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In this paper, we establish the q-analogue of Hermite-Hadamard inequalities for some convex type functions.

1. Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, I is an interval, is said to be a convex function on I if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0,1]$. If the reversed inequality in (1) holds, then f is said to be concave. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{2}$$

is known in the literature as Hermite-Hadamard inequality for convex mappings [7].

Some basic definitions can be given as follows:

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Definition 1.1 ([8])**.** Let $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex in the second sense if

$$f(tx+(1-t)y) \leq t^s f(x)+(1-t)^s f(y) \tag{3}$$

for all $x,y \in [0,\infty)$ and $t \in [0,1]$. This class of s-convex functions is usually denoted by K_s^2.

Definition 1.2 ([20])**.** Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function and $[0,1] \subset J$. We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is h-convex function, or that f belong to the class $SX(h,I)$, if f is nonnegative and for all $x,y \in I$ and $\alpha \in (0,1)$ we have

$$f(\alpha x+(1-\alpha)y) \leq h(\alpha)f(x)+h(1-\alpha)f(y). \tag{4}$$

If inequality (4) is reversed, then f is said to be h-concave, i.e. $f \in SV(h,I)$. Obviously, if $h(t)=t$, then all nonnegative convex functions belong to $SX(h,I)$ and all nonnegative concave functions belong to $SV(h,I)$; and if $h(t)=t^s$, where $s \in (0,1)$, then $SX(h,I) \supseteq K_s^2$.

In [19], G. H. Toader defined the concept of m-convexity as the following:

Definition 1.3. The function $f : [0,b] \to \mathbb{R}$ is said to be m-convex, where $m \in [0,1]$, if for every $x,y \in [0,b]$ and $t \in [0,1]$ we have:

$$f(tx+m(1-t)y) \leq tf(x)+m(1-t)f(y). \tag{5}$$

Denote by $K_m(b)$ the set of the m-convex functions on $[0,b]$.

In [12], V. G. Miheşan introduced the class of (s,m)-convex functions as the following:

Definition 1.4. The function $f : [0,b] \to \mathbb{R}$ is said to be (s,m)-convex, where $(s,m) \in (0,1]^2$, if for every $x,y \in [0,b]$ and $t \in [0,1]$ we have

$$f(tx+m(1-t)y) \leq t^s f(x)+m(1-t^s)f(y). \tag{6}$$

Denote by $K_m^s(b)$ the set of the (s,m)-convex functions on $[0,b]$. If we choose $(s,m)=(1,m)$, it can be easily seen that (s,m)-convexity reduces to m-convexity and for $(s,m)=(1,1)$, we have ordinary convex functions on $[0,b]$. For the recent results based on the above definition see the papers ([1, 12–15, 18]). In [4], S. S. Dragomir and N. M. Ionescu introduced the following class of functions:

Definition 1.5. Let $g : I \to \mathbb{R}$ be a given convex function on the interval I from $\mathbb{R}$. The real function $f : I \to \mathbb{R}$ is called g-convex dominated on I if the following condition is satisfied:

$$\begin{aligned} &|tf(x)+(1-t)f(y)-f(tx+(1-t)y)| \\ &\leq tg(x)+(1-t)g(y)-g(tx+(1-t)y) \end{aligned} \tag{7}$$

for all $x,y \in I$ and $t \in [0,1]$.

The following theorems are some known results obtained in recent years:In [3], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex functions in the second sense:

Theorem 1.6 ([3]). *Suppose that $f:[0,\infty)\to[0,\infty)$ is an s-convex function in the second sense, where $s\in(0,1]$ and let $a,b\in[0,\infty)$, $a<b$. If $f\in L^1[0,1]$, then the following inequalities hold:*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{s+1}, \tag{8}$$

the constant $k=\frac{1}{s+1}$ is the best possible in the second inequality in (8). The above inequalities are sharp.

