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AN ANALYSIS OF TWO DIMENSIONAL INTEGRAL EQUATIONS OF THE SECOND KIND

M. M. EL-BORAI - M. A. ABDOU - M. BASSEEM

In this article, A numerical method is used to solve the two dimensional Fredholm integral equation of the second kind with weak singular kernel using the Toeplitz matrix and product Nystrom method. The numerical results given in this paper are computed using maple 8. The error, in each case, is computed.

1. Introduction.

Singular integral equations arise in many problems of mathematical physics. Its applications are in many important fields like fracture mechanics, aerodynamics, the theory of porous filtering, antenna problems in electromagnetic theory and others. The solutions of their applications can be obtained analytically, using the theory developed by Muskhelishvili [1], but in practice approximate methods are needed to solve the Fredholm integral equation in one dimensional problem with different kernels. The direct numerical are preferred, which attack the equation as it is written, without transforming it beforehand into a Fredholm equation. Among

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these, Galerkin method [2], block by block method, Nystrom method [3] and Toeplitz matrix method [4]. Afterwards, many numerical methods, which solve problems in one dimensional integral equations, can be found in the books [5], [6] and [7].

Here, in this work, the existence and uniqueness solution of the singular Fredholm integral equation in two dimensional are proved. Also, we use the Toeplitz matrix and product Nystrom method, as famous methods for solving singular integral equations, to obtain numerically the solution of the integral equation. The error, in each case, is computed when the kernel of Fredholm takes a logarithmic and Carleman form.

2. Existence and uniqueness solution.

We consider the two dimensional Fredholm integral equation

$$\mu\Phi(x,y) - \lambda\int_a^b\int_c^d k(x,u;y,v)\Phi(u,v)dvdu = f(x,y), \tag{2.1}$$

where μ is a constant defined the kind of the integral equation, for $\mu = 0$ and $\mu = constant \neq 0$, we have, respectively, the Fredholm first and second kind, while λ is a constant, may be complex, that has many physical meaning. The known functions $k(x,u;y,v)$ and $f(x,y)$ represent respectively, the discontinuous kernel of the integral equation and its free term. While $\Phi(x,y)$ represents the unknown function.

Assume the following conditions:

(i) The general kernel $k(x,u;y,v)$ satisfies the condition

(2.2)

$$\{\int_a^b\int_a^b\int_c^d\int_c^d |k(x,u;y,v)|^2 dxdudydv\}^{\frac{1}{2}} \leq C,\ \ C \text{ is small enough.}$$

(ii) The given function $f(x,y)$ and its partial derivatives with respect to $x,\ y$ are continuous and its normality in $L_2[a,b]\times L_2[c,d]$ is given by

$$\| f(x,y) \|_{L_2[a,b]\times L_2[c,d]} = [\int_a^b\int_c^d |f(x,y)|^2 dxdy]^{\frac{1}{2}} = D \tag{2.3}$$

(iii) The unknown function $\Phi(x,y)$ satisfies Lipschitz condition for the arguments x, y, where its norm is considered in $L_2[a,b]\times L_2[c,d]$.

Theorem 2.1 *Using the previous conditions, where* $|\lambda| < \frac{|\mu|}{C}$*, the solution of (2.1) is exist and unique .*

Proof. To prove that the solution of Eq.(2.1) is exist, we use the Picard method, by picking up any real continuous function $\phi_0(x, y)$ in $L_2[a, b] \times L_2[c, d]$, then constructing a sequence $\phi_n(x, y)$ to have

$$\mu\phi_n(x, y) = f(x, y) + \lambda\int_a^b\int_c^d k(x, u; y, v)\phi_{n-1}(u, v)dvdu, \quad n = 1, 2, 3, ...$$

$$\mu\phi_0(x, y) = f(x, y). \tag{2.4}$$

It is convenient to introduce:

$$\begin{aligned} \mu\psi_n(x, y) &= \mu[\phi_n(x, y) - \phi_{n-1}(x, y)] \\ &= \lambda\int_a^b\int_c^d k(x, u; y, v)(\phi_{n-1}(u, v) - \phi_{n-2}(u, v))dvdu, \end{aligned} \tag{2.5}$$

then

$$\phi_n(x, y) = \sum_{i=0}^{n} \psi_i(x, y). \tag{2.6}$$

Using (2.5), we get

$$\mu\psi_n(x, y) = \lambda \int_a^b \int_c^d k(x, u; y, v)\psi_{n-1}(u, v)dvdu, \quad \mu\psi_0 = f(x, y). \tag{2.7}$$

Taking the norm of Eq.(2.7), we obtain

$$|\mu| \parallel \psi_n(x, y) \parallel = |\lambda| \parallel \int_a^b \int_c^d k(x, u; y, v)\psi_{n-1}(u, v)dvdu \parallel . \tag{2.8}$$

For $n = 1$, then using Cauchy-Schwarz inequality, we have

$$\begin{aligned} \parallel \psi_1(x, y) \parallel \leq |\frac{\lambda}{\mu}|\{\int_a^b \int_a^b \int_c^d \int_c^d |k(x, u; y, v)|^2 \cdot \\ \cdot\, dxdudydv\}^{\frac{1}{2}} \parallel \psi_0(x, y) \parallel . \end{aligned} \tag{2.9}$$

Hence, we get

$$\parallel \psi_1(x, y) \parallel \leq |\frac{\lambda}{\mu}|CD. \tag{2.10}$$

By induction, one has

$$\| \psi_n(x, y) \| \leq \alpha^n D, \qquad \alpha = |\frac{\lambda}{\mu}|C. \tag{2.11}$$

This bound under the used condition $|\lambda| < \frac{|\mu|}{C}$ makes the sequence $\psi_n(x, y)$ uniformly convergent. Hence, we get

$$\Phi(x, y) = \sum_{i=0}^{\infty} \psi_i(x, y). \tag{2.12}$$

Since each of $\psi_i(x, y)$ in (2.12) is continuous, therefore $\phi_i(x, y)$ is also continuous, convergent and represents the existence of the solution of Eq. (2.1).

To prove $\Phi(x, y)$ is the unique solution of Eq. (2.1), assume $\tilde{\Phi}(x, y)$ is another solution, hence, we get

(2.13)

$$\Phi(x, y) - \tilde{\Phi}(x, y) = \frac{\lambda}{\mu} \int_a^b \int_c^d k(x, u; y, v)[\Phi(u, v) - \tilde{\Phi}(u, v)]dvdu.$$

Applying Cauchy-Schwarz inequality and conditions (i) and (iii), we get

$$\| \Phi(x, y) - \tilde{\Phi}(x, y) \| \leq \alpha \| \Phi(x, y) - \tilde{\Phi}(x, y) \|, \qquad \alpha < 1, \tag{2.14}$$

which leads to $\Phi = \tilde{\Phi}$.

3. Integral operator.

The normality and continuity of the integral operator are very important to prove the existence and uniqueness solution of the integral equation of the first kind or for a homogeneous integral equation, where the Picard method fails.

For this, assume the integral operator

$$W\Phi = \frac{f}{\mu} + K\Phi \tag{2.15}$$

where

$$K\Phi = \frac{\lambda}{\mu} \int_a^b \int_c^d k(x, u; y, v)\Phi(u, v)dvdu. \tag{2.16}$$

Hence, the formula (2.1), yields

$$W\Phi = \Phi \tag{3.17}$$

The normality of K can be showed as follows

$$\| K\Phi \| = |\frac{\lambda}{\mu}| \, \| \int_a^b \int_c^d k(x, u; y, v)\Phi(u, v)dvdu \|, \tag{3.18}$$

then, using Cauchy-Schwarz inequality and condition (i), we have

$$\| K\Phi \| \leq |\frac{\lambda}{\mu}| C \, \| \Phi \| = \alpha \, \| \Phi \|, \;\; \alpha = |\frac{\lambda}{\mu}| C. \tag{3.19}$$

So by the condition $\alpha < 1$,K is bounded , which leads to say that W is also bounded operator.

