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HERMITE-HADAMARD TYPE FRACTIONAL INTEGRAL INEQUALITIES FOR GEOMETRIC-GEOMETRIC CONVEX FUNCTIONS

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By utilizing two fractional integral identities and elementary inequalities via geometric-geometric (*GG* for short) convex functions, we derive new type Hermite-Hadamard inequalities involving Hadamard fractional integrals. Some applications to special means of real numbers are given.

1. Introduction

Fractional calculus originally appeared in the letter between L'Hospital and Leibniz. Since 1695, Riemann, Liouville, Caputo, Hadamard and other famous mathematicians paid attention to study such a branch of mathematical analysis. Meanwhile, fractional calculus have been widely applied to the fields of electricity, biology, economics and signal and image processing [1–8].

The classical Hermite-Hadamard inequality was firstly discovered by Hermite in 1881 in the journal Mathesis, which provides a lower and an upper estimations for the integral average of any convex function defined on a compact

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interval, involving the midpoint and the endpoints of the domain. Very recently, many authors pay attention to study Hermite-Hadamard type inequalities involving Riemann-Liouville and Hadamard fractional integrals. For more recent results which generalize, improve, and extend the classical Hermite-Hadamard inequality, one can see [9–23] and the references therein.

For $f \in L[a,b]$, the Hadamard fractional integrals [1] $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha \in R^{+}$ with $a \geq 0$ are defined by

$$({}_HJ^{\alpha}_{a^+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t},\ (0<a<x\leq b),$$

and

$$({}_HJ^{\alpha}_{b^-}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \left(\ln\frac{t}{x}\right)^{\alpha-1} f(t)\frac{dt}{t},\ (0<a\leq x<b),$$

where $\Gamma(\cdot)$ is the Gamma function.

Recently, Wang et al. [17, 18] established the following two powerful fractional integral identities involving Hadamard fractional integrals.

Lemma 1.1 (see [17]). *Let $f:[a,b]\to R$ be a differentiable mapping on (a,b) with $0<a<b$. If $f'\in L[a,b]$, then the following equality for fractional integrals holds*

$$\begin{aligned}
&\frac{(\ln x-\ln a)^{\alpha}+(\ln b-\ln x)^{\alpha}}{\ln b-\ln a}f(x)-\frac{\Gamma(\alpha+1)}{\ln b-\ln a}[{}_HJ^{\alpha}_{x^-}f(a)+{}_HJ^{\alpha}_{x^+}f(b)]\\
&=\frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^{\alpha}e^{t\ln x+(1-t)\ln a}f'(e^{t\ln x+(1-t)\ln a})dt\\
&-\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^{\alpha}e^{t\ln x+(1-t)\ln b}f'(e^{t\ln x+(1-t)\ln b})dt\\
&=\frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^{\alpha}x^ta^{1-t}f'\left(x^ta^{1-t}\right)dt\\
&-\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^{\alpha}x^tb^{1-t}f'\left(x^ta^{1-t}\right)dt,
\end{aligned}$$

for any $x\in(a,b)$.

Lemma 1.2 (see [18]). *Let $f:[a,b]\to R$ be a differentiable mapping on (a,b) with $0<a<b$. If $f'\in L[a,b]$, then the following equality for fractional integrals holds:*

$$\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}[{}_HJ^{\alpha}_{a^+}f(b)+{}_HJ^{\alpha}_{b^-}f(a)]-f\left(\sqrt{ab}\right)=$$

$$=\frac{\ln b-\ln a}{2}\left[\int_0^1 ke^{t\ln a+(1-t)\ln b}f'(e^{t\ln a+(1-t)\ln b})dt\right.$$

$$-\int_0^1 [(1-t)^\alpha - t^\alpha] e^{t\ln a+(1-t)\ln b} f'(e^{t\ln a+(1-t)\ln b})dt\Big]$$

$$= \frac{\ln b - \ln a}{2}\Big[\int_0^1 k a^t b^{1-t} f'(a^t b^{1-t})dt - \int_0^1 [(1-t)^\alpha - t^\alpha] a^t b^{1-t} f'(a^t b^{1-t})dt\Big],$$

where

$$k = \begin{cases} 1, & 0 \le t < \frac{1}{2}, \\ -1, & \frac{1}{2} \le t < 1. \end{cases}$$

Meanwhile, Wang et al. [17] introduced a new classes of functions satisfying "s-e-condition".

Definition 1.3. A function $f : I \subset (0,\infty) \to R$ is said to satisfy s-e-condition if

$$f(e^{\lambda x+(1-\lambda)y}) \le \lambda^s f(e^x) + (1-\lambda)^s f(e^y)$$

for all $x, y \in I$, $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$.

With the help of the new concept of "s-e-condition", monotonicity and Lemma 1.1 and Lemma 1.2, some new Hermite-Hadamard inequalities involving Hadamard fractional integrals are established [17].

Motivated by our previous works [17, 18], we do not use the concept of "s-e-condition" and use the concept of GG-convex functions (see Definition 2.1), which maybe more suitable for solving such problem in some sense. We will use Lemma 1.1, Lemma 1.2 and elementary equalities and inequalities via GG-convex functions to derive new type Hermite-Hadamard inequalities involving Hadamard fractional integrals.

2. Definitions, elementary equalities and inequalities

Definition 2.1 (see [24, 25])**.** Let $f : I \subseteq R^+ \to R^+$. A function f is said to be GG-convex on I if for every $x, y \in I$ and $\lambda \in [0,1]$, we have

$$f(x^\lambda y^{1-\lambda}) \le [f(x)]^\lambda [f(y)]^{1-\lambda}. \tag{1}$$

Remark 2.2. By the arithmetic-geometric mean inequality, we have

$$[f(x)]^\lambda [f(y)]^{1-\lambda} \le \lambda f(x) + (1-\lambda) f(y). \tag{2}$$

Linking (1) and (2), we obtain

$$f(x^\lambda y^{1-\lambda}) \le \lambda f(x) + (1-\lambda) f(y),$$

which appears in the standard definition of geometric-arithmetic (GA for short) convex function [24]. So GG-convex function is GA-convex function.