In[17], Sarikaya, Saglam and Yildrim proved some Hadamard-type inequalities for h-convex functions:

Theorem 1.7 ([17]). *Let $f\in SX(h,I)$, $a,b\in I$, with $a<b$ and $f\in L^1[a,b]$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le[f(a)+f(b)]\int_0^1 h(\alpha)d\alpha. \tag{9}$$

In [6], the following inequality of Hermite-Hadamard type for m-convex functions holds:

Theorem 1.8 ([6]). *Let $f:[0,\infty)\to\mathbb{R}$ be a m-convex function with $m\in(0,1]$. If $0\le a<b<\infty$ and $f\in L^1[a,b]$, then one has the inequality:*

$$\frac{1}{b-a}\int_a^b f(x)dx\le\min\left\{\frac{f(a)+mf\left(\frac{b}{m}\right)}{2},\frac{f(b)+mf\left(\frac{a}{m}\right)}{2}\right\}. \tag{10}$$

For $0<q<1$, the q-Jackson integral from 0 to b is defined by [10]

$$\int_0^b f(x)d_qx=(1-q)b\sum_{n=0}^{\infty}f(bq^n)q^n \tag{11}$$

provided the sum converge absolutely. The q-Jackson integral in a generic interval $[a,b]$ is given by [10]

$$\int_a^b f(x)d_qx=\int_0^b f(x)d_qx-\int_0^a f(x)d_qx. \tag{12}$$

In [16], the authors presented a Riemann-type q-integral by

$$R_q(f;a,b)=(b-a)(1-q)\sum_{k=0}^{\infty}f(a+(b-a)q^k)q^k \tag{13}$$

The aim of this work is to divide the interval in two parts, then we can get another definition from the Riemman-type q-integral:

$$\begin{aligned}
&\frac{2}{b-a}\int_a^b f(x)d_q^R x \\
&= (1-q)\sum_{k=0}^{\infty}\left(f\left(\frac{a+b}{2}+q^k\left(\frac{b-a}{2}\right)\right)+f\left(\frac{a+b}{2}-q^k\left(\frac{b-a}{2}\right)\right)\right)q^k
\end{aligned}$$

From the q-Jackson integral we can write:

$$\begin{aligned}
\frac{2}{b-a}\int_a^b f(x)d_q^R x &= \int_{-1}^1 f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)d_q t \\
&= \int_{-1}^1 f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)d_q t.
\end{aligned}$$

In recent years, many authors have studied several inequalities connected to this famous integral inequality (2). For some results which generalize, improve and extend the inequality (2) (see [2], [3], [5], [6], [17], [18]). S.S. Dragomir ([2],[6]) proved several Hermite-Hadamrd type inequalities for m-convex functions. S.S. Dragomir and S. Fitzpatrick [3] proved a variant of Hadamard's inequality which holds for s-convex functions in the second sense. S.S Dragomir, C.E.M. Pearce and J.E. Pečarić [5] proved the Hermite-Hadamrd's inequality for convex-dominated functions. M.Z. Sarikaya, A. Saglam and H. Yildirim [17] proved some Hadamard-type inequalities for h-convex functions. E. Set, M. Sardari, M.E. Ozdemir and J. Rooin [18] proved the Hermite-Hadamard for (s,m)-convex functions. For more inequalities on convex functions see also the references in the above cited papers.

The main purpose of this paper is to establish the q-analogue of the Hermite-Hadamard inequality for some inequalities proved in ([2], [3], [5], [6], [17], [18]).

2. Main Results

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function. Then one has the inequalities:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)d_q^R t\leq\frac{f(a)+f(b)}{2}.$$

Proof. According to the definition of convex function we have

$$f\left(\frac{x+y}{2}\right)\leq\frac{f(x)}{2}+\frac{f(y)}{2}$$

for all $x \in [a,b]$.
Choose $x = \frac{1-t}{2}a + \frac{1+t}{2}b$ and $y = \frac{1+t}{2}a + \frac{1-t}{2}b$, $t \in [-1,1]$. We obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) + \frac{1}{2}f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)$$

for all $t \in [-1,1]$.
Integrating over $t \in [-1,1]$, we obtain

$$\begin{aligned} &f\left(\frac{a+b}{2}\right)\int_{-1}^{1} d_qt \\ &\leq \frac{1}{2}\int_{-1}^{1} f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) d_qt + \frac{1}{2}\int_{-1}^{1} f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) d_qt \end{aligned}$$

then

$$2f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_a^b f(t)d_q^R t$$

which proves the first inequality. The proof of the second inequality is given by

$$f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \leq \left(\frac{1-t}{2}\right)f(a) + \left(\frac{1+t}{2}\right)f(b).$$

We integrate t over $[-1,1]$, we get

$$\frac{2}{b-a}\int_a^b f(t)d_q^R t \leq f(a)\int_{-1}^{1}\frac{1-t}{2}d_qt + f(b)\int_{-1}^{1}\frac{1+t}{2}d_qt.$$

Then

$$\frac{2}{b-a}\int_a^b f(x)d_q^R x \leq f(a) + f(b).$$

The proof of Theorem 2.1 is completed. □

Theorem 2.2. *Let $f : [a,b] \to \mathbb{R}$ be a s-convex function. Then one has the following:*

$$2^{s-1}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)d_q^R t \leq \frac{(2+q)^s + q^s}{2^{s+1}[2]_q^s}(f(a)+f(b)),$$

where $[2]_q = 1+q$.