The continuity of the integral operator W, given by (3.15), can be proved by assuming $\Phi_n(x, y), \;\; \Phi_m(x, y)$ satisfy Eq.(3.15) then

$$\| W\Phi_n - W\Phi_m \| = \| K\Phi_n - K\Phi_m \| \leq \alpha \, \| \Phi_n - \Phi_m \| \; . \tag{3.20}$$

Therefore, if $\| \Phi_n - \Phi_m \| \to 0$, then $\| W\Phi_n - W\Phi_m \| \to 0$, which yields W is continuous operator.

Hence, W is a contraction mapping, then, by Banach fixed point theorem, the Eq. (2.1) has a unique solution.

4. Method of Solution.

Here, we discuss the solution of integral equation, in two dimensional problem, using two different methods.

4.1.-1. Toeplitz matrix method.

Toeplitz matrix method is used to obtain the numerical solution of two dimensional integral equation of the second kind with singular kernel. The idea of this method is to obtain, in general, a system of $(2N+1)\times(2M+1)$ linear algebraic equation, where $2N + 1$ and $2M + 1$ are the numbers of discretization points used in x and y dimensions respectively. The coefficients matrix is expressed as the sum of two matrices, one of them is the Toeplitz matrix and the other is a matrix with zero elements except the first and last rows and columns.

For this, we assume

$$\begin{aligned}(4.21)\quad &\int_a^b \int_c^d k(x,u;y,v)\Phi(u,v)dvdu0\\ &= \sum_{n=-N}^{N-1}\sum_{l=-M}^{M-1}\int_{a=nh}^{(n+1)h}\int_{c=lh'}^{(l+1)h'} k(x,u;y,v)\Phi(u,v)dvdu,\end{aligned}$$

where $h = \dfrac{b-a}{N}$ and $h' = \dfrac{d-c}{M}$. The integral of the right hand side of Eq.(4.21) can be evaluated by assuming $x = mh$, increasing by a rate h, and $y = ph'$, increasing by a rate h'. Therefore,

$$\begin{aligned}(4.22)\quad &\int_{nh}^{(n+1)h}\int_{lh'}^{(l+1)h'} k(x,u;y,v)\Phi(u,v)dvdu\\ &= A(x,y)\Phi(nh,lh') + B(x,y)\Phi(nh,lh'+h')\\ &+C(x,y)\Phi(nh+h,lh') + D(x,y)\Phi(nh+h,lh'+h') + R.\end{aligned}$$

where the weights of the integration $A,\ B,\ C\ and\ D$ are functions of $x,\ y$ will be determined , and R is the error term.

For the principal of Toeplitz matrix method, to solve Eq.(4.22), we assume $\Phi(x,y) = 1,\ x,\ y,\ xy,$ in this case $R = 0$, which lead to

$$\begin{aligned}
&I_1 = \int_{nh}^{nh+h}\int_{lh'}^{lh'+h'} k(x,u;y,v)dvdu = A + B + C + D,\\
&I_2 = \int_{nh}^{nh+h}\int_{lh'}^{lh'+h'} uk(x,u;y,v)dvdu = Anh + Bnh\\
&\qquad + C(nh+h) + D(nh+h),\\
(4.23)\quad &I_3 = \int_{nh}^{nh+h}\int_{lh'}^{lh'+h'} vk(x,u;y,v)dvdu = Alh' + B(lh'+h')\\
&\qquad + Clh' + D(lh'+h'),\\
&I_4 = \int_{nh}^{nh+h}\int_{lh'}^{lh'+h'} uvk(x,u;y,v)dvdu = Anhlh'\\
&\qquad + Bnh(lh'+h') + Clh'(nh+h) + D(nh+h)(lh'+h').
\end{aligned}$$

In fact, one can easily evaluate weights explicity, but we prefer to evaluate such integral numerically, then the values of A, B , C and D are directly obtained, where

$$
\begin{aligned}
A &= (nl+n+l+1)I_1 - \frac{l+1}{h}I_2 - \frac{n+1}{h'}I_3 + \frac{1}{hh'}I_4, \\
B &= -(nl+l)I_1 + \frac{l}{h}I_2 + \frac{n+1}{h'}I_3 - \frac{1}{hh'}I_4, \\
C &= -(nl+n)I_1 + \frac{l+1}{h}I_2 + \frac{n}{h'}I_3 - \frac{1}{hh'}I_4, \\
D &= nlI_1 - \frac{l}{h}I_2 - \frac{n}{h'}I_3 + \frac{1}{hh'}I_4.
\end{aligned} \tag{4.24}
$$

Let $x = mh,\ \ y = ph'$, so Eq.(4.21) becomes
(4.25)

$$\int_a^b \int_c^d k(x,u;y,v)\Phi(u,v)dvdu = \sum_{n=-N}^{N}\sum_{l=-M}^{M} \chi_{n,m}^{l,p}\Phi_{N,M}(nh,lh'),$$

where

$$
\chi_{n,m}^{l,p} = \begin{cases}
C_{n-1,l} + D_{n-1,l-1}, & \text{if } n = N \\
B_{n,l-1} + D_{n-1,l-1}, & \text{if } l = M \\
A_{n,l}+B_{n,l-1}+C_{n-1,l}+D_{n-1,l-1}, & \text{if } -M+1 \le l \le M-1, \\
 & \text{if } -N+1 \le n \le N-1 \\
A_{n,l} + B_{n,l-1}, & \text{if } n = -N \\
A_{n,l} + C_{n-1,l}, & \text{if } l = -M
\end{cases} \tag{4.26}
$$

and $\Phi_{N,M}$ is the numerical approximate solution, which satisfy the following formula

$$\| \Phi(x,y) - \Phi_{N,M} \| \to 0 \qquad as\ N,\ \ M \to \infty. \tag{4.27}$$

Hence, Eq.(2.1) is approximately equivalent to the following
(4.28)

$$\mu\Phi_{N,M}(mh,\, ph') - \lambda \sum_{n=-N}^{N}\sum_{l=-M}^{M} \chi_{n,m}^{l,k}\Phi_{N,M}(nh,lh') = f(mh,\, ph')$$

which represents a system of linear algebraic equations.

The matrix $\chi_{n,m}^{l,p}$ can be written as

$$\chi_{n,m}^{l,p} = \chi_{n-m}^{l-p} - G_{n,m}^{l,p}, \tag{4.29}$$

where

$$\chi_{n-m}^{l-p} = |!A_{n,l} + B_{n,l-1} + C_{n-1,l} + D_{n-1,l-1}, \quad , -M \leq l \leq M, \quad , -N \leq n \leq N, \tag{4.30}$$

which is the Toeplitz matrix of order $(2N+1) \times (2M+1)$ and

$$G_{n,m}^{l,p} = \begin{cases} A_{n-1,l} + B_{n,l-1} + C_{n-1,l-1}, & n = N, \ l = M \\ A_{n-1,l} + C_{n-1,l-1}, & l = M \\ A_{n-1,l} + B_{n,l-1}, & n = N \\ 0, & -N+1 \leq n \leq N-1, \\ & -M+1 \leq l \leq M-1 \\ B_{n,l-1} + D_{n-1,l-1}, & l = -M \\ C_{n-1,l-1} + D_{n-1,l-1}, & n = -N \\ C_{n-1,l} + B_{n,l-1} + D_{n-1,l-1}, & l = -M, n = -N \end{cases} \tag{4.31}$$

The formula (4.30) represents the elements of a Toeplitz matrix of order $(2N+1) \times (2M+1)$,while (4.31) is a matrix of order $(2N+1) \times (2M+1)$ whose elements are zeros except the first and last rows and columns.