Remark 2.3 (see [26])**.** The functions $f(x) = \cosh x$ and $g(x) = e^x$ are GG-convex on $(0,+\infty)$.

It follows Lemma 2.1 and Lemma 2.2 in [14, 17] that we give two singular integral equalities.

Lemma 2.4. *For $\alpha > 0$ and $k > 0$, we have*

$$I_f(\alpha,x,y) = \int_0^1 t^{\alpha-1} T_f^t(x,y)dt = T_f(x,y)\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln T_f(x,y))^{i-1}}{(\alpha)_i} < +\infty$$

where T is defined by

$$T_f(x,y) = \frac{x|f'(x)|}{y|f'(y)|},$$

and

$$(\alpha)_i = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+i-1).$$

Lemma 2.5. *For $\alpha > 0$ and $k > 0, z > 0$,we have*

$$I_f(a,x,y,z) = \int_0^z t^{\alpha-1} T_f^t(x,y)dt = z^\alpha T_f^z(x,y)\sum_{i=0}^{\infty}\frac{(-z\ln T_f(x,y))^{i-1}}{(\alpha)_i} < +\infty.$$

The following two inequalities are also needed.

Lemma 2.6 (see [14])**.** *For $A > B > 0$, it holds*

$$(A-B)^\theta \le A^\theta - B^\theta \quad \text{when } \theta \ge 1,$$
$$(A-B)^\theta \ge A^\theta - B^\theta \quad \text{when } 0 < \theta \le 1.$$

Lemma 2.7 (see [19])**.** *For $t \in [0,1]$, we have*

$$(1-t)^n \le 2^{1-n} - t^n \quad for\ n \in [0,1],$$
$$(1-t)^n \ge 2^{1-n} - t^n \quad for\ n \in [1,\infty).$$

3. Main results

In this section, we will use the results via *GG*-convex functions in Section 2 to derive our main results in this paper.

Theorem 3.1. *Let $f : [0,b] \to R$ be a differentiable mapping. If $|f'|$ is measurable and $|f'|$ is GG-convex on $[0,b]$ for some fixed $\alpha \in (0,\infty)$, $0 \le a < b$, then the following inequality for fractional integrals holds:*

$$\left|\frac{(\ln x - \ln a)^\alpha + (\ln b - \ln x)^\alpha}{\ln b - \ln a} f(x) - \frac{\Gamma(\alpha+1)}{\ln b - \ln a}\left[{}_HJ^\alpha_{x^-} f(a) + {}_HJ^\alpha_{x^+} f(b)\right]\right|$$

$$\leq \frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\cdot a\cdot|f'(a)|\cdot T_f(x,a)\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln T_f(x,a))^{i-1}}{(\alpha+1)_i}$$
$$+\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\cdot b\cdot|f'(b)|\cdot T_f(x,b)\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln T_f(x,b))^{i-1}}{(\alpha+1)_i},$$

for any $x\in(a,b)$.

Proof. By using Definition 2.1 and Lemma 2.4, we have

$$\int_0^1 t^\alpha x^t a^{1-t}|f'(x^t a^{1-t})|dt \leq \int_0^1 t^\alpha x^t a^{1-t}|f'(x)|^t|f'(a)|^{1-t}dt$$
$$= a|f'(a)|I_f(\alpha+1,x,a).$$

Similarly, one has

$$\int_0^1 t^\alpha x^t b^{1-t}|f'(x^t b^{1-t})|dt = b|f'(b)|I_f(\alpha+1,x,b).$$

By using Lemma 1.1, we obtain

$$\left|\frac{(\ln x-\ln a)^{\alpha}+(\ln b-\ln x)^{\alpha}}{\ln b-\ln a}f(x)-\frac{\Gamma(\alpha+1)}{\ln b-\ln a}\left[{}_HJ^{\alpha}_{x^-}f(a)+{}_HJ^{\alpha}_{x^+}f(b)\right]\right|$$
$$=\left|\frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^\alpha x^t a^{1-t}f'\left(x^t a^{1-t}\right)dt\right.$$
$$\left.-\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^\alpha x^t b^{1-t}f'\left(x^t a^{1-t}\right)dt\right|$$
$$\leq \frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^\alpha x^t a^{1-t}|f'(x^t a^{1-t})|dt$$
$$+\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^\alpha x^t b^{1-t}|f'(x^t b^{1-t})|dt$$
$$\leq \frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}a|f'(a)|I_f(\alpha+1,x,a)$$
$$+\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}b|f'(b)|I_f(\alpha+1,x,b)$$
$$= \frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\cdot a\cdot|f'(a)|\cdot T_f(x,a)\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln T_f(x,a))^{i-1}}{(\alpha+1)_i}$$
$$+\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\cdot b\cdot|f'(b)|\cdot T_f(x,b)\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln T_f(x,b))^{i-1}}{(\alpha+1)_i}.$$