Proof. According to the definition of s-convex function with $x = \frac{1-t}{2}a + \frac{1+t}{2}b$ and $y = \frac{1+t}{2}a + \frac{1-t}{2}b$, $t \in [-1,1]$, we obtain

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) + \left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)}{2}\right)$$
$$\leq \frac{1}{2^s} f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) + \frac{1}{2^s} f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)$$

Integrating with respect t over $[-1,1]$, we obtain

$$2f\left(\frac{a+b}{2}\right) \leq \frac{1}{2^s}\frac{4}{b-a}\int_a^b f(t)d_q^R t$$

the first inequality follows.
Secondly, we have

$$f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \leq \left(\frac{1-t}{2}\right)^s f(a) + \left(\frac{1+t}{2}\right)^s f(b)$$

then, integrating this inequality over $t \in [-1,1]$, we obtain

$$\frac{2}{b-a}\int_a^b f(t)d_q^R t \leq \frac{f(a)}{2^s}\int_{-1}^1 (1-t)^s d_q t + \frac{f(b)}{2^s}\int_{-1}^1 (1+t)^s d_q t.$$

Moreover,

$$\int_{-1}^1 (1+t)^s d_q t = \int_{-1}^0 (1+t)^s d_q t + \int_0^1 (1+t)^s d_q t$$

using the q-Hölder's inequality, we get

$$\frac{1}{b-a}\int_a^b f(t)d_q^R t \leq \frac{(2+q)^s + q^s}{2^{s+1}[2]_q^s}(f(a)+f(b))$$

The result is thus proved. □

Remark 2.3. Applying Theorem 2.2 for $s = 1$, we obtain Theorem 2.1.

Theorem 2.4. *Let $f \in SX(h,I)$, $a,b \in I$, with $a < b$ and $f \in L^1[a,b]$, then*

$$\frac{1}{h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_a^b f(t)d_q^R t \leq (f(a)+f(b))\int_0^1 h(t)d_q^R t.$$

Proof. Since $f \in SX(h,I)$ with $x = \frac{1-t}{2}a + \frac{1+t}{2}b$, $y = \frac{1+t}{2}a + \frac{1-t}{2}b$, and $\alpha = \frac{1}{2}$, we obtain

$$f\left(\frac{a+b}{2}\right) \le h\left(\frac{1}{2}\right)\left[f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)+f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right].$$

Integrating over $t \in [-1,1]$, we obtain

$$\begin{aligned}&2f\left(\frac{a+b}{2}\right)\\ &\le h\left(\frac{1}{2}\right)\left[\int_{-1}^{1} f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)d_qt+\int_{-1}^{1} f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)d_qt\right].\end{aligned}$$

Then

$$f\left(\frac{a+b}{2}\right) \le h\left(\frac{1}{2}\right)\frac{2}{b-a}\int_a^b f(t)d_q^R t$$

and the first inequality is proved. Now the proof of second inequality is given by

$$f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right) \le h\left(\frac{1-t}{2}\right)f(a)+h\left(\frac{1+t}{2}\right)f(b).$$

We integrate t on $[-1,1]$, we obtain

$$\begin{aligned}&\int_{-1}^{1} f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)d_qt\\ &\le f(a)\int_{-1}^{1} h\left(\frac{1-t}{2}\right)d_qt+f(b)\int_{-1}^{1} h\left(\frac{1+t}{2}\right)d_qt.\end{aligned}$$

However,

$$\int_{-1}^{1} h\left(\frac{1-t}{2}\right)d_qt=\int_{-1}^{1} h\left(\frac{1+t}{2}\right)d_qt=\int_0^1 h(t)d_q^R t$$

then

$$\frac{2}{b-a}\int_a^b f(t)d_q^R t \le (f(a)+f(b))\int_0^1 h(t)d_q^R t. \tag{14}$$

Theorem 2.4 is thus proved. □

Remark 2.5. Applying Theorem 2.4 for $h(t) = t^s$, we obtain Theorem 2.2.

