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# SUFFICIENCY CRITERIA FOR A CLASS OF $p$-VALENT ANALYTIC FUNCTIONS OF COMPLEX ORDER

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In this paper we obtain extensions of sufficient conditions for analytic functions $f(z)$ in the open unit disk $\mathcal{U}$ to be starlike and convex of complex order. Our results unify and extend some starlikeness and convexity conditions for analytic functions discussed by Mocanu [4], Uyanik et al. [6], Goyal et al. [2] and others.

## 1. Introduction

Let $\mathcal{A}_p(n)$ be the class of functions of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \qquad (p,n \in N := \{1,2,3,\ldots\}) \tag{1}$$

which are analytic in the open unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \ and \ |z| < 1\}$$

In particular $\mathcal{A}_1(n) = \mathcal{A}(n)$ and $\mathcal{A}_1(1) = \mathcal{A}$
A function $f(z) \in \mathcal{A}_p(n)$ is said to be starlike of complex order $b$ in $\mathcal{U}$ if and

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only if it satisfies the condition

$$\operatorname{Re}\left[1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-p\right)\right]>0 \qquad (p\in\mathbb{N}, b\in\mathbb{C}\backslash\{0\}) \tag{2}$$

we denote by $\mathcal{S}_p^*(n;b)$, the subclass of $\mathcal{A}_p(n)$ consisting of all functions $f(z)$ which are starlike of complex order $b$ in $\mathcal{U}$ and in particular

$\mathcal{S}_1^*(1;b)=\mathcal{S}^*(b)$ is the subclass of starlike functions of complex order $b$ in $\mathcal{A}$ and $\mathcal{S}_1^*(n;1)=\mathcal{S}^*$ is the subclass of starlike functions.

A function $f(z)\in\mathcal{A}_p(n)$ is said to be convex of complex order $b$ in $\mathcal{U}$ if and only if it satisfies the condition

$$\operatorname{Re}\left[1+\frac{1}{b}\left(\frac{zf''(z)}{f'(z)}-p+1\right)\right]>0 \tag{3}$$

we denote by $\mathcal{C}_p(n;b)$, the subclass of $\mathcal{A}_p(n)$ consisting of all functions $f(z)$ which are convex of complex order $b$ in $\mathcal{U}$ and in particular

$\mathcal{C}_1(1;b)=\mathcal{C}(b)$ is the subclass of convex functions of complex order $b$ in $\mathcal{A}$;
$\mathcal{C}_1(n;1)=\mathcal{C}$ is the subclass of convex functions.

## 2. Conditions for starlikeness of complex order $b$

In order to consider the starlikeness of complex order $b$ for $f(z)\in\mathcal{A}_p(n)$. We need the following lemmas.

**Lemma 2.1** (see [5]). *If $f(z)\in\mathcal{A}(n)$ satisfies the condition*

$$\left|f'(z)-\frac{f(z)}{z}\right|<\frac{n}{n+1}, \qquad z\in\mathcal{U},\ n\in\mathbb{N} \tag{4}$$

*then*

$$f(z)\in\mathcal{S}^*(n;1)$$

**Lemma 2.2** ([3]). *If $f(z)\in\mathcal{A}(n)$ satisfies the condition*

$$\left|\arg f'(z)\right|<\frac{\pi}{2}\delta_n \qquad (z\in\mathcal{U})$$

*where $\delta_n$ is the unique root of the equation*

$$2\tan^{-1}\left[n(1-\delta_n)\right]+\pi(1-2\delta_n)=0 \tag{5}$$

*then*

$$f(z)\in\mathcal{S}^*(n;1)$$

## 3. Main Results

**Theorem 3.1.** *If* $f(z) \in \mathcal{A}_p(n)$ *satisfies*

$$\left| \left( \frac{f(z)}{z} \right)^{\frac{1}{b}} \left( z^{\frac{1-p+b}{b}} \frac{f'(z)}{f(z)} - pz^{\frac{1-p}{b}} \right) \right| < \frac{n}{n+1} |b|$$

*for some* $b \in \mathbb{C} \setminus \{0\}$, *then* $f(z) \in \mathcal{S}_p^*(n;b)$.