The proof is completed. □

Theorem 3.2. *Let $f : [0,b] \to R$ be a differentiable mapping and $1 < q < \infty$. If $|f'|^q$ is measurable and $|f'|^q$ is GG-convex on $[0,b]$ for some fixed $\alpha \in (0,\infty)$, $0 \le a < b$, then the following inequality for fractional integrals holds:*

$$\left|\frac{(\ln x-\ln a)^{\alpha}+(\ln b-\ln x)^{\alpha}}{\ln b-\ln a}f(x)-\frac{\Gamma(\alpha+1)}{\ln b-\ln a}\left[{}_HJ^{\alpha}_{x^-}f(a)+{}_HJ^{\alpha}_{x^+}f(b)\right]\right|$$
$$\le \frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\cdot a\cdot|f'(a)|\cdot\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\cdot\left(\frac{T_f^q(x,a)-1}{q\ln(T_f(x,a))}\right)^{\frac{1}{q}}$$
$$+\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\cdot b\cdot|f'(b)|\cdot\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\cdot\left(\frac{T_f^q(x,b)-1}{q\ln(T_f(x,b))}\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Definition 2.1, Hölder inequality, we obtain

$$\int_0^1 t^{\alpha}x^t a^{1-t}|f'(x^t a^{1-t})|dt$$
$$\le\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1 (x^t a^{1-t}|f'(x^t a^{1-t})|)^q dt\right)^{\frac{1}{q}}$$
$$\le\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\int_0^1 x^{qt}a^{q(1-t)}|f'(x)|^{qt}|f'(a)|^{q(1-t)}dt\right)^{\frac{1}{q}}$$
$$\le\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\cdot a\cdot|f'(a)|\cdot\left(\int_0^1 T_f^q(x,a)dt\right)^{\frac{1}{q}}$$
$$\le\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\cdot a\cdot|f'(a)|\cdot\left(\frac{T_f^q(x,a)-1}{q\ln(T_f(x,a))}\right)^{\frac{1}{q}}.$$

Similarly, one has

$$\int_0^1 t^{\alpha}x^t b^{1-t}|f'(x^t b^{1-t})|dt\le\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\cdot b\cdot|f'(b)|\cdot\left(\frac{T_f^q(x,b)-1}{q\ln(T_f(x,b))}\right)^{\frac{1}{q}}.$$

By using Lemma 1.1, we have

$$\left|\frac{(\ln x-\ln a)^{\alpha}+(\ln b-\ln x)^{\alpha}}{\ln b-\ln a}f(x)-\frac{\Gamma(\alpha+1)}{\ln b-\ln a}\left[{}_HJ^{\alpha}_{x^-}f(a)+{}_HJ^{\alpha}_{x^+}f(b)\right]\right|$$
$$=\left|\frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^{\alpha}x^t a^{1-t}f'(x^t a^{1-t})dt-\right.$$

$$
\begin{aligned}
&- \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^\alpha x^t b^{1-t} f'(x^t b^{1-t}) dt \Big| \\
&\le \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^\alpha x^t a^{1-t} |f'(x^t a^{1-t})| dt \\
&+ \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^\alpha x^t b^{1-t} |f'(x^t b^{1-t})| dt \\
&\le \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \cdot a \cdot |f'(a)| \cdot \left(\frac{1}{\alpha p + 1} \right)^{\frac{1}{p}} \cdot \left(\frac{T_f^q(x,a) - 1}{q \ln(T_f(x,a))} \right)^{\frac{1}{q}} \\
&+ \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \cdot b \cdot |f'(b)| \cdot \left(\frac{1}{\alpha p + 1} \right)^{\frac{1}{p}} \cdot \left(\frac{T_f^q(x,b) - 1}{q \ln(T_f(x,b))} \right)^{\frac{1}{q}}.
\end{aligned}
$$

The proof is done. □

Theorem 3.3. *Let $f : [0,b] \to R$ be a differentiable mapping . If $|f'|$ is measurable and $|f'|$ is GG-convex on $[0,b]$ for some fixed $\alpha \in (0,\infty)$, $0 \le a < b$, then the following inequality for fractional integrals holds:*

$$
\begin{aligned}
&\left| \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha} \left[{}_H J_{b^-}^\alpha f(a) + {}_H J_{a^+}^\alpha f(b) \right] - f(\sqrt{ab}) \right| \\
&\le \max \Bigg\{ \frac{\ln b - \ln a}{2} b |f'(b)| \Big[\frac{2T_f(a,b) - 1 + (2^{1-\alpha}+1) T_f^{\frac{1}{2}}(a,b)}{\ln(T_f(a,b))} \\
&- 4 I_f(\alpha+1, a, b, \frac{1}{2}) + 2 I_f(\alpha+1, a, b, a) \Big], \\
&\frac{\ln b - \ln a}{2} b |f'(b)| \Big[\frac{2T_f^{\frac{1}{2}}(a,b) - 1}{\ln(T_f(a,b))} - 4 I_f(\alpha+1, a, b, \frac{1}{2}) \\
&+ 2 I_f(\alpha+1, a, b, a) \Big] \Bigg\}.
\end{aligned}
$$

Proof. To achieve our aim, we divide our proof into two cases.

Case 1: $\alpha \in (0,1)$. By using Definition 2.1, Lemma 1.2, Lemma 2.4, Lemma 2.5, Lemma 2.6, Lemma 2.7 via Hölder inequality, we have

$$
\begin{aligned}
&\left| \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha} \left[{}_H J_{b^-}^\alpha f(a) + {}_H J_{a^+}^\alpha f(b) \right] - f(\sqrt{ab}) \right| \\
&= \Bigg| \frac{\ln b - \ln a}{2} \Big[\int_0^1 k a^t b^{1-t} f'(a^t b^{1-t}) dt \\
&- \int_0^1 [(1-t)^\alpha - t^\alpha] a^t b^{1-t} f'(a^t b^{1-t}) dt \Big] \Bigg| \le
\end{aligned}
$$