Theorem 2.6. *Let $f : [0,\infty) \to \mathbb{R}$ be an m-convex function with $m \in [0,1]$. If $0 \le a < b < \infty$ and $f \in L^1[a,b]$, then one has the inequality:*

$$\frac{1}{b-a}\int_a^b f(t)d_q^R t \le \min\left\{\frac{f(a)+mf\left(\frac{b}{m}\right)}{2},\frac{f(b)+mf\left(\frac{a}{m}\right)}{2}\right\}.$$

Proof. Since f is m-convex, we have

$$f\left(\frac{1-t}{2}x+m\frac{1+t}{2}y\right)\leq\left(\frac{1-t}{2}\right)f(x)+m\left(\frac{1+t}{2}\right)f(y),$$

for all $x,y\geq 0$ and $t\in[-1,1]$, which gives

$$f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\leq\left(\frac{1-t}{2}\right)f(a)+m\left(\frac{1+t}{2}\right)f\left(\frac{b}{m}\right)$$

and

$$f\left(\frac{1-t}{2}b+\frac{1+t}{2}a\right)\leq\left(\frac{1-t}{2}\right)f(b)+m\left(\frac{1+t}{2}\right)f\left(\frac{a}{m}\right)$$

then, by integrating both sides with respect t over $[-1,1]$, we obtain

$$\frac{1}{b-a}\int_a^b f(t)d_q^R t\leq\frac{f(a)+mf\left(\frac{b}{m}\right)}{2}$$

and

$$\frac{1}{b-a}\int_a^b f(t)d_q^R t\leq\frac{f(b)+mf\left(\frac{a}{m}\right)}{2}.$$

So the proof is completed. □

Theorem 2.7. *Let $f:[0,\infty)\to\mathbb{R}$ be a m-convex function with $m\in(0,1]$ and $0\leq a<b$. If $f\in L^1[a,b]$, then one has the inequalities*

$$\begin{aligned}f\left(\frac{a+b}{2}\right)&\leq\frac{1}{b-a}\int_a^b\frac{f(t)+mf\left(\frac{t}{m}\right)}{2}d_q^R t\\&\leq\frac{1}{4}\left[f(a)+mf\left(\frac{b}{m}\right)+mf\left(\frac{a}{m}\right)+m^2 f\left(\frac{b}{m^2}\right)\right]\end{aligned}$$

Proof. By the m-convexity of f we have

$$f\left(\frac{x+y}{2}\right)\leq\frac{1}{2}\left[f(x)+mf\left(\frac{y}{m}\right)\right]$$

for all $x,y\in[0,\infty)$.
Choose $x=\frac{1-t}{2}a+\frac{1+t}{2}b,\ \ y=\frac{1+t}{2}a+\frac{1-t}{2}b$, we obtain

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{2}\left[f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)+mf\left(\left(\frac{1+t}{2}\right)\left(\frac{a}{m}\right)+\left(\frac{1-t}{2}\right)\left(\frac{b}{m}\right)\right)\right]$$

for all $t \in [-1,1]$.
Integrating on $[-1,1]$

$$2f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\int_{-1}^{1} f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right) d_q t + \frac{m}{2}\int_{-1}^{1} f\left(\left(\frac{1+t}{2}\right)\left(\frac{a}{m}\right)+\left(\frac{1-t}{2}\right)\left(\frac{b}{m}\right)\right) d_q t$$

then

$$2f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left[\frac{2}{b-a}\int_{a}^{b} f(t) d_q^R t + \frac{2m}{b-a}\int_{a}^{b} f\left(\frac{t}{m}\right) d_q^R t\right]$$

and we get the proof of the first inequality. Now, by using the m-convexity of f, we get

$$\frac{1}{2}\left[f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)+mf\left(\left(\frac{1+t}{2}\right)\left(\frac{a}{m}\right)+\left(\frac{1-t}{2}\right)\left(\frac{b}{m}\right)\right)\right] \leq \frac{1}{2}\left[\left(\frac{1-t}{2}\right)f(a)+m\left(\frac{1+t}{2}\right)f\left(\frac{b}{m}\right) + m\left(\frac{1+t}{2}\right)f\left(\frac{a}{m}\right)+m^2\left(\frac{1-t}{2}\right)f\left(\frac{b}{m^2}\right)\right]$$