However, the linear algebraic system of (4.28) can be reduced to the following matrix form

$$[\mu I - \lambda(\chi - G)]\Phi = F, \qquad \| \mu I - \lambda(\chi - G) \| \neq 0. \tag{4.32}$$

Definition 4.1. The estimate local error $R_{N,M}$ can be determined by the following equation

$$\begin{aligned} &\Phi(x, y) - \Phi_{N,M}(x, y) = \\ &\sum_{n=-N}^{N} \sum_{l=-M}^{M} \chi_{n-m}^{l-p} [\Phi(nh, lh') - \Phi_{N,M}(nh, lh')] + R_{N,M}, \end{aligned} \tag{4.33}$$

where

$$\begin{aligned} R_{N,M} = | &\int_a^b \int_c^d k(x, u; y, v)\Phi(u, v) dv du \\ &- \sum_{n=-N}^{N} \sum_{l=-M}^{M} \chi_{n-m}^{l-p} \Phi(nh, lh')|. \end{aligned} \tag{4.34}$$

Definition 4.2. The Toeplitz matrix method is said to be convergent of order $r_1 + r_2$ in the domain $[a, b] \times [c, d]$, if and only if for large N, M, there exist a constant $D > 0$ independent of N, M such that

$$\| \Phi(x, y) - \Phi_{N,M}(x, y) \| \leq D N^{-r_1} M^{-r_2} \tag{4.35}$$

5. Convergence of algebraic system.

Theorem 5.1. *The linear algebraic system of Eq. (4.28), when $N, M \to \infty$ is bounded and has a unique solution under the following conditions*

$$\text{(i)}\ \| f(x, y) \| = | \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f^2(nh, lh')|^{\frac{1}{2}} < H, \quad H \text{ is a constant.} \tag{5.36}$$

$$\text{(ii)} | \sum_{l=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} (\chi_{n-m}^{l-p})^2 |^{\frac{1}{2}} < Q, \quad Q \text{ is a constant.} \tag{5.37}$$

$$\text{(iii)}\quad |\lambda| < \frac{|\mu|}{Q}. \tag{5.38}$$

Proof. The convergence of the linear system (4.28) can be proved by considering the two bounded sets $\bar{x}, \ \bar{y} \in \{\bar{s}\}$, where $\{\bar{s}\}$ is the family set of all bounded vectors.

Here, the metric space will defined by

$$\rho(\bar{x}, \bar{y}) = \sup_{l} |\bar{x}_l - \bar{y}_l|, \tag{5.39}$$

and

$$\begin{gathered} \bar{x} = (x_1, x_2), \quad \bar{y} = (y_1, y_2), \\ x_i = \{x_l^{(i)}\}_{l=-\infty}^{\infty}, \quad y_i = \{y_l^{(i)}\}_{l=-\infty}^{\infty}, \end{gathered} \qquad i = 1, 2. \tag{5.40}$$

Set the operator sum $L \subset \{\bar{s}\} : R^2 \to R^2$ such that

$$\bar{z} = L\bar{x}, \qquad \bar{z} = (z_1, z_2) \in R^2. \tag{5.41}$$

Let

$$\bar{z} = \sum_{n,l=-\infty}^{\infty} L_{l,n} \bar{x} + \bar{C}, \tag{5.42}$$

where

$$\bar{C} = (c_1, c_2), \quad c_i = \{c_l^{(i)}\}_{l=-\infty}^{\infty} \in R, \quad i = 1, 2. \tag{5.43}$$

The system of Eq.(4.28), when $N, M \to \infty$, can be written in the form

$$x_{m,p} = \frac{1}{\mu} f_l^{m,p} + \frac{\lambda}{\mu} \sum_{l,n=-\infty}^{\infty} \chi_{n-m}^{l-p} x_{l,n}. \tag{5.44}$$

From the two linear system of Eq.(5.42) and (5.44), we take

$$S_{m,p} = \frac{\lambda}{\mu} \sum_{l,n=-\infty}^{\infty} \chi_{n-m}^{l-p} = \frac{\lambda}{\mu} \sum_{l,n=-\infty}^{\infty} K_{l,n}. \tag{5.45}$$

Apply Cauchy-Minkowski inequality, with the aid of condition (ii) of theorem (5.1), we get

$$S_{m,p} < |\frac{\lambda}{\mu}| Q < 1. \tag{5.46}$$

Therefore, under the condition $|\lambda| < \frac{|\mu|}{Q}$ we have a unique solution of (5.44), and the value of $|x_{m,p}|$ satisfies

$$|x_{m,p}| < |\frac{H}{\mu - \lambda}|, \qquad \mu \neq \lambda. \tag{5.47}$$

To prove $x_{m,p}$ is the unique solution of Eq. (5.44), assume $x_{m,p}^*$ is another solution. Hence, we get

$$x_{m,p} - x_{m,p}^* = \frac{\lambda}{\mu} \sum_{l,n=-\infty}^{\infty} \chi_{n-m}^{l-p} [x_{l,n} - x_{m,p}^*]. \tag{5.48}$$

Applying Cauchy-Schwarz inequality and conditions (i) and (ii) of theorem (5.1), we get

$$\| x_{m,p} - x_{m,p}^* \| \leq |\frac{\lambda}{\mu}| Q \| x_{m,p} - x_{m,p}^* \| . \tag{5.49}$$

Therefore, we have $x_{m,p} = \tilde{x}_{m,p}$.

Corollary 5.1. *Assume that, conditions of Theorem (5.1) are verified, then*

$$\lim_{N,M \to \infty} R_{N,M} = 0. \tag{5.50}$$

Proof. The formula (4.33), yields

$$R_{N,M} = [\Phi(x,y) - \Phi_{N,M}(x,y)] - \sum_{n=-N}^{N} \sum_{m=-M}^{M} \chi_{n-m}^{l-p}[\Phi(nh, lh') - \Phi_{N,M}(nh, lh')]. \tag{5.51}$$

Hence, we have

$$\| R_{N,M} \| \leq \| \Phi(x,y) - \Phi_{N,M}(x,y) \| + \left| \sum_{n=-N}^{N} \sum_{m=-M}^{M} \chi_{n-m}^{l-p}[\Phi(nh, lh') - \Phi_{N,M}(nh, lh')] \right|, \tag{5.52}$$

therefore

$$\| R_{N,M} \| < \| \Phi(x,y) - \Phi_{N,M}(x,y) \| \, (1+Q). \tag{5.54}$$

Since $\| \Phi(x,y) - \Phi_{N,M}(x,y) \| \to 0$ as $N,\ M \to \infty$, then $\| R_{n,m} \| \to 0$.

6. The product Nystrom method.

In this section, we discuss the product Nystrom method [4] by considering the integral equation

$$\mu\Phi(x,y) - \lambda \int_a^b \int_c^d p(x,u;y,v)\bar{k}(x,u;y,v)\Phi(u,v)dvdu = f(x,y) \tag{6.54}$$

where p and $\bar{k}$ are respectively 'well behaved' and 'badly behaved' functions of their arguments, and $f(x,y)$ is a given function, while $\Phi(x,y)$ is the unknown function.

The formula (6.54) can be written in the form

$$\mu\Phi_{N,M}(x_i, y_k) - \lambda \sum_{j=0}^{N} \sum_{l=0}^{M} w_{ijkl}\, p(x_i, u_j; y_k, v_l)\Phi_{N,M}(u_j, v_l) = f(x_i, y_k), \tag{6.55}$$

where $x_i = u_i = a + ih, \quad i = 0, 1, 2, ..., N$ with $h = \dfrac{b-a}{N}, \quad N$ *even*, and $y_k = v_k = c + kh', \quad k = 0, 1, 2, ..., M$ with $h' = \dfrac{d-c}{M}, \quad M$ *even*, and the approximate numerical solution $\Phi_{N,M}$ satisfy

$$\| \Phi(x, y) - \Phi_{N,M}(x, y) \| \to 0, \qquad as\ N,\ M \to \infty. \tag{6.56}$$

Here, the estimate error $R_{N,M}$ can be defined from the following

$$\begin{aligned} \Phi(x, y) - \Phi_{N,M}(x, y) = \sum_{j=0}^{N} \sum_{l=0}^{M} w_{ijkl}\, p(x_i, u_j; y_k, v_l)\cdot \\ \cdot [\Phi(u_j, v_l) - \Phi_{N,M}(u_j, v_l)] + R_{N,M}, \end{aligned} \tag{6.57}$$

$$\begin{aligned} R_{N,M} = \Big| \int_a^b \int_c^d k(x, u; y, v)\Phi(u, v) dv du \\ - \sum_{j=0}^{N} \sum_{l=0}^{M} w_{ijkl}\, p(x_i, u_j; y_k, v_l)\Phi(u_j, v_l) \Big|. \end{aligned} \tag{6.58}$$

Therefore, in the light of corollary (5.1), we can prove $R_{N,M} \to 0$ as $N,\ M \to \infty$.