$$\leq \frac{\ln b - \ln a}{2}\Big[\int_0^1 a^t b^{1-t}|f'(a^t b^{1-t})|dt$$
$$+\int_0^1 |(1-t)^\alpha - t^\alpha|\cdot a^t b^{1-t}\cdot|f'(a^t b^{1-t})|dt\Big]$$
$$\leq \frac{\ln b - \ln a}{2}\Big[\int_0^1 a^t b^{1-t}|f'(a)|^t|f'(b)|^{1-t}dt$$
$$+\int_0^1 |(1-t)^\alpha - t^\alpha|a^t b^{1-t}|f'(a)|^t|f'(b)|^{1-t}dt\Big]$$
$$= \frac{\ln b - \ln a}{2}\Big[b|f'(b)|\int_0^1 T_f^t(a,b)dt + b|f'(b)|\int_0^1 |(1-t)^\alpha - t^\alpha|T_f^t(a,b)dt\Big]$$
$$= \frac{\ln b - \ln a}{2}b|f'(b)|\Big[\int_0^1 T_f^t(a,b)dt + \int_0^1 |(1-t)^\alpha - t^\alpha|T_f^t(a,b)dt\Big]$$
$$= \frac{\ln b - \ln a}{2}b|f'(b)|\Big[\int_0^1 T_f^t(a,b)dt + \int_0^{\frac{1}{2}} ((1-t)^\alpha - t^\alpha)T_f^t(a,b)dt$$
$$+\int_{\frac{1}{2}}^1 (-(1-t)^\alpha + t^\alpha)T_f^t(a,b)dt\Big]$$
$$\leq \frac{\ln b - \ln a}{2}b|f'(b)|\Big[\int_0^1 T_f^t(a,b)dt + \int_0^{\frac{1}{2}} (1-t)^\alpha T_f^t(a,b)dt$$
$$-\int_0^{\frac{1}{2}} t^\alpha T_f^t(a,b)dt - \int_{\frac{1}{2}}^1 (1-t)^\alpha T_f^t(a,b)dt + \int_{\frac{1}{2}}^1 t^\alpha T_f^t(a,b)dt\Big]$$
$$\leq \frac{\ln b - \ln a}{2}b|f'(b)|\Big[\int_0^1 T_f^t(a,b)dt + \int_0^{\frac{1}{2}} (2^{1-\alpha} - t^\alpha)T_f^t(a,b)dt$$
$$-\int_0^{\frac{1}{2}} t^\alpha T_f^t(a,b)dt - \int_{\frac{1}{2}}^1 (1-t^\alpha)T_f^t(a,b)dt + \int_{\frac{1}{2}}^1 t^\alpha T_f^t(a,b)dt\Big]$$
$$= \frac{\ln b - \ln a}{2}b|f'(b)|\Big[\int_0^1 T_f^t(a,b)dt$$
$$+\int_0^{\frac{1}{2}} (2^{1-\alpha} - 2t^\alpha)T_f^t(a,b)dt + \int_{\frac{1}{2}}^1 (2t^\alpha - 1)T_f^t(a,b)dt\Big]$$
$$= \frac{\ln b - \ln a}{2}b|f'(b)|\Big[\int_0^1 T_f^t(a,b)dt + \int_0^{\frac{1}{2}} 2^{1-\alpha}T_f^t(a,b)dt$$
$$-2\int_0^{\frac{1}{2}} t^\alpha T_f^t(a,b)dt - \int_{\frac{1}{2}}^1 T_f^t(a,b)dt + 2\int_{\frac{1}{2}}^1 t^\alpha T_f^t(a,b)dt\Big]$$
$$= \frac{\ln b - \ln a}{2}b|f'(b)|\Big[\int_0^1 T_f(a,b)dt + 2^{1-\alpha}\int_0^{\frac{1}{2}} T_f(a,b)dt$$
$$-\int_{\frac{1}{2}}^1 T_f(a,b)dt - 4\int_0^{\frac{1}{2}} t^\alpha T_f(a,b)dt + 2\int_0^1 t^\alpha T_f(a,b)dt\Big] =$$

$$= \frac{\ln b - \ln a}{2} b|f'(b)| \Big[\frac{2T_f(a,b) - 1 + (2^{1-\alpha}+1)T_f^{\frac{1}{2}}(a,b)}{\ln(T_f(a,b))}$$
$$- 4I_f(\alpha+1,a,b,\frac{1}{2}) + 2I_f(\alpha+1,a,b,a)\Big].$$

Case 2: $\alpha \in [1,\infty)$. By using Definition 2.1, Lemma 1.2, Lemma 2.4, Lemma 2.6, Lemma 2.7, Hölder inequality and Lemma 2.5 again, we have

