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# ON CONVOLUTION PROPERTIES FOR CERTAIN CLASSES OF p-VALENT MEROMORPHIC FUNCTIONS DEFINED BY LINEAR OPERATOR

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In this paper, by making use of convolution, we obtain some interesting results for certain family of meromorphic p-valent functions defined by new linear operator.

## 1. Introduction

Let $\Sigma_p$ denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \quad (p \in \mathbb{N} = \{1, 2, \dots\}), \tag{1}$$

which are analytic and p-valent in the punctured unit disc $U^* = \{z : z \in \mathbb{C}$ and $0 < |z| < 1\}$. If $f(z)$ and $g(z)$ are analytic in $U = U^* \cup \{0\}$, we say that $f(z)$ is subordinate to $g(z)$, written $f \prec g$ or $f(z) \prec g(z) \quad (z \in U)$, if there exists a Schwarz function $w(z)$ in $U$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z)), \ (z \in U)$. Furthermore, if $g(z)$ is univalent in $U$, then the following equivalence relationship holds true (see [6]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

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For functions $f(z) \in \sum_p$, given by (1) and $g(z) \in \sum_p$ defined by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p} \quad (p \in \mathbb{N}), \tag{2}$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} b_{k-p} z^{k-p} = (g * f)(z). \tag{3}$$

Using the operator $Q^{\alpha}_{\beta,p} : \sum_p \to \sum_p (\alpha \geq 0; \beta > -1; p \in \mathbb{N})$ defined by Aqlan et al. [1], where:

$$Q^{\alpha}_{\beta,p} f(z) = \begin{cases} z^{-p} + \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} \sum\limits_{k=1}^{\infty} \frac{\Gamma(k+\beta)}{\Gamma(k+\beta+\alpha)} a_{k-p} z^{k-p} & (\alpha > 0) \\ f(z) & (\alpha = 0) \end{cases},$$

Mostafa [8] (see also [9]) defined the operator $H^{\alpha}_{p,\beta,\mu} : \Sigma_p \to \Sigma_p$ as follows:
For $G^{\alpha}_{\beta,p}$, given by

$$G^{\alpha}_{\beta,p} = z^{-p} + \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} \sum_{k=1}^{\infty} \frac{\Gamma(k+\beta)}{\Gamma(k+\beta+\alpha)} z^{k-p} \tag{4}$$

let $G^{\alpha *}_{\beta,p,\mu}$ be defined by

$$G^{\alpha}_{\beta,p}(z) * G^{\alpha *}_{\beta,p,\mu}(z) = \frac{1}{z^p (1-z)^{\mu}} \quad (\mu > 0; p \in \mathbb{N}). \tag{5}$$

Then

$$H^{\alpha}_{p,\beta,\mu} f(z) = G^{\alpha *}_{\beta,p}(z) * f(z) \ \ (f \in \Sigma_p). \tag{6}$$

Using (4)-(6), we have

$$H^{\alpha}_{p,\beta,\mu} f(z) = z^{-p} + \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \sum_{k=1}^{\infty} \frac{\Gamma(k+\beta+\alpha)(\mu)_k}{\Gamma(k+\beta)(1)_k} a_{k-p} z^{k-p}, \tag{7}$$

where $f \in \Sigma_p$ is in the form (1) and $(\nu)_n$ denotes the Pochhammer symbol given by

$$(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1 & (n = 0) \\ \nu(\nu+1)\ldots(\nu+n-1) & (n \in \mathbb{N}). \end{cases}$$

It is readily verified from (7) that

$$z(H^{\alpha}_{p,\beta,\mu} f(z))' = (\alpha+\beta) H^{\alpha+1}_{p,\beta,\mu} f(z) - (\alpha+\beta+p) H^{\alpha}_{p,\beta,\mu} f(z) \tag{8}$$

and

$$z(H^{\alpha}_{p,\beta,\mu}f(z))' = \mu H^{\alpha}_{p,\beta,\mu+1}f(z) - (\mu+p)H^{\alpha}_{p,\beta,\mu}f(z)\,. \tag{9}$$