The product Nystrom method is said to be convergent of order r_1+r_2 in the domain $[a, b] \times [c, d]$, if and only if for large N, M, there exist a constant $D > 0$ independent of N, M such that

$$\| \Phi(x, y) - \Phi_{N,M}(x, y) \| \leq D N^{-r_1} M^{-r_2} \tag{6.59}$$

The weights functions w_{ijkl} are constructed by insisting that the rule in (6.55) be exact when $p(x_i, u; y_k, v)\Phi(u, v)$ is a polynomial of degree $\leq r_1 r_2$, say.

According to the product Nystrom method, we approximate the integral term in (6.54), when $x = x_i,\ y = y_k$, by a product integration form such as Simpson rule, therefore we write

$$\begin{aligned} \int_a^b \int_c^d p(x_i, u; y_k, v)\bar{k}(x_i, u; y_k, v)\Phi(u, v) dv du \\ = \sum_{j=0}^{\frac{N-2}{2}} \sum_{l=0}^{\frac{M-2}{2}} \int_{u_{2j}}^{u_{2j+2}} \int_{v_{2l}}^{v_{2l+2}} p(x_i, u; y_k, v)\bar{k}(x_i, u; y_k, v)\Phi(u, v) dv du. \end{aligned} \tag{6.60}$$

Now, if we approximate the nonsingular part of the integrand over each interval $[y_{2j}, y_{2j+2}]$ by the second degree Lagrange interpolation polynomial which interpolates it at the points y_{2j}, y_{2j+1}, y_{2j+2}, we obtain

$$\begin{aligned}\int_a^b \int_c^d p(u_i, u; v_k, v)\bar{k}(u_i, u; v_k, v)\Phi(u, v)dvdu \\ = \sum_{j=0}^{N}\sum_{l=0}^{M} w_{ijkl}\, p(u_i, u_j; v_k, v_l)\Phi(u_j, v_l),\end{aligned} \tag{6.61}$$

where,

$$\begin{aligned}
w_{i,k,0,0} &= \beta_2(i, k, 1, 1),\\
w_{i,k,2j+1,0} &= -2\beta_3(i, k, j+1, 1),\\
w_{i,k,2j,0} &= \beta_1(i, k, j, 1) + \beta_2(i, k, j+1, 1),\\
w_{i,k,N,0} &= \beta_1(i, k, N/2, 1),\\
w_{i,k,0,2l+1} &= -2\gamma_2(i, k, 1, l+1),\\
w_{i,k,2j+1,2l+1} &= 4\gamma_3(i, k, j+1, l+1),\\
w_{i,k,2j,2l+1} &= \gamma_1(i, k, j, l+1) + \gamma_2(i, k, j+1, l+1),\\
w_{i,k,N,2l+1} &= \gamma_1(i, k, N/2, l+1),\\
w_{i,k,0,2l} &= \alpha_2(i, k, 1, l) + \beta_2(i, k, 1, l+1),\\
w_{i,k,2j+1,2l} &= -2[\alpha_3(i, k, j+1, l) + \beta_3(i, k, j+1, l+1)],\\
w_{i,k,2j,2l} &= \alpha_1(i, k, j, l) + \beta_2(i, k, j+1, l+1),\\
w_{i,k,N,2l} &= \alpha_1(i, k, N/2, l) + \beta_1(i, k, N/2, l+1),\\
w_{i,k,0,M} &= \alpha_2(i, k, 1, M/2),\\
w_{i,k,2j+1,M} &= -2\alpha_3(i, k, j+1, M/2),\\
w_{i,k,2j,M} &= \alpha_1(i, k, j, M/2) + \alpha_2(i, k, j+1, M/2),\\
w_{i,k,N,M} &= \alpha_1(i, k, N/2, M/2),
\end{aligned} \tag{6.62}$$

such that

$$\alpha_1(i,k,j,l) = \frac{1}{4h^2h'^2}\int_{v_{2l-2}}^{v_{2l}}\int_{u_{2j-2}}^{u_{2j}}\bar{k}(u_i,u;v_k,v)(u-u_{2j-2})\cdot \\ \cdot(u-u_{2j-1})(v-v_{2l-2})(v-v_{2l-1})dudv,$$

$$\beta_1(i,k,j,l) = \frac{1}{4h^2h'^2}\int_{v_{2l-2}}^{v_{2l}}\int_{u_{2j-2}}^{u_{2j}}\bar{k}(u_i,u;v_k,v)(u-u_{2j-2})\cdot \\ \cdot(u-u_{2j-1})(v-v_{2l-1})(v-v_{2l})dudv,$$

$$\gamma_1(i,k,j,l) = \frac{1}{4h^2h'^2}\int_{v_{2l-2}}^{v_{2l}}\int_{u_{2j-2}}^{u_{2j}}\bar{k}(u_i,u;v_k,v)(u-u_{2j-2})\cdot \\ \cdot(u-u_{2j-1})(v-v_{2l-2})(v-v_{2l})dudv,$$

$$\alpha_2(i,k,j,l) = \frac{1}{4h^2h'^2}\int_{v_{2l-2}}^{v_{2l}}\int_{u_{2j-2}}^{u_{2j}}\bar{k}(u_i,u;v_k,v)(u-u_{2j-1})\cdot \\ \cdot(u-u_{2j})(v-v_{2l-2})(v-v_{2l-1})dudv,$$

$$\beta_2(i,k,j,l) = \frac{1}{4h^2h'^2}\int_{v_{2l-2}}^{v_{2l}}\int_{u_{2j-2}}^{u_{2j}}\bar{k}(u_i,u;v_k,v)(u-u_{2j-1})\cdot \\ \cdot(u-u_{2j})(v-v_{2l-1})(v-v_{2l})dudv, \tag{6.63}$$

$$\gamma_2(i,k,j,l) = \frac{1}{4h^2h'^2}\int_{v_{2l-2}}^{v_{2l}}\int_{u_{2j-2}}^{u_{2j}}\bar{k}(u_i,u;v_k,v)(u-u_{2j-1})\cdot \\ \cdot(u-u_{2j})(v-v_{2l-2})(v-v_{2l})dudv,$$

$$\alpha_3(i,k,j,l) = \frac{1}{4h^2h'^2}\int_{v_{2l-2}}^{v_{2l}}\int_{u_{2j-2}}^{u_{2j}}\bar{k}(u_i,u;v_k,v)(u-u_{2j-2})\cdot \\ \cdot(u-u_{2j})(v-v_{2l-2})(v-v_{2l-1})dudv,$$

$$\beta_3(i,k,j,l) = \frac{1}{4h^2h'^2}\int_{v_{2l-2}}^{v_{2l}}\int_{u_{2j-2}}^{u_{2j}}\bar{k}(u_i,u;v_k,v)(u-u_{2j-2})\cdot \\ \cdot(u-u_{2j})(v-v_{2l-1})(v-v_{2l})dudv,$$

$$\gamma_3(i,k,j,l) = \frac{1}{4h^2h'^2}\int_{v_{2l-2}}^{v_{2l}}\int_{u_{2j-2}}^{u_{2j}}\bar{k}(u_i,u;v_k,v)(u-u_{2j-2})\cdot \\ \cdot(u-u_{2j})(v-v_{2l-2})(v-v_{2l})dudv.$$

Therefore, the integral equation (6.54) is reduced to the following system of linear algebraic equations

$$\mu\Phi_{N,M}(x_i, y_k) - \lambda \sum_{j=0}^{N}\sum_{l=0}^{M} w_{ijkl}\, p(x_i, u_j; y_k, v_l)\Phi_{N,M}(u_j, v_l) = f(x_i, y_k), \tag{6.64}$$

which can be written in matrix form.