$$\left| \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha} [{}_HJ^\alpha_{b^-} f(a) +_H J^\alpha_{a^+} f(b)] - f(\sqrt{ab}) \right|$$
$$= \left| \frac{\ln b - \ln a}{2} \Big[\int_0^1 k a^t b^{1-t} f'(a^t b^{1-t}) dt \right.$$
$$\left. - \int_0^1 [(1-t)^\alpha - t^\alpha] a^t b^{1-t} f'(a^t b^{1-t}) dt \Big] \right|$$
$$\leq \frac{\ln b - \ln a}{2} \Big[\int_0^1 a^t b^{1-t} |f'(a^t b^{1-t})| dt$$
$$+ \int_0^1 |(1-t)^\alpha - t^\alpha| a^t b^{1-t} |f'(a^t b^{1-t})| dt \Big]$$
$$\leq \frac{\ln b - \ln a}{2} \Big[\int_0^1 a^t b^{1-t} |f'(a)|^t |f'(b)|^{1-t} dt$$
$$+ \int_0^1 |(1-t)^\alpha - t^\alpha| a^t b^{1-t} |f'(a)|^t |f'(b)|^{1-t} dt \Big]$$
$$= \frac{\ln b - \ln a}{2} \Big[b|f'(b)| \int_0^1 T_f^t(a,b) dt$$
$$+ b|f'(b)| \int_0^1 |(1-t)^\alpha - t^\alpha| T_f^t(a,b) dt \Big]$$
$$= \frac{\ln b - \ln a}{2} b|f'(b)| \Big[\int_0^1 T_f^t(a,b) dt + \int_0^1 |(1-t)^\alpha - t^\alpha| T_f^t(a,b) dt \Big]$$
$$= \frac{\ln b - \ln a}{2} b|f'(b)| \Big[\int_0^1 T_f^t(a,b) dt + \int_0^{\frac{1}{2}} ((1-t)^\alpha - t^\alpha) T_f^t(a,b) dt$$
$$+ \int_{\frac{1}{2}}^1 (-(1-t)^\alpha + t^\alpha) T_f^t(a,b) dt \Big]$$
$$= \frac{\ln b - \ln a}{2} b|f'(b)| \Big[\int_0^1 T_f^t(a,b) dt + \int_0^{\frac{1}{2}} (1-t)^\alpha T_f^t(a,b) dt$$
$$- \int_0^{\frac{1}{2}} t^\alpha T_f^t(a,b) dt - \int_{\frac{1}{2}}^1 (1-t)^\alpha T_f^t(a,b) dt + \int_{\frac{1}{2}}^1 t^\alpha T_f^t(a,b) dt \Big]$$
$$\leq \frac{\ln b - \ln a}{2} b|f'(b)| \Big[\int_0^1 T_f^t(a,b) dt + \int_0^{\frac{1}{2}} (1-t^\alpha) T_f^t(a,b) dt -$$

$$-\int_0^{\frac{1}{2}} t^{\alpha} T_f^t(a,b)dt - \int_{\frac{1}{2}}^1 (2^{1-\alpha} - t^{\alpha}) T_f^t(a,b)dt + \int_{\frac{1}{2}}^1 t^{\alpha} T_f^t(a,b)dt\Big]$$

$$= \frac{\ln b - \ln a}{2} b\Big[\int_0^1 T_f^t(a,b)dt + \int_0^{\frac{1}{2}} (1-2t^{\alpha}) T_f^t(a,b)dt$$

$$+ \int_{\frac{1}{2}}^1 (-1+2t^{\alpha}) T_f^t(a,b)dt\Big]$$

$$= \frac{\ln b - \ln a}{2} b|f'(b)|\Big[\int_0^1 T_f^t(a,b)dt + \int_0^{\frac{1}{2}} T_f^t(a,b)dt - \int_{\frac{1}{2}}^1 T_f^t(a,b)dt$$

$$-2\int_0^{\frac{1}{2}} t^{\alpha} T_f^t(a,b)dt + 2\int_0^1 t^{\alpha} T_f^t(a,b)dt - 2\int_0^{\frac{1}{2}} t^{\alpha} T_f^t(a,b)dt\Big]$$

$$= \frac{\ln b - \ln a}{2} b|f'(b)|\Big[\int_0^1 T_f^t(a,b)dt + \int_0^{\frac{1}{2}} T_f^t(a,b)dt$$

$$-\int_{\frac{1}{2}}^1 T_f^t(a,b)dt - 4\int_0^{\frac{1}{2}} t^{\alpha} T_f^t(a,b)dt + 2\int_0^1 t^{\alpha} T_f^t(a,b)dt\Big]$$

$$= \frac{\ln b - \ln a}{2} b|f'(b)|\Big[\frac{2T_f^{\frac{1}{2}}(a,b) - 1}{\ln(T_f(a,b))} - 4I_f(\alpha+1,a,b,\frac{1}{2}) + 2I_f(\alpha+1,a,b,a)\Big].$$

The proof is done. □

Corollary 3.4. *Under the conditions of Theorem 3.3, if $x|f'(x)|$ is decreasing, then*

$$\left|\frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^{\alpha}}\left[{}_HJ_{b^-}^{\alpha} f(a) + {}_HJ_{a^+}^{\alpha} f(b)\right] - f(\sqrt{ab})\right| \leq$$

$$\leq \max\Big\{\frac{\ln b - \ln a}{2} b|f'(b)|\Big[\frac{2T_f(a,b) - 1 + (2^{1-\alpha}+1)T_f^{\frac{1}{2}}(a,b)}{\ln(T_f(a,b))}$$

$$-4I_f(\alpha+1,a,b,\frac{1}{2}) + 2\frac{T_f(a,b)}{\alpha+1}\Big],$$

$$\frac{\ln b - \ln a}{2} b|f'(b)|\Big[\frac{2T_f^{\frac{1}{2}}(a,b) - 1}{\ln(T_f(a,b))} - 4I_f(\alpha+1,a,b,\frac{1}{2}) + 2\frac{T_f(a,b)}{\alpha+1}\Big]\Big\}.$$

Proof. Note that $T_f^t(a,b) \leq T_f(a,b)$ for $t \in [0,1]$ gives

$$\int_0^1 t^{\alpha} T_f^t(a,b)dt \leq \int_0^1 t^{\alpha} T_f(a,b)dt = \frac{T_f(a,b)}{\alpha+1}.$$