for all $t \in [-1,1]$. We integrate over t on $[-1,1]$, we obtain

$$\frac{1}{b-a}\left(\int_{a}^{b} f(t) d_q^R t + m\int_{a}^{b} f\left(\frac{t}{m}\right) d_q^R t\right) \leq \frac{1}{2}\left[f(a)+mf\left(\frac{b}{m}\right)+mf\left(\frac{a}{m}\right)+m^2 f\left(\frac{b}{m^2}\right)\right]. \qquad \square$$

Theorem 2.8. *Let $f : [0,\infty) \to \mathbb{R}$ be a m-convex function with $m \in (0,1]$. If $f \in L^1[am,b]$ where $0 \leq a < b$, then one has the inequality:*

$$\frac{1}{mb-a}\int_{a}^{mb} f(t) d_q^R t + \frac{1}{b-ma}\int_{ma}^{b} f(t) d_q^R t \leq (m+1)\frac{f(a)+f(b)}{2}$$

Proof. By the m-convexity of f we get

$$f\left(\frac{1-t}{2}a+m\left(\frac{1+t}{2}\right)b\right)+f\left(\frac{1+t}{2}a+m\left(\frac{1-t}{2}\right)b\right) + f\left(\frac{1-t}{2}b+m\left(\frac{1+t}{2}\right)a\right)+f\left(\frac{1+t}{2}b+m\left(\frac{1-t}{2}\right)a\right) \leq (m+1)\left(f(a)+f(b)\right).$$

Integrating over $t \in [-1,1]$, we obtain

$$\begin{aligned}&\int_{-1}^{1} f\left(\frac{1-t}{2}a+m\left(\frac{1+t}{2}\right)b\right)d_qt+\int_{-1}^{1} f\left(\frac{1+t}{2}a+m\left(\frac{1-t}{2}\right)b\right)d_qt\\&+\int_{-1}^{1} f\left(\frac{1-t}{2}b+m\left(\frac{1+t}{2}\right)a\right)d_qt+\int_{-1}^{1} f\left(\frac{1+t}{2}b+m\left(\frac{1-t}{2}\right)a\right)d_qt\\&\leq 2(m+1)\left(f(a)+f(b)\right).\end{aligned}$$

then

$$\begin{aligned}\frac{1}{mb-a}\int_a^{mb} f(t)d_q^Rt+\frac{1}{mb-a}\int_a^{mb} f(t)d_q^Rt+\frac{1}{b-ma}\int_{ma}^{b} f(t)d_q^Rt\\+\frac{1}{b-ma}\int_{ma}^{b} f(t)d_q^Rt\leq (m+1)\left(f(a)+f(b)\right)\qquad\square\end{aligned}$$

and the result is proved.

Theorem 2.9. *Let $f:[0,\infty)\to\mathbb{R}$ be an (s,m)-convex function with $(s,m)\in(0,1)^2$. If $0\leq a<b<\infty$ and $f\in L^1[a,b]$, then one has the inequality:*

$$\frac{2}{b-a}\int_a^b f(t)d_q^Rt\leq C_s\min\{L(a,b),L(b,a)\}$$

where $C_s=\dfrac{q^s+(2+q)^s}{2^s[2]_q^s}$, $\quad L(a,b)=f(a)-mf\left(\frac{b}{m}\right)+\frac{2m}{C_s}f\left(\frac{b}{m}\right)$.

Proof. Since f is an (s,m)-convex function on $[a,b]$, we know that for any $t\in[-1,1]$

$$f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\leq\left(\frac{1+t}{2}\right)^s f(a)+m\left(1-\left(\frac{1+t}{2}\right)^s\right)f\left(\frac{b}{m}\right)\tag{15}$$

and

$$f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\leq\left(\frac{1+t}{2}\right)^s f(b)+m\left(1-\left(\frac{1+t}{2}\right)^s\right)f\left(\frac{a}{m}\right).\tag{16}$$