By following the same way of Toeplitz matrix, the existence and uniqueness solution of the linear algebraic system (6.64) can be proved under the following condition

$$|\lambda| < \frac{|\mu|}{P}, \quad P > |\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}(w_{ijkl}\, p(x_i, u_j; y_k, v_l))^2|^{\frac{1}{2}}. \tag{6.65}$$

7. Numerical results.

In this section, we apply the Toeplitz method and Nystrom method, for different cases. The error, in each case, is computed.

Example 1: (Logarithmic kernels).

Consider the two dimensional Fredholm integral equation

$$\mu\Phi(x, y) - \lambda \int_{-1}^{1}\int_{-1}^{1} ln|x-u|ln|y-v|\Phi(u, v)dvdu = f(x, y), \tag{7.66}$$

where the exact solution is $\Phi(x, y) = 2x^2y^2 + xy$, $\mu = 1$, $\lambda = 0.01$, and $f(x, y)$ is given by

$$\begin{aligned} f(x, y) = \mu(2x^2y^2 + xy) - \frac{2\lambda}{9}&[(1-x^3)ln|1-x| \\ &+ (1+x^3)ln|1+x| - 2x^2 - \frac{2}{3}]\cdot \\ &\cdot [(1-y^3)ln|1-y| + (1+y^3)ln|1+y| - 2y^2 - \frac{2}{3}] \\ &- \frac{\lambda}{4}[(1-x^2)(ln|1-x| - ln|1+x|) - 2x]\cdot \\ &\cdot [(1-y^2)(ln|1-y| - ln|1+y|) - 2y]. \end{aligned} \tag{7.67}$$

Discussion

(i) For $N = M$: In Tables 1 - 3, the numerical solution Φ^T and its error R^T for Toeplitz matrix method, in Eq.(4.28), are obtained for $N = M = 1,\ 2$ *and* 3. Also, we calculate Φ^N and evaluate R^N for the same corresponding points when $N = M = 2,\ 4$ *and* 6, in Eq. (6.64).

We find that the error is decreasing in the two methods for increasing $N = M$, see Figures 1-3.

Table 1

x	y	ϕ	ϕ^T	R^T	ϕ^N	R^N	R^N–R^T
-1	-1	3	2.994911	0.005089	3.003469	0.003469	-0.00162
-1	0	0	0.001005	0.001005	0.002187	0.002187	0.001182
-1	1	1	1.008181	0.008181	1.001501	0.001501	-0.00668
0	-1	0	0.001005	0.001005	-0.00079	0.000789	-0.00022
0	0	0	0.003425	0.003425	-0.00012	0.000123	-0.0033
0	1	0	0.005408	0.005408	0.000315	0.000315	-0.00509
1	-1	1	1.008181	0.008181	0.993484	0.006516	-0.00166
1	0	0	0.005408	0.005408	-0.00266	0.002658	-0.00275
1	1	3	2.997506	0.002494	2.999721	0.000279	-0.00222

Table 2

x	y	ϕ	ϕ^T	R^T	ϕ^N	R^N	R^N–R^T
-1	-1	3	2.998118	0.001882	2.992166	0.007834	0.005952
-1	-0.5	1	0.999326	0.000674	0.991031	0.008969	0.008296
-1	0	0	0.000155	0.000155	0.002165	0.002165	-0.002011
-1	0.5	0	0.001603	0.001603	0.011073	0.011073	0.00947
-1	1	1	1.004291	0.004291	1.005565	0.005565	0.001274
-0.5	-1	1	0.999326	0.000674	0.991605	0.008395	0.007722
-0.5	-0.5	0.375	0.375287	0.000287	0.36599	0.00901	0.008722
-0.5	0	0	0.000598	0.000598	0.001014	0.001014	0.000416
-0.5	0.5	-0.125	-0.12346	0.001536	-0.11758	0.007422	0.005886
-0.5	1	0	0.003206	0.003206	0.003367	0.003367	0.000161
0	-1	0	0.000155	0.000155	0.005472	0.005472	0.005317
0	-0.5	0	0.000598	0.000598	0.002082	0.002082	0.001483
0	0	0	0.000453	0.000453	0.001462	0.001462	0.001009
0	0.5	0	0.000686	0.000686	0.000152	0.000152	-0.00053
0	1	0	0.001992	0.001992	-0.00487	0.004866	0.002874
0.5	-1	0	0.001603	0.001603	0.013062	0.013062	0.011458
0.5	-0.5	-0.125	-0.12346	0.001536	-0.11572	0.00928	0.007744
0.5	0	0	0.000686	0.000686	0.002371	0.002371	0.001685
0.5	0.5	0.375	0.373949	0.001051	0.369557	0.005443	0.004392
0.5	1	1	0.999424	0.000576	0.988385	0.011615	0.011039
1	-1	1	1.004291	0.004291	1.007541	0.007541	0.00325
1	-0.5	0	0.003206	0.003206	0.006904	0.006904	0.003698
1	0	0	0.001992	0.001992	0.000104	0.000104	-0.00189
1	0.5	1	0.999424	0.000576	0.991117	0.008883	0.008307
1	1	3	2.996566	0.003434	2.990885	0.009115	0.005681

Table 3

x	y	ϕ	ϕ^T	R^T	ϕ^N	R^N	$R^N - R^T$
-1	-1	3	2.998959	0.001041	2.991851	0.008149	0.007108
-1	-0.66667	1.555556	1.555042	0.000514	1.544832	0.010723	0.01021
-1	-0.33333	0.555556	0.555435	0.00012	0.552459	0.003096	0.002976
-1	0	0	7.24E-05	7.24E-05	0.007074	0.007074	0.007002
-1	0.33333	-0.11111	-0.11073	0.000383	-0.10118	0.009935	0.009552
-1	0.666667	0.2222222	0.223519	0.001297	0.229929	0.007707	0.00641
-1	1	1	1.002733	0.002733	1.003696	0.003696	0.000964
-0.66667	-1	1.555556	1.555042	0.000514	1.54486	0.010696	0.010182
-0.66667	-0.66667	0.839506	0.839489	1.68E-05	0.825562	0.013944	0.013928
-0.66667	-0.33333	0.320988	0.321194	0.000206	0.315269	0.005718	0.005512
-0.66667	0	0	0.000278	0.000278	0.004502	0.004502	0.004224
-0.66667	0.33333	-1.2346	-0.123	0.000459	-0.1149	0.00856	0.008101
-0.66667	0.666667	-0.04938	-0.04834	0.001044	-0.04239	0.006997	0.005953
-0.66667	1	0.22222	0.224185	0.001963	0.225709	0.003487	0.001524
-0.33333	-1	0.555556	0.555435	0.00012	0.553895	0.00166	0.00154
-0.33333	-0.66667	0.320988	0.321194	0.000206	0.315602	0.005386	0.005179
-0.33333	-0.33333	0.135802	0.136056	0.000253	0.134572	0.001231	0.000978
-0.33333	0	0	0.000226	0.000226	0.005823	0.005823	0.005597
-0.33333	0.33333	-0.08642	-0.08612	0.000302	-0.07923	0.007192	0.00689
-0.33333	0.666667	-0.12346	-0.12281	0.000644	-0.11989	0.003562	0.002918
-0.33333	1	-0.11111	-0.10996	0.001149	-0.11157	0.000454	-0.00069
0	-1	0	7.24E-05	7.24E-05	0.007228	0.007228	0.007156
0	-0.66667	0	0.000278	0.000278	0.004744	0.004744	0.004466
0	-0.33333	0	0.000226	0.000226	0.006473	0.006473	0.006248
0	0	0	0.000179	0.000179	0.011326	0.011326	0.011147
0	0.333333	0	0.000208	0.000208	0.007282	0.007282	0.007073
0	0.666667	0	0.000325	0.000325	-0.00035	0.000351	2.61E-05
0	1	0	0.001033	0.001033	-0.00427	0.004266	0.003234
0.33333	-1	-0.11111	-0.11073	0.000383	-0.10185	0.009259	0.008876
0.33333	-0.66667	-0.12346	-0.123	0.000459	-0.11518	0.008278	0.007819
0.33333	-0.33333	-0.08642	-0.08612	0.000302	-0.07959	0.006833	0.006531
0.33333	0	0	0.000208	0.000208	0.004894	0.004894	0.004685
0.33333	0.33333	0.135802	0.135881	7.85E-05	0.134745	0.001057	0.000979
0.33333	0.666667	0.320988	0.320652	0.000336	0.312021	0.008967	0.008631
0.33333	1	0.555556	0.556209	0.000654	0.545466	0.01009	0.009435
0.66667	-1	0.222222	0.223519	0.001297	0.230714	0.008492	0.007195
0.66667	-0.66667	-0.04938	-0.04834	0.001044	-0.04081	0.008569	0.007525
0.66667	-0.33333	-0.12346	-0.12281	0.000644	-0.11729	0.006166	0.005522
0.66667	0	0	0.000325	0.000325	0.00035	0.00035	2.49E-05
0.66667	0.33333	0.320988	0.320652	0.000336	0.312815	0.008172	0.007837
0.66667	0.666667	0.839506	0.837571	0.0019356	0.822811	0.016695	0.01476
0.66667	1	1.555556	1.554177	0.001378	1.54091	0.014646	0.013268
1	-1	1	1.002733	0.002733	1.006329	0.006329	0.003569
1	-0.66667	0.222222	0.224185	0.001963	0.229404	0.007181	0.005219
1	-0.33333	-0.11111	-0.10996	0.001149	-0.10614	0.004973	0.003825
1	0	0	0.001033	0.001033	-0.00102	0.001025	-8E-06
1	0.33333	0.555556	0.556209	0.000654	0.547613	0.007943	0.007289
1	0.666667	1.555556	1.554177	0.001387	1.542374	0.013181	0.011803
1	1	3	2.99721	0.00279	2.990482	0.009518	0.006728