Then, the result can be derived immediately. □

Theorem 3.5. *Let $f:[0,b]\to R$ be a differentiable mapping and $1<q<\infty$. If $|f'|^q$ is measurable and $|f'|^q$ is GG-convex on $[0,b]$ for some fixed $\alpha\in(0,\infty)$, $0\le a<b$, then the following inequality for fractional integrals holds*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}\left[{}_HJ^{\alpha}_{b^-}f(a)+{}_HJ^{\alpha}_{a^+}f(b)\right]-f(\sqrt{ab})\right|\\ &\le\max\left\{\frac{\ln b-\ln a}{2}\cdot b\cdot|f'(b)|\cdot\left(\frac{(1+2^{1-\alpha})^p}{2}-\frac{2^p(1-2^{-p\alpha})}{p\alpha+1}\right)^{\frac{1}{p}}\right.\\ &\cdot\left(\frac{T_f^q(a,b)}{q\ln T_f(a,b)}\right)^{\frac{1}{q}},\frac{\ln b-\ln a}{2}\cdot b\cdot|f'(b)|\cdot\left(2^p-\frac{2^{p+1}-2^{p(1-\alpha)}}{p\alpha+1}\right)^{\frac{1}{p}}\\ &\left.\cdot\left(\frac{T_f^q(a,b)}{q\ln T_f(a,b)}\right)^{\frac{1}{q}}\right\},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. To achieve our aim, we divide into two cases.

Case 1: $\alpha\in(0,1)$. For

$$k=\begin{cases}1, & 0\le t<\frac{1}{2},\\ -1, & \frac{1}{2}\le t<1.\end{cases}$$

By using Lemma 2.6, Lemma 2.7, we obtain

$$\begin{aligned}&\int_0^1|k-(1-t)^{\alpha}+t^{\alpha}|^p dt\\ &=\int_0^{\frac{1}{2}}(1-(1-t)^{\alpha}+t^{\alpha})^p dt+\int_{\frac{1}{2}}^1(1-t^{\alpha}+(1-t)^{\alpha})^p dt\\ &\le\int_0^{\frac{1}{2}}[(1+t^{\alpha})-(1-t^{\alpha})]^p dt+\int_{\frac{1}{2}}^1[1-t^{\alpha}+2^{1-\alpha}-t^{\alpha}]^p dt\\ &=\int_0^{\frac{1}{2}}2^pt^{p\alpha}dt+\int_{\frac{1}{2}}^1[(1+2^{1-\alpha})^p-2^pt^{p\alpha}]dt\\ &=\frac{(1+2^{1-\alpha})^p}{2}-\frac{2^p(1-2^{-p\alpha})}{p\alpha+1}.\end{aligned}$$

Case 2: $\alpha\in[1,\infty)$. By using Lemma 2.6 and Lemma 2.7, we obtain

$$\int_0^1|k-(1-t)^{\alpha}+t^{\alpha}|^p dt=$$

$$
\begin{aligned}
&= \int_0^{\frac{1}{2}} (1-(1-t)^{\alpha}+t^{\alpha})^p dt + \int_{\frac{1}{2}}^1 (1-t^{\alpha}+(1-t)^{\alpha})^p dt \\
&= 2\int_{\frac{1}{2}}^1 (1-t^{\alpha}+(1-t)^{\alpha})^p dt \ \le \ 2\int_{\frac{1}{2}}^1 (1-t^{\alpha}+1-t^{\alpha})^p dt \\
&\le 2^{p+1}\int_{\frac{1}{2}}^1 (1-t^{\alpha})^p dt \ \le \ 2^{p+1}\int_{\frac{1}{2}}^1 (1-t^{\alpha p}) dt \\
&= 2^{p+1}\left(\frac{1}{2}-\int_{\frac{1}{2}}^1 t^{p\alpha} dt\right) \ \le \ 2^p - \frac{2^{p+1}-2^{p(1-\alpha)}}{p\alpha+1}.
\end{aligned}
$$

Since $|f'(x)|^q$ is GG-convex on $[0,b]$, from Definition 2.1, we drive

$$
\begin{aligned}
\int_0^1 (a^t b^{1-t}|f'(a^t b^{1-t})|)^q dt &\le \int_0^1 (a^t b^{1-t})^q |f'(a)|^{tq}|f'(b)|^{q(1-t)} dt \\
&= \big(b\cdot|f'(b)|\big)^q \int_0^1 T_f^q(a,b) dt \\
&= \big(b\cdot|f'(b)|\big)^q \frac{T_f^q(a,b)}{q\ln T_f(a,b)}.
\end{aligned}
$$

Therefore, by Lemma 2.7 and Hölder inequality, we have

$$
\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}\left[{}_HJ^{\alpha}_{b^-}f(a)+{}_HJ^{\alpha}_{a^+}f(b)\right]-f(\sqrt{ab})\right| \\
&= \left|\frac{\ln b-\ln a}{2}\Big[\int_0^1 (k-(1-t)^{\alpha}+t^{\alpha})a^t b^{1-t} f'(a^t b^{1-t}) dt\Big]\right| \\
&\le \frac{\ln b-\ln a}{2}\int_0^1 |k-(1-t)^{\alpha}+t^{\alpha}|\cdot a^t b^{1-t}\cdot|f'(a^t b^{1-t})| dt \\
&\le \frac{\ln b-\ln a}{2}\left(\int_0^1 |k-(1-t)^{\alpha}+t^{\alpha}|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \big(a^t b^{1-t}|f'(a^t b^{1-t})|\big)^q dt\right)^{\frac{1}{q}} \\
&\le \max\left\{\frac{\ln b-\ln a}{2}\cdot b\cdot|f'(b)|\cdot\left(\frac{(1+2^{1-\alpha})^p}{2}-\frac{2^p(1-2^{-p\alpha})}{p\alpha+1}\right)^{\frac{1}{p}}\right. \\
&\cdot\left(\frac{T_f^q(a,b)}{q\ln T_f(a,b)}\right)^{\frac{1}{q}}, \frac{\ln b-\ln a}{2}\cdot b\cdot|f'(b)|\cdot\left(2^p-\frac{2^{p+1}-2^{p(1-\alpha)}}{p\alpha+1}\right)^{\frac{1}{p}} \\
&\left.\cdot\left(\frac{T_f^q(a,b)}{q\ln T_f(a,b)}\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