It is noticed that, putting $\mu = 1$ in (7), we obtain the operator

$$H^{\alpha}_{p,\beta,1}f(z) = H^{\alpha}_{p,\beta}f(z) = z^{-p} + \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\sum_{k=1}^{\infty}\frac{\Gamma(k+\alpha+\beta)}{\Gamma(k+\beta)}a_{k-p}z^{k-p}, \tag{10}$$

and

$$H^{0}_{p,\beta,1}f(z) = f(z).$$

For $-1 \leq A < B \leq 1, B \geq 0$ and $z \in U^*$, Mogra [8] defined the class

$$S^*_p[A,B] = \left\{f \in \sum\nolimits_p : -\frac{zf'(z)}{f(z)} \prec p\frac{1+Az}{1+Bz}, z \in U^*\right\} \tag{11}$$

and Srivastava et al. [12] defined the class

$$\sum K_p[A,B] = \left\{f \in \sum\nolimits_p : -[1+\frac{zf''(z)}{f'(z)}] \prec p\frac{1+Az}{1+Bz}, z \in U^*\right\}. \tag{12}$$

It is clear that

$$f(z) \in \sum K_p[A,B] \Leftrightarrow -\frac{zf'(z)}{p} \in S^*_p[A,B], \tag{13}$$

$$S^*_1[2\alpha-1,1] = \sum S^*(\alpha) \quad \text{(see Juneja and Reddy [5])},$$

$$K_1[2\alpha-1,1] = \sum K(\alpha)\ (0 \leq \alpha < 1) \quad \text{(see Srivastava et al. [12])},$$

$$S^*_p[\frac{2\alpha}{p}-1,1] = \sum S^*_p(\alpha)\ (0 \leq \alpha < p) \quad \text{(see Aouf and Hossen [2])}$$

and

$$\sum K_p[\frac{2\alpha}{p}-1,1] = \sum K_p(\alpha)\ (0 \leq \alpha < p) \quad \text{(see Aouf and Srivastava [3])}.$$

Using the operator $H^{\alpha}_{p,\beta,\mu}$ and for $-1 \leq B < A \leq 1, \alpha \geq 0, \beta > -1, \mu > 0$ and $z \in U^*$ we define the classes $S^*_p(\alpha,\beta,\mu,A,B)$ and $K_p(\alpha,\beta,\mu,A,B)$ as follows:

$$S^*_p(\alpha,\beta,\mu,A,B) = \left\{f \in \sum\nolimits_p : H^{\alpha}_{p,\beta,\mu}f(z) \in \sum[p,A,B], z \in U\right\}, \tag{14}$$

and

$$K_p(\alpha,\beta,\mu,A,B) = \left\{f \in \sum\nolimits_p : H^{\alpha}_{p,\beta,\mu}f(z) \in \sum K_p[A,B], z \in U\right\}. \tag{15}$$

We notice that

$$f(z) \in K_p(\alpha,\beta,\mu,A,B) \Leftrightarrow -\frac{zf'(z)}{p} \in S^*_p(\alpha,\beta,\mu,A,B). \tag{16}$$

## 2. Main Results

Unless otherwise mentioned, we shall assume in this paper that $-1 \leq A < B \leq 1$, $0 \leq B < 1$, $\alpha \geq 0$, $\beta > -1$, $(\alpha+\beta) \neq 0$, $\mu > 0$, $0 < \theta < 2\pi$, $p \in \mathbb{N}$ and $z \in U^*$.

To prove our results we need the following lemmas.