(ii) For $N \neq M$:

If $N \neq M$, for example $N = 2$, $M = 1$ with respect to Toeplitz matrix method and its corresponding values of Nystrom method, the error is bigger than the previous results when $N = M$ (shown by Table 4 and Figure 4).

Table 4

x	y	ϕ	ϕ^T	R^T	ϕ^N	R^N	$R^N – R^T$
-1	-1	3	2.988415	0.011585	2.994564	0.005436	-0.00615
-1	0	0	-0.00292	0.002922	-0.001912	0.001912	-0.00101
-1	1	1	1.008598	0.008598	0.998524	0.001476	-0.00712
-0.5	-1	1	0.990137	0.009863	0.992707	0.007293	-0.00257
-0.5	0	0	-0.002313	0.002313	-0.002375	0.002375	6.2E-05
-0.5	1	0	0.007229	0.007229	-0.001624	0.001624	-0.0056
0	-1	0	-0.001959	0.001959	0.00881	0.00881	0.006851
0	0	0	0.000336	0.000336	3.01E-05	3.01E-05	-0.00031
0	1	0	0.003449	0.003449	-0.009653	0.009653	0.006154
0.5	-1	0	0.002605	0.002605	0.013828	0.013828	0.011223
0.5	0	0	0.001975	0.001975	-0.001028	0.001028	-0.00095
0.5	1	1	1.00107	0.00107	0.982969	0.017031	0.016924
1	-1	1	1.00882	0.00882	1.001752	0.001752	-0.00707
1	0	0	0.004798	0.004798	-0.002227	0.002227	-0.00257
1	1	3	2.995932	0.004068	2.992215	0.007785	0.003717

Table 5

x	y	ϕ	ϕ^T	R^T	ϕ^N	R^N	$R^N – R^T$
-1	-1	3	3.017511	0.017511	3.000137	0.000137	-0.01737
-1	0	0	0.017593	0.017593	7.17E-05	7.17E-05	-0.01752
-1	1	1	1.017571	0.017571	1.000029	2.95E-05	-0.01754
0	-1	0	0.017606	0.017606	0.00013	0.00013	-0.01748
0	0	0	0.017681	0.017681	6.77E-05	6.77E-05	-0.01761
0	1	0	0.017647	0.017647	2.8E-05	2.8E-05	-0.01762
1	-1	1	1.017579	1.017579	1.000119	0.000119	-0.01746
1	0	0	0.017642	0.017642	6.21E-05	6.21E-05	-0.01758
1	1	3	3.017587	0.017587	3.000026	2.57E-05	-0.01756

Example 2: (Carleman kernels)

Consider the two dimensional Fredholm integral equation

$$\mu\Phi(x,y)-\lambda\int_{-1}^{1}\int_{-1}^{1}|x-u|^{-\nu_1}|y-v|^{-\nu_2}\Phi(u,v)dvdu = f(x,y), \tag{7.68}$$

where the exact solution is $\Phi(x,y)=2x^2y^2+xy$ and $f(x,y)$ is given by

$$\begin{aligned}
f(x,y) &= \mu(2x^2y^2+xy)-2\lambda[\frac{1}{1-\nu_1}(|1-x|^{1-\nu_1}-|1+x|^{1-\nu_1})\\
&-\frac{2}{(1-\nu_1)(2-\nu_1)}(|1-x|^{2-\nu_1}+|1+x|^{2-\nu_1})\\
&+\frac{2}{(1-\nu_1)(2-\nu_1)(3-\nu_1)}(|1-x|^{3-\nu_1}-|1+x|^{3-\nu_1})]\\
&[\frac{1}{1-\nu_2}(|1-y|^{1-\nu_2}-|1+y|^{1-\nu_2})-\frac{2}{(1-\nu_2)(2-\nu_2)}(|1-y|^{2-\nu_2}+|1+y|^{2-\nu_2})\\
&+\frac{2}{(1-\nu_2)(2-\nu_2)(3-\nu_2)}(|1-y|^{3-\nu_2}-|1+y|^{3-\nu_2})]\\
&-\lambda[\frac{1}{1-\nu_1}(|1-x|^{1-\nu_1}+|1+x|^{1-\nu_1})\\
&-\frac{1}{(1-\nu_1)(2-\nu_1)}(|1-x|^{2-\nu_1}-|1+x|^{2-\nu_1})][\frac{1}{1-\nu_2}(|1-y|^{1-\nu_2}+|1+y|^{1-\nu_2})-\\
&\frac{2}{(1-\nu_2)(2-\nu_2)}(|1-y|^{2-\nu_2}+|1+y|^{2-\nu_2})].
\end{aligned}$$

Discussion

(i) For $N=M$: Let $\mu=1,\ \lambda=0.01, \nu_1=0.05, \nu_2=0.04,$ then the approximate solution Φ^T and the error E^T of the integral equation (7.68), using Toeplitz matrix method by taking $N=M=1,\ 2\ and\ 3$ in Eq.(4.28), is displayed by Table 4, 5 and 6 respectively. Also, it displays the values of the approximate solution Φ^N and the error E^N at the same points for the same integral equation but by using the product Nystrom method with $N=M=2,\ 4,\ and\ 6$ in Eq.(6.64). The errors of Toeplitz matrix method and Nystrom method are decreasing for increasing $M\ and\ N$, see Figures 5-7.