The proof is done. □

4. Applications to special means

Consider the following special means (see [27]) for arbitrary real numbers $x, y, x \neq y$ as follows:

(i) $A(x,y) = \frac{x+y}{2}, x, y \in R$.
(ii) $H(x,y) = \frac{2}{\frac{1}{x}+\frac{1}{y}}, x, y \in R \setminus \{0\}$.
(iii) $G(x,y) = \sqrt{xy}$.
(iv) $L(x,y) = \frac{y-x}{\ln|y|-\ln|x|}, |x| \neq |y|, xy \neq 0$.
(v) $L_n(x,y) = \left[\frac{y^{n+1}-x^{n+1}}{(n+1)(y-x)}\right]^{\frac{1}{n}}, n \in Z \setminus \{-1, 0\}, x, y \in R, x \neq y$.

Using the results obtained in Section 3, we give some applications to special means of real numbers.

Proposition 4.1. *Let $a, b \in R^+ \setminus \{0\}, 0 \leq a < b, x \in [0,b]$. Then*

$$\left|A(x,x) - L(a,b)\right| \leq \frac{(2\ln x - \ln a - \ln b - 2)x + a + b}{\ln b - \ln a},$$

$$\left|A(x,x) - L(a,b)\right| \leq \frac{(\ln x - \ln a)^2}{\ln b - \ln a} a \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{\left(\frac{x}{a}\right)^q - 1}{q(\ln x - \ln a)}\right)^{\frac{1}{q}}$$
$$+ \frac{(\ln b - \ln x)^2}{\ln b - \ln a} b \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{\left(\frac{x}{b}\right)^q - 1}{q(\ln x - \ln b)}\right)^{\frac{1}{q}},$$

$$\left|L(a,b) - [A(a,b)H(a,b)]^{\frac{1}{2}}\right| \leq a - b - \frac{(\sqrt{a}+\sqrt{b})^2}{\ln a - \ln b},$$

$$\left|L(a,b) - [A(a,b)H(a,b)]^{\frac{1}{2}}\right| \leq \frac{a(\ln b - \ln a)}{2(q(\ln a - \ln b))^{\frac{1}{q}}} \left(2^p - \frac{2^{p+1}-1}{p+1}\right)^{\frac{1}{p}}.$$

Proof. Applying Theorems 3.1, 3.2, 3.3, and 3.5 for $f(x) = x$ and $\alpha = 1$, one can obtain the results immediately. □

Proposition 4.2. *Let $a, b \in R^+ \setminus \{0\}, 0 \leq a < b, x \in [0,b], n \geq 2$. Then*

$$\left|A(x^n,x^n) - L(a,b)L_{n-1}^{n-1}(a,b)\right| \leq \left(1 - \frac{2}{n(\ln b - \ln a)}\right) x^n - \frac{a^n + b^n}{n(\ln b - \ln a)},$$

$$\left|A(x^n,x^n) - L(a,b)L_{n-1}^{n-1}(a,b)\right| \leq \frac{n\left(\frac{1}{p+1}\right)^{\frac{1}{p}}}{\ln b - \ln a} \left[\left(\frac{(\ln x - \ln a)^{2q-1}(x^{qn} - a^{qn})}{qn}\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\frac{(\ln x - \ln b)^{2q-1}(x^{qn} - b^{qn})}{qn}\right)^{\frac{1}{q}}\right],$$

$$\left|L(a^n,b^n)-(H(a,b)A(a,b))^{\frac{n}{2}}\right| \leq -nb^n\left(\left(\frac{a}{b}\right)^{\frac{n}{2}}-1+\frac{2\left(\frac{a}{b}\right)^{\frac{n}{2}}-\left(\frac{a}{b}\right)^n-1}{n^2(\ln a-\ln b)}\right),$$

$$\left|L(a^n,b^n)-(H(a,b)A(a,b))^{\frac{n}{2}}\right| \leq \frac{\ln b-\ln a}{2(qn(\ln a-\ln b))^{\frac{1}{q}}}na^n\left(2^p-\frac{2^{p+1}-1}{p+1}\right)^{\frac{1}{p}}.$$

Proof. Applying Theorems 3.1, 3.2, 3.3, and 3.5 for $f(x)=x^n$ and $\alpha=1$, one can obtain the results immediately. □

Proposition 4.3. *Let $a,b\in R^+\setminus\{0\}, 0\leq a<b, x\in[0,b], n\geq 2$. Then*

$$\left|A\left(\frac{1}{x},\frac{1}{x}\right)-\frac{1}{2}L\left(\frac{1}{a},\frac{1}{b}\right)\right| \leq \frac{2(\ln a+\ln b)-4\ln x-4}{x(\ln b-\ln a)}-\frac{2}{a(\ln b-\ln a)},$$

$$\left|A\left(\frac{1}{x},\frac{1}{x}\right)-\frac{1}{2}L\left(\frac{1}{a},\frac{1}{b}\right)\right| \leq \frac{2(\ln x-\ln a)^2}{a(\ln b-\ln a)}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{\left(\frac{a}{x}\right)^q-1}{q(\ln a-\ln x)}\right)^{\frac{1}{q}}$$
$$+\frac{2(\ln b-\ln x)^2}{b(\ln b-\ln a)}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{\left(\frac{b}{x}\right)^q-1}{q(\ln b-\ln x)}\right)^{\frac{1}{q}},$$

$$\left|L\left(\frac{1}{b},\frac{1}{a}\right)-G\left(\frac{1}{a},\frac{1}{b}\right)\right| \leq -\frac{1}{2b}\left(-1+\frac{b}{a}+\frac{2\left(\frac{b}{a}\right)^{\frac{1}{2}}-\frac{b}{a}-1}{\ln b-\ln a}\right),$$

$$\left|L\left(\frac{1}{b},\frac{1}{a}\right)-G\left(\frac{1}{a},\frac{1}{b}\right)\right| \leq \frac{\ln b-\ln a}{4a(q(\ln b-\ln a))^{\frac{1}{q}}}\left(2^p-\frac{2^{p+1}-1}{p+1}\right)^{\frac{1}{p}}.$$