By integrating both side (15) and (16) with respect to t over $[-1,1]$, we get

$$\begin{aligned}&\int_{-1}^{1} f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)d_qt\\&\qquad\leq f(a)\int_{-1}^{1}\left(\frac{1+t}{2}\right)^s d_qt+mf\left(\frac{b}{m}\right)\int_{-1}^{1}\left(1-\left(\frac{1+t}{2}\right)^s\right)d_qt\end{aligned}$$

and

$$\int_{-1}^{1} f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right) d_q t$$
$$\leq f(b)\int_{-1}^{1}\left(\frac{1+t}{2}\right)^s d_q t+mf\left(\frac{a}{m}\right)\int_{-1}^{1}\left(1-\left(\frac{1+t}{2}\right)^s\right) d_q t.$$

Moreover,

$$\int_{-1}^{1}\left(\frac{1+t}{2}\right)^s d_q t=\int_{-1}^{0}\left(\frac{1+t}{2}\right)^s d_q t+\int_{0}^{1}\left(\frac{1+t}{2}\right)^s d_q t.$$

Then by the q-Hölder's inequality, we have

$$\frac{2}{b-a}\int_a^b f(t)d_q^R t \leq \left(f(a)-mf\left(\frac{b}{m}\right)\right)\left(\frac{q^s+(2+q)^s}{2^s[2]_q^s}\right)+2mf\left(\frac{b}{m}\right)$$

and

$$\frac{2}{b-a}\int_a^b f(t)d_q^R t \leq \left(f(b)-mf\left(\frac{a}{m}\right)\right)\left(\frac{q^s+(2+q)^s}{2^s[2]_q^s}\right)+2mf\left(\frac{a}{m}\right).$$

The proof of Theorem 2.9 is completed. □

Remark 2.10. Applying Theorem 2.9 for $s=1$, we obtain Theorem 2.6.

Theorem 2.11. *Let $g: I \to \mathbb{R}$ be a convex mapping on I and $f: I \to \mathbb{R}$ a g-convex-dominated mapping. Then, for all $a,b \in I$ with $a<b$, one has the inequalities:*

$$\frac{1}{b-a}\int_a^b (g+f)(t)d_q^R t \leq \frac{(f+g)(a)+(f+g)(b)}{2}$$

and

$$\frac{1}{b-a}\int_a^b (g-f)(t)d_q^R t \leq \frac{(g-f)(a)+(g-f)(b)}{2}.$$

Proof. From (7), we can write

$$g\left(\frac{1-t}{2}x+\frac{1+t}{2}y\right)-\left(\frac{1-t}{2}\right)g(x)-\left(\frac{1+t}{2}\right)g(y)$$
$$\leq \left(\frac{1-t}{2}\right)f(x)+\left(\frac{1+t}{2}\right)f(y)-f\left(\frac{1-t}{2}x+\frac{1+t}{2}y\right)$$
$$\leq \left(\frac{1-t}{2}\right)g(x)+\left(\frac{1+t}{2}\right)g(y)-g\left(\frac{1-t}{2}x+\frac{1+t}{2}y\right)$$

for all $x,y \in I$ and $t \in [-1,1]$, or additionally, with

$$f\left(\frac{1-t}{2}x+\frac{1+t}{2}y\right)+g\left(\frac{1-t}{2}x+\frac{1+t}{2}y\right) \leq \left(\frac{1-t}{2}\right)(f(x)+g(x))+\left(\frac{1+t}{2}\right)(f(y)+g(y))$$

and

$$g\left(\frac{1-t}{2}x+\frac{1+t}{2}y\right)-f\left(\frac{1-t}{2}x+\frac{1+t}{2}y\right) \leq \left(\frac{1-t}{2}\right)(g(x)-f(x))+\left(\frac{1+t}{2}\right)(g(y)-f(y))$$

for all $x,y \in I$ and $t \in [-1,1]$. Then, by using $x=a$, $y=b$ and integrating with respect t over $[-1,1]$, we get

$$\int_{-1}^{1} f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)d_qt+\int_{-1}^{1} g\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)d_qt \leq (f(a)+g(a))\int_{-1}^{1}\left(\frac{1-t}{2}\right)d_qt+(f(b)+g(b))\int_{-1}^{1}\left(\frac{1+t}{2}\right)d_qt$$

and

$$\int_{-1}^{1} g\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)d_qt-\int_{-1}^{1} f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)d_qt \leq (g(a)-f(a))\int_{-1}^{1}\left(\frac{1-t}{2}\right)d_qt+(g(b)-f(b))\int_{-1}^{1}\left(\frac{1+t}{2}\right)d_qt$$