*Proof.* Let us define a function $h(z)$ by

$$h(z) = \left( \frac{f(z)}{z^{p-b}} \right)^{\frac{1}{b}} = z + \frac{a_{p+n}}{b} z^{n+1} + \dots \tag{6}$$

for $f(z) \in \mathcal{A}_p(n)$. Then $h(z) \in \mathcal{A}(n)$.
Differentiating (6) logarithmically, we find that

$$\frac{h'(z)}{h(z)} = \frac{1}{b} \left[ \frac{f'(z)}{f(z)} - \frac{(p-b)}{z} \right] \tag{7}$$

which gives

$$\left| h'(z) - \frac{h(z)}{z} \right| = \left| \frac{1}{b} \left( \frac{f(z)}{z} \right)^{\frac{1}{b}} \left( z^{\frac{1-p+b}{b}} \frac{f'(z)}{f(z)} - pz^{\frac{1-p}{b}} \right) \right|. \tag{8}$$

Thus using the condition given with the theorem, we get

$$\left| h'(z) - \frac{h(z)}{z} \right| < \frac{n}{n+1} \qquad (z \in \mathcal{U}). \tag{9}$$

Hence using the Lemma 2.1, we have $h(z) \in \mathcal{S}^*(n;1)$. From (7), we infer that

$$\frac{zh'(z)}{h(z)} = 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - p \right) \tag{10}$$

Since

$$h(z) \in S^*(n;1) \Rightarrow \mathrm{Re} \left( \frac{zh'(z)}{h(z)} \right) > 0$$

therefore from (10), we get

$$\mathrm{Re} \left[ 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - p \right) \right] > 0 \qquad (b \in \mathbb{C} \setminus \{0\},\ z \in \mathcal{U})$$

Thus $f(z) \in \mathcal{S}_p^*(n;b)$ which completes the proof of the theorem. □

**Theorem 3.2.** *Let a function $h(z)$ be defined by*

$$h(z) = \left(\frac{f(z)}{z^{p-b}}\right)^{\frac{1}{b}} (b \in \mathbb{C}\backslash\{0\}, z \in \mathcal{U})$$

*for $f(z) \in \mathcal{A}_p(n)$. If $h(z)$ satisfies*

$$\left|h''(z)\right| \leq \frac{2n}{n+1} (b \in \mathbb{C}\backslash\{0\}, z \in \mathcal{U})$$

*then $f(z) \in \mathcal{S}_p^*(n;b)$.*

*Proof.* From (6), we have $h(z) \in \mathcal{A}(n)$. Also

$$\begin{aligned}\left|h'(z) - \frac{h(z)}{z}\right| &= \left|\frac{1}{z}\int_0^z th''(t)dt\right| \\ &\leq \frac{1}{|z|}\int_0^{|z|} \left|re^{i\theta} \cdot h''(re^{i\theta})\right|dr \\ &\leq \frac{2n}{(n+1)\,|z|}\int_0^{|z|} rdr \\ &< \frac{n}{n+1}\end{aligned}$$

This shows that $h(z)$ satisfies the condition of Lemma 2.1, thus $h(z) \in \mathcal{S}^*(n;1)$ which implies that $f(z) \in \mathcal{S}_p^*(n;b)$. □

**Theorem 3.3.** *If $f(z) \in \mathcal{A}_p(n)$ satisfies*

$$\left|\arg\left(\frac{f(z)}{z^p}\right)^{\frac{1}{b}} + \arg\left\{\frac{1}{b}\left(\frac{zf'(z)}{f(z)} - p + b\right)\right\}\right| < \frac{\pi}{2}\delta_n \tag{11}$$

*where $\delta_n$ is unique root of (5), then $f(z) \in \mathcal{S}_p^*(n;b)$.*

*Proof.* Let us define the function $h(z)$ by

$$h(z) = \left(\frac{f(z)}{z^{p-b}}\right)^{\frac{1}{b}} dt = z + \frac{a_{p+n}}{b}z^{n+1} + \ldots$$

for $f(z) \in \mathcal{A}_p(n)$. Then $h(z) \in \mathcal{A}(n)$ and

$$\arg h'(z) = \arg\left[\left(\frac{f(z)}{z^p}\right)^{\frac{1}{b}} \cdot \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - p + b\right)\right]$$
$$= \arg\left(