Proof. Applying Theorems 3.1, 3.2, 3.3, and 3.5 for $f(x)=\frac{1}{x}$ and $\alpha=1$, one can obtain the results immediately. □

Proposition 4.4. *Let $a,b\in R^+\setminus\{0\}, 0\leq a<b, x\in[0,b], n\geq 2$. Then*

$$\left|A(x,x)-L(b^{-1},a^{-1})\right|$$
$$\leq \frac{(2\ln x+\ln a+\ln b-2)x+a^{-1}+b^{-1}}{\ln b-\ln a},$$
$$\left|A(x,x)-L(b^{-1},a^{-1})\right|$$
$$\leq \frac{(\ln x+\ln b)^2}{\ln b-\ln a}b^{-1}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{(xb)^q-1}{q(\ln x+\ln b)}\right)^{\frac{1}{q}}$$
$$+\frac{(\ln a+\ln x)^2}{\ln b-\ln a}a^{-1}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{(xa)^q-1}{q(\ln x+\ln a)}\right)^{\frac{1}{q}},$$

$$\left|L(b^{-1},a^{-1})-[A(b^{-1},a^{-1})H(b^{-1},a^{-1})]^{\frac{1}{2}}\right|$$
$$\leq -a^{-1}+b^{-1}-\frac{(\sqrt{a^{-1}}+\sqrt{b^{-1}})^2}{\ln a-\ln b},$$
$$\left|L(b^{-1},a^{-1})-[A(b^{-1},a^{-1})H(b^{-1},a^{-1})]^{\frac{1}{2}}\right|$$
$$\leq \frac{\ln b-\ln a}{2}b^{-1}\left(2^p-\frac{2^{p+1}-1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{q(\ln a-\ln b)}\right)^{\frac{1}{q}}.$$
$$\left|A(x^n,x^n)-L(b^{-1},a^{-1})L_{n-1}^{n-1}(b^{-1},a^{-1})\right|$$
$$\leq \left(1-\frac{2}{n(\ln b-\ln a)}\right)x^n-\frac{a^{-n}+b^{-n}}{n(\ln b-\ln a)},$$
$$\left|A(x^n,x^n)-L(b^{-1},a^{-1})L_{n-1}^{n-1}(b^{-1},a^{-1})\right|$$
$$\leq \frac{n\left(\frac{1}{p+1}\right)^{\frac{1}{p}}}{\ln b-\ln a}\left[\left((\ln x+\ln b)^{2q-1}(x^{qn}-b^{-qn})\cdot\frac{1}{qn}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left((\ln x+\ln a)^{2q-1}(x^{qn}-a^{-qn})\cdot\frac{1}{qn}\right)^{\frac{1}{q}}\right],$$
$$\left|L(b^{-n},a^{-n})-(H(b^{-1},a^{-1})A(b^{-1},a^{-1}))^{\frac{n}{2}}\right|$$
$$\leq -na^{-n}\left(\left(\frac{a}{b}\right)^{\frac{n}{2}}-1+\frac{2\left(\frac{a}{b}\right)^{\frac{n}{2}}-\left(\frac{a}{b}\right)^{n}-1}{n^2(\ln a-\ln b)}\right),$$
$$\left|L(b^{-n},a^{-n})-(H(b^{-1},a^{-1})A(b^{-1},a^{-1}))^{\frac{n}{2}}\right|$$
$$\leq \frac{\ln b-\ln a}{2}nb^{-n}\left(2^p-\frac{2^{p+1}-1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{qn(\ln a-\ln b)}\right)^{\frac{1}{q}}.$$
$$\left|A\left(\frac{1}{x},\frac{1}{x}\right)-\frac{1}{2}L(b,a)\right|\leq\frac{-2(\ln a+\ln b)-4\ln x-4}{x(\ln b-\ln a)}-\frac{2b}{\ln b-\ln a},$$
$$\left|A\left(\frac{1}{x},\frac{1}{x}\right)-\frac{1}{2}L(b,a)\right|\leq\frac{2b(\ln x+\ln b)^2}{(\ln b-\ln a)(p+1)^{\frac{1}{p}}}\left(\frac{\left(\frac{1}{xb}\right)^q-1}{q(-\ln b-\ln x)}\right)^{\frac{1}{q}}$$
$$+\frac{2a(\ln a+\ln x)^2}{(\ln b-\ln a)(p+1)^{\frac{1}{p}}}\left(\frac{\left(\frac{1}{xa}\right)^q-1}{q(-\ln a-\ln x)}\right)^{\frac{1}{q}},$$

$$\left|L(a,b)-G(b,a)\right| \leq -2a\left(-1+\frac{b}{a}+\frac{2\left(\frac{b}{a}\right)^{\frac{1}{2}}-\frac{b}{a}-1}{\ln b-\ln a}\right),$$

$$\left|L(a,b)-G(b,a)\right| \leq \frac{b(\ln b-\ln a)}{4(q(\ln b-\ln a))^{\frac{1}{q}}}\left(2^{p}-\frac{2^{p+1}-1}{p+1}\right)^{\frac{1}{p}}.$$

Proof. Making the substitutions $a \to b^{-1}$, $b \to a^{-1}$ in the Proposition 4.1, Proposition 4.2 and Proposition 4.3, one can obtain desired inequalities respectively. □

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