therefore

$$\frac{1}{b-a}\int_a^b (f+g)(t)d_q^Rt \leq \frac{(f+g)(a)+(f+g)(b)}{2}$$

and

$$\frac{1}{b-a}\int_a^b (g-f)(t)d_q^Rt \leq \frac{(g-f)(a)+(g-f)(b)}{2}.$$

Then, the proof of Theorem 2.11 is completed. □

Theorem 2.12. *Let $g : I \to \mathbb{R}$ be a convex mapping on I and $f : I \to \mathbb{R}$ a g-convex-dominated mapping. Then, for all $a,b \in I$ with $a < b$, one has the inequalities:*

$$\left|\frac{1}{b-a}\int_a^b f(t)d_q^Rt - f\left(\frac{a+b}{2}\right)\right| \leq \frac{1}{b-a}\int_a^b g(t)d_q^Rt - g\left(\frac{a+b}{2}\right) \tag{17}$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)d_q^R t\right| \le \frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)d_q^R t. \tag{18}$$

Proof. Since f is g-convex dominated, we have

$$\left|\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right| \le \frac{g(x)+g(y)}{2}-g\left(\frac{x+y}{2}\right)$$

for all $x,y\in[a,b]$.
Choose $x=\frac{1-t}{2}a+\frac{1+t}{2}b,\ y=\frac{1+t}{2}a+\frac{1-t}{2}b$, $t\in[-1,1]$. Then we get

$$\left|\frac{f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)+f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)}{2}-f\left(\frac{a+b}{2}\right)\right| \le \frac{g\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)+g\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)}{2}-g\left(\frac{a+b}{2}\right).$$

Now, we integrate t over $[-1,1]$ we obtain

$$\left|\frac{\int_{-1}^1 f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)d_qt+\int_{-1}^1 f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)d_qt}{2}-2f\left(\frac{a+b}{2}\right)\right| \le \frac{\int_{-1}^1 g\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)d_qt+\int_{-1}^1 g\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)d_qt}{2}-2g\left(\frac{a+b}{2}\right),$$

therefore

$$\left|\frac{1}{b-a}\left(\int_a^b f(t)d_q^R t+\int_a^b f(x)d_q^R t\right)-2f\left(\frac{a+b}{2}\right)\right| \le \frac{1}{b-a}\left(\int_a^b g(t)d_q^R t+\int_a^b g(t)d_q^R t\right)-2g\left(\frac{a+b}{2}\right)$$

and the inequality (17) is proved.
Secondly, the proof of the second inequality is given by

$$\left|\left(\frac{1-t}{2}\right)f(x)+\left(\frac{1+t}{2}\right)f(y)-f\left(\frac{1-t}{2}x+\frac{1+t}{2}y\right)\right| \le \left(\frac{1-t}{2}\right)g(x)+\left(\frac{1+t}{2}\right)g(y)-g\left(\frac{1-t}{2}x+\frac{1+t}{2}y\right).$$

Then, by integrating over $t \in [-1,1]$ with $x = a$ and $y = b$, we obtain

$$\left| f(a)\int_{-1}^{1}\left(\frac{1-t}{2}\right)d_qt + f(b)\int_{-1}^{1}\left(\frac{1+t}{2}\right) - \int_{-1}^{1} f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)d_qt\right|$$
$$\leq g(a)\int_{-1}^{1}\left(\frac{1-t}{2}\right)d_qt + g(b)\int_{-1}^{1}\left(\frac{1+t}{2}\right)d_qt - \int_{-1}^{1} g\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)d_qt.$$

Therefore,

$$\left| f(a) + f(b) - \frac{2}{b-a}\int_a^b f(t)d_q^Rt\right| \leq g(a) + g(b) - \frac{2}{b-a}\int_a^b g(t)d_q^Rt$$

then, we deduce the inequalities (18). □

Remark 2.13. From Theorem 2.11 and Theorem 2.12, we deduce

$$\begin{aligned}(f+g)\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b (f+g)(t)d_q^Rt\\ &\leq \frac{(f+g)(a)+(f+g)(b)}{2}\end{aligned}$$

and

$$\begin{aligned}(g-f)\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b (g-f)(t)d_q^Rt\\ &\leq \frac{(g-f)(a)+(g-f)(b)}{2}.\end{aligned}$$

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