**Lemma 2.1** ([10]). *The function $f(z)$ defined by* (1.1) *is in the class* $\sum[p,A,B]$ *if and only if*

$$z^p \left[ f(z) * \frac{1+(D-1)z}{z^p(1-z)^2} \right] \neq 0, \tag{17}$$

*where*

$$D = \frac{e^{-i\theta}+B}{p(A-B)}. \tag{18}$$

**Lemma 2.2** ([10]). *The function $f(z)$ defined by* (1) *is in the class* $\sum K_p[A,B]$ *if and only if*

$$z^p \left\{ f(z) * \left[ \frac{p-\{2+p-(p-1)(D-1)\}z-(p+1)(D-1)z^2}{pz^p(1-z)^3} \right] \right\} \neq 0.$$

**Lemma 2.3** ([4]). *Let $h$ be convex (univalent) in $U$, with $\Re[\beta h(z)+\gamma] > 0$ for all $z \in U$. If $p$ is analytic in $U$, with $p(0) = h(0)$, then*

$$p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} \prec h(z) \Rightarrow p(z) \prec h(z). \tag{19}$$

**Theorem 2.4.** *The function $f(z)$ defined by* (1) *is in the class* $S_p^*(\alpha,\beta,\mu,A,B)$ *if and only if*

$$1 + \sum_{k=1}^{\infty} \left[ \frac{ke^{-i\theta}+pA+(k-p)B}{p(A-B)} \right] \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \frac{\Gamma(k+\alpha+\beta)}{\Gamma(k+\beta)} a_k z^k \neq 0. \tag{20}$$

*Proof.* From Lemma 2.1, we find that $f(z) \in S_p^*(\alpha,\beta,\mu,A,B)$ if and only if

$$z^p \left[ H_{p,\beta,\mu}^{\alpha} f(z) * \frac{1+(D-1)z}{z^p(1-z)^2} \right] \neq 0.$$

Expanding $\frac{1+(D-1)z}{z^p(1-z)^2}$, we have (20) which completes the proof of Theorem 2.4. □

**Theorem 2.5.** *The function $f(z)$ defined by* (1) *is in the class* $K_p(\alpha,\beta,\mu,A,B)$*if and only if*

$$1 - \sum_{k=1}^{\infty} \left[ \frac{k\left[ke^{-i\theta}+pA+(k-p)B\right]}{p^2(A-B)} \right] \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \frac{\Gamma(k+\alpha+\beta)}{\Gamma(k+\beta)} a_k z^k \neq 0. \tag{21}$$

*Proof.* From Lemma 2.2, we find that $f(z) \in K_p(\alpha,\beta,\mu,A,B)$ if and only if

$$z^p\left\{H^{\alpha}_{p,\beta,\mu}f(z) * \left[\frac{p-\{2+p-(p-1)(D-1)\}z-(p+1)(D-1)z^2}{pz^p(1-z)^3}\right]\right\} \neq 0. \quad (22)$$

Now it can be easily shown that

$$z^{-p}(1-z)^{-3} = z^{-p} + \sum_{k=1}^{\infty}\frac{(k+1)(k+2)}{2}z^{k-p}, \quad (23)$$

$$z^{1-p}(1-z)^{-3} = \sum_{k=1}^{\infty}\frac{k(k+1)}{2}z^{k-p}, \quad (24)$$

$$z^{2-p}(1-z)^{-3} = \sum_{k=1}^{\infty}\frac{k(k-1)}{2}z^{k-p}. \quad (25)$$

Using (23)-(25) and (18) in (22), we have the desired result which completes the proof of Theorem 2.5. □

**Theorem 2.6.** *If the function $f$ defined by* (1) *belongs to the class $S^*_p(\alpha,\beta,\mu,A,B)$, then*

$$\sum_{k=1}^{\infty}[k+pA+(k-p)B]\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{\Gamma(k+\alpha+\beta)}{\Gamma(k+\beta)}|a_k| \leq p(A-B). \quad (26)$$

*Proof.* Since

$$\left|\frac{ke^{-i\theta}+pA+(k-p)B}{p(A-B)}\right| = \frac{\left|ke^{-i\theta}+pA+(k-p)B\right|}{p(A-B)} \leq \frac{k+pA+(k-p)B}{p(A-B)}$$

and

$$\left|1+\sum_{k=1}^{\infty}\frac{\left[ke^{-i\theta}+pA+(k-p)B\right]}{p(A-B)}\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{\Gamma(k+\alpha+\beta)}{\Gamma(k+\beta)}\left|a_kz^k\right|\right|$$
$$> 1-\sum_{k=1}^{\infty}\left|\frac{\left[ke^{-i\theta}+pA+(k-p)B\right]}{p(A-B)}\right|\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{\Gamma(k+\alpha+\beta)}{\Gamma(k+\beta)}|a_k|$$

the result follows from Theorem 2.4. □

Using the same technique, we can also prove the following theorem.