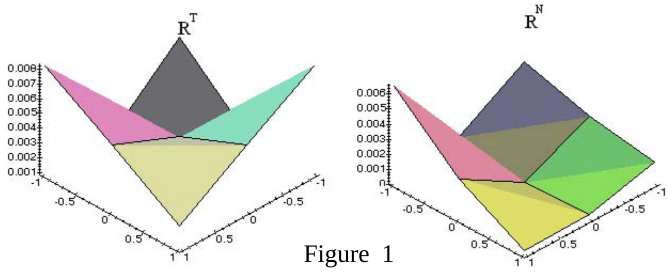

Figure 1

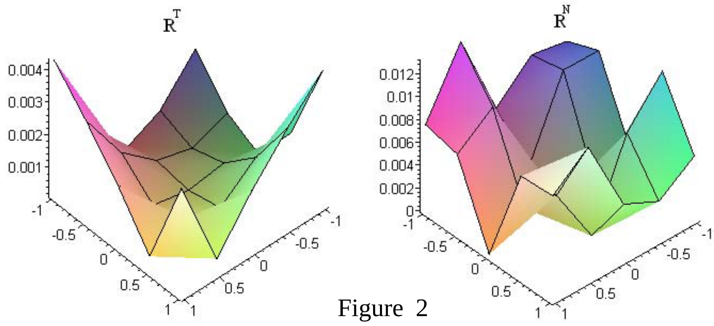

Figure 2

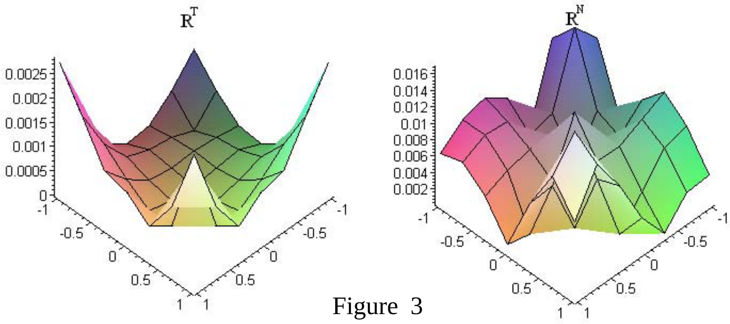

Figure 3

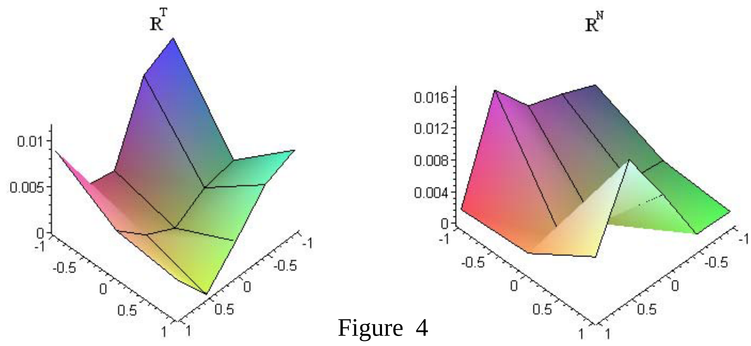

Figure 4

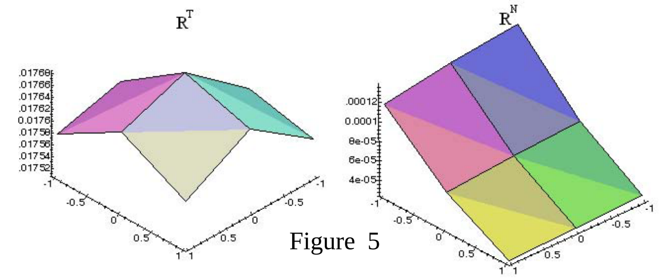

Figure 5

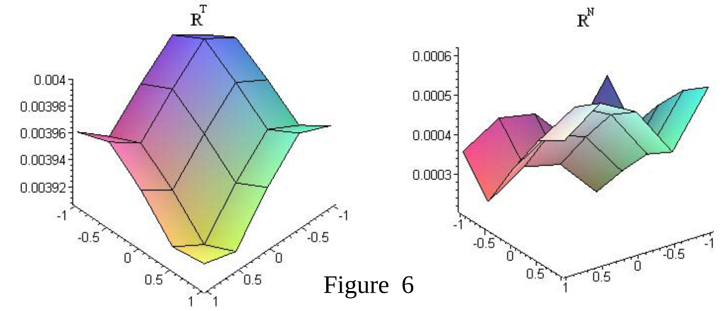

Figure 6

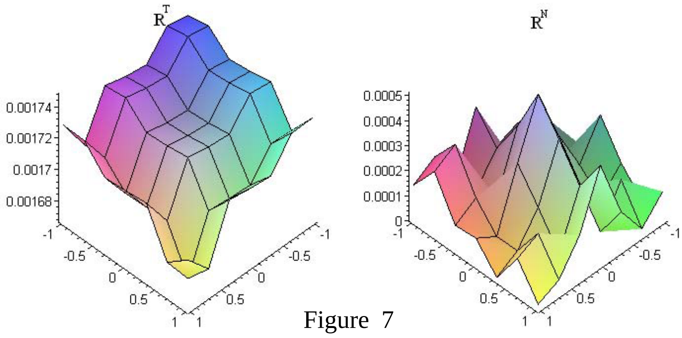

Figure 7

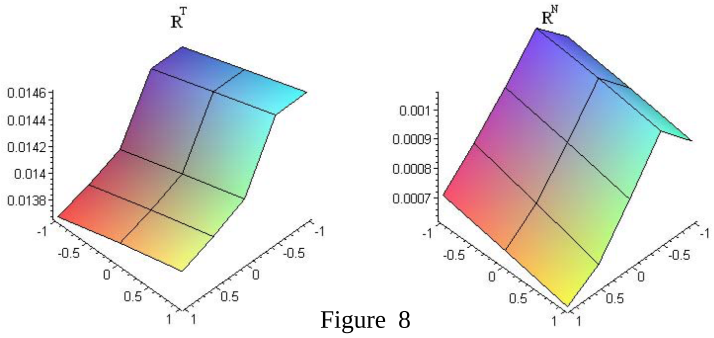

Figure 8

Table 6

x	y	ϕ	ϕ^T	R^T	ϕ^N	R^N	$R^N–R^T$
-1	-1	3	3.00397	0.00397	2.999562	0.000438	-0.00353
-1	-0.5	1	1.003983	0.003983	0.999649	0.000351	-0.00363
-1	0	0	0.003968	0.003968	-0.00044	0.00044	-0.00352
-1	0.5	0	0.00395	0.00395	-0.000522	0.000522	-0.00343
-1	1	1	1.003965	0.003965	0.999429	0.000571	-0.00339
-0.5	-1	1	1.003985	0.003985	0.9997	0.0003	-0.00369
-0.5	-0.5	0.375	0.378998	0.003998	0.374789	0.000211	-0.00379
-0.5	0	0	0.003982	0.003982	-0.0003	0.000304	-0.00368
-0.5	0.5	-0.125	-0.12104	0.003963	-0.12539	0.000388	-0.00357
-0.5	1	0	0.003976	0.003976	-0.000433	0.000433	-0.00354
0	-1	0	0.003965	0.003965	-0.000397	0.000397	-0.00357
0	-0.5	0	0.003977	0.003977	-0.000309	0.000309	-0.00367
0	0	0	0.003959	0.003959	-0.000419	0.000419	0.00354
0	0.5	0	0.003939	0.003939	-0.000516	0.000516	-0.00342
0	1	0	0.003951	0.003951	-0.000564	0.000564	-0.00339
0.5	-1	0	0.003942	0.003942	-0.000415	0.000415	-0.00353
0.5	-0.5	-0.125	-0.12105	0.003952	-0.12533	0.000331	-0.00362
0.5	0	0	0.003933	0.003933	-0.000455	0.000455	-0.00348
0.5	0.5	0.375	0.378909	0.003909	0.374436	0.000564	-0.00334
0.5	1	1	1.003919	0.003919	0.999386	0.000614	-0.00331
1	-1	1	1.003961	0.003961	0.999642	0.000358	-0.0036
1	-0.5	0	0.003969	0.003969	-0.00027	0.00027	-0.0037
1	0	0	0.003949	0.003949	-0.00039	0.000396	-0.00355
1	0.5	1	1.003923	0.003923	0.999493	0.000507	-0.00342
1	1	3	3.003926	0.003926	2.999454	0.000546	-0.00338

Table 7

x	y	ϕ	ϕ^T	R^T	ϕ^N	R^N	$R^N–R^T$
-1	-1	3	3.001742	0.001742	2.999959	4.13E-05	-0.0017
-1	-0.66667	1.555556	1.5573	0.001745	1.55562	6.44E-05	-0.00168
-1	-0.33333	0.555556	0.557288	0.001733	0.555483	7.23E-05	-0.00166
-1	0	0	0.001734	0.001734	-0.00025	0.000248	-0.00149
-1	0.33333	-0.11111	-0.10937	0.001737	-0.11123	0.000115	-0.00162
-1	0.666667	0.2222222	0.223938	0.001715	0.222244	2.22E-05	-0.00169
-1	1	1	1.001733	0.001733	0.999883	0.00017	-0.00162
-0.66667	-1	1.555556	1.5573	0.001745	1.555654	9.81E-05	-0.00165
-0.66667	-0.66667	0.839506	0.841254	0.001747	0.839709	0.000203	-0.00154
-0.66667	-0.33333	0.320988	0.322723	0.001735	0.321056	6.78E-05	-0.00167
-0.66667	0	0	0.001736	0.001736	-0.00011	0.000108	-0.00163
-0.66667	0.33333	-1.2346	-0.12172	0.001739	-0.12343	2.73E-05	-0.00171
-0.66667	0.666667	-0.04938	-0.04767	0.001716	-0.04922	0.000166	-0.00155
-0.66667	1	0.22222	0.223955	0.001732	0.22225	2.79E-05	-0.0017
-0.33333	-1	0.555556	0.557285	0.001729	0.55549	6.51E-05	-0.00166
-0.33333	-0.66667	0.320988	0.322719	0.001732	0.321029	4.14E-05	-0.00169
-0.33333	-0.33333	0.135802	0.137522	0.001719	0.135703	9.96E-05	-0.00162

segue: Table 7

x	y	ϕ	ϕ^T	R^T	ϕ^N	R^N	$R^N – R^T$
-0.33333	0	0	0.00172	0.00172	-0.000281	0.000281	-0.00144
-0.33333	0.33333	-0.08642	-0.0847	0.001723	-0.08657	0.000151	-0.00157
-0.33333	0.666667	-0.12346	-0.12176	0.001699	-0.12347	1.56E-05	-0.00168
-0.33333	1	-0.11111	-0.1094	0.001715	-0.11126	0.000154	-0.00156
0	-1	0	0.001731	0.001731	-0.00028	0.000279	-0.00145
0	-0.66667	0	0.001733	0.001733	-0.00017	0.000173	-0.00156
0	-0.33333	0	0.00172	0.00172	-0.00032	0.000319	-0.0014
0	0	0	0.001722	0.001722	-0.0005	0.000504	-0.0012
0	0.333333	0	0.001724	0.001724	-0.000385	0.000385	-0.00134
0	0.666667	0	0.0017	0.0017	-0.00026	0.000258	-0.00144
01	0	0.001716	0.001716	-0.0004	0.000397	-0.00132	
0.33333	-1	-0.11111	-0.10938	0.001735	-0.11116	4.41E-05	-0.00169
0.33333	-0.66667	-0.12346	-0.12172	0.001736	-0.12339	6.33E-05	-0.00167
0.33333	-0.33333	-0.08642	-0.0847	0.001724	-0.08651	9.05E-05	-0.00163
0.33333	0	0	0.001725	0.001725	-0.000289	0.000289	-0.00144
0.33333	0.33333	0.135802	0.137529	0.001726	0.135631	0.000172	-0.00155
0.33333	0.666667	0.320988	0.322689	0.001702	0.320943	4.46E-05	-0.00166
0.33333	1	0.555556	0.557274	0.001718	0.555377	0.000178	-0.00154
0.66667	-1	0.222222	0.223929	0.001706	0.222419	0.000196	-0.00151
0.66667	-0.66667	-0.04938	-0.04768	0.001707	-0.04908	0.000306	-0.0014
0.66667	-0.33333	-0.12346	-0.12176	0.001694	-0.12331	0.00015	-0.00154
0.66667	0	0	0.001695	0.001695	-5.56E-05	5.56E-05	-0.00164
0.66667	0.33333	0.320988	0.322683	0.001695	0.321045	5.75E-05	-0.00164
0.66667	0.666667	0.839506	0.841174	0.001668	0.839691	0.000185	-0.00148
0.66667	1	1.555556	1.557239	0.001683	1.555613	5.77E-05	-0.00163
1	-1	1	1.001728	0.001728	1.000144	0.000144	-0.00158
1	-0.66667	0.222222	0.22395	0.001728	0.222477	0.000255	-0.00147
1	-0.33333	-0.11111	-0.1094	0.001713	-0.11101	9.96E-05	-0.00161
1	0	0	0.001714	0.001714	-0.00011	0.000107	-0.00161
1	0.33333	0.555556	0.557271	0.001716	0.555565	9.26E-06	-0.00171
1	0.666667	1.555556	1.557243	0.001688	1.555699	0.000143	-0.00154
1	1	3	3.001698	0.001698	3.00002	1.97E-05	-0.00168

(ii) For $N \neq M$:

When $N \neq M$, for example $N = 2,\ M = 1$ with respect to Toeplitz matrix method and its corresponding values, $N = 4,\ M = 2$, of Nystrom method, the error is bigger than the previous results in which $N = M$ (see Table 8 and Figure 8).

Table 8

x	y	ϕ	ϕ^T	R^T	ϕ^N	R^N	R^N–R^T
-1	-1	3	3.014286	0.014286	3.00095	0.00095	-0.01334
-1	0	0	0.014445	0.014445	0.000924	0.000924	-0.01352
-1	1	1	1.014603	0.014603	1.000893	0.000893	-0.01371
-0.5	-1	1	1.014287	0.014287	1.001053	0.001053	-0.01323
-0.5	0	0	0.014444	0.014444	0.001032	0.001032	-0.01341
-0.5	1	0	0.014597	0.014597	0.001003	0.001003	-0.01359
0	-1	0	0.013851	0.013851	0.000926	0.000926	-0.01293
0	0	0	0.013997	0.013997	0.000889	0.000889	-0.01311
0	1	0	0.014136	0.014136	0.000845	0.000845	-0.01329
0.5	-1	0	0.013752	0.013752	0.000812	0.000812	-0.01294
0.5	0	0	0.013892	0.013892	0.000757	0.000757	-0.01313
0.5	1	1	1.014022	0.014022	1.000699	0.000699	-0.01332
1	-1	1	1.013678	0.013678	1.000709	0.000709	-0.01297
1	0	0	0.01381	0.01381	0.000671	0.000671	-0.01314
1	1	3	3.013925	0.013925	3.00063	0.00063	-0.0133

Conclusion:

1 - The error function of Toeplitz matrix method is smaller than the corresponding error of Nystrom method in the most cases of logarithmic kernel form except for the first values of $N,\ M$, see Tables 1-4.

2 - In Carleman kernels form $(\nu_1 = 0.05,\ \nu_2 = 0.04)$, Nystrom method is better than Toeplitz matrix method.

3 - In the following Table, we take different values of $\lambda = \{0.1,\ 0.01,\ 0.001\}$, where $N = M = 2$ (with respect to Toeplitz matrix method). The error, in both methods, is decreasing whenever λ decreases.

4 - In Carleman kernel form, when ν_1 *or* ν_2 approaches 1, for example $\nu_2 = 0.04$ and $\nu_1 = \{0.05,\ 0.5,\ 0.8\}$, the error is increasing in two methods (see Table 10).

Table 9

λ	max R^T	max R^N	max R^T	max R^N
0.001	0.000424	0.001321	0.000384	6.02E-05
0.01	0.004291	0.013062	0.003998	0.000614
0.1	0.048918	0.121001	0.068467	0.007835

Table 10

λ	max R^T	max R^N
0.05	0.003998	0.000614
0.5	0.007243	0.010904
0.8	0.021443	0.028935

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M. M. El-Borai
Alexandria University
Faculty of Science
Department of Mathematics
Egypt
E-mail: m_m_elborai@ yahoo.com

M. A. Abdou
Alexandria University
Faculty of Education
Department of Mathematics
Egypt
E-mail: abdella_77@ yahoo.com

M. Basseem
Alexandria
Tharwat
9 Mohammed Eqbal St.
Egypt
E-mail: basseem777@ yahoo.com