**Theorem 2.7.** *If the function $f$ defined by* (1) *belongs to the class $K_p(\alpha,\beta,\mu,A,B)$, then*

$$\sum_{k=1}^{\infty}k\,[k+pA+(k-p)B]\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{\Gamma(k+\alpha+\beta)}{\Gamma(k+\beta)}|a_k| \leq p^2(A-B). \quad (27)$$

**Theorem 2.8.** *Let the function* $f(z)$ *be defined by* (1). *If*

$$\frac{1+AB+(A+B)\cos\theta}{1+B^2+2B\cos\theta} \leq \alpha+\beta+p \tag{28}$$

*and* $f \in S_p^*(\alpha+1,\beta,\mu,A,B)$ *with* $H_{p,\beta,\mu}^{\alpha} f(z) \neq 0$*, then* $f \in S_p^*(\alpha,\beta,\mu,A,B)$.

*Proof.* Let $f(z) \in S_p^*(\alpha+1,\beta,\mu,A,B)$ and define the function

$$P(z) = -\frac{z\left(H_{p,\beta,\mu}^{\alpha} f(z)\right)'}{H_{p,\beta,\mu}^{\alpha} f(z)}, \tag{29}$$

we see that $P$ is analytic in U with $P(0)=1$. Using the identity (8) in (29), we have

$$\frac{H_{p,\beta,\mu}^{\alpha+1} f(z)}{H_{p,\beta,\mu}^{\alpha} f(z)} = -\frac{1}{\alpha+\beta}P(z) + \frac{\alpha+\beta+p}{\alpha+\beta}. \tag{30}$$

Differentiating (30) logarithmically and using (29), we have

$$-\frac{z\left(H_{p,\beta,\mu}^{\alpha+1} f(z)\right)'}{H_{p,\beta,\mu}^{\alpha+1} f(z)} = p(z) + \frac{zP'(z)}{-P(z)+\alpha+\beta+p} \prec \frac{1+Az}{1+Bz} = h(z). \tag{31}$$

Simple computations show that the inequalty $\Re\{-h(z)+\alpha+\beta+p\} > 0$ can be written in the form

$$\Re\frac{1+Az}{1+Bz} - (\alpha+\beta+p) < 0,$$

which is equivalent to (28). Since the function $h(z)$ is a convex function, then applying Lemma 2.3, we see that the subordination (31) implies $P(z) \prec h(z)$. This completes the proof of Theorem 2.8. □

**Theorem 2.9.** *Let the function* $f(z)$ *be defined by* (1). *If*

$$\frac{1+AB+(A+B)\cos\theta}{1+B^2+2B\cos\theta} \leq \mu+p \tag{32}$$

*and* $f \in S_p^*(\alpha,\beta,\mu+1,A,B)$ *with* $H_{p,\beta,\mu}^{\alpha} f(z) \neq 0$*, then* $f \in S_p^*(\alpha,\beta,\mu,A,B)$.

*Proof.* The proof follows in the same steps as that used in Theorem 2.8 and using the identity (9) instead of (8). □

Using (16) and the fact that

$$H_{p,\beta,\mu}^{\alpha}(-zf')(z) = -z\left(H_{p,\beta,\mu}^{\alpha} f(z)\right)',$$

Theorem 2.8 yields the following theorem.

**Theorem 2.10.** *Let the function* $f(z)$ *be defined by* (1). *If* (28) *holds and* $f(z) \in K_p(\alpha+1,\beta,\mu,A,B)$ *with* $H^{\alpha}_{p,\beta,\mu} f(z) \neq 0$*, then* $f(z) \in K_p(\alpha,\beta,\mu,A,B)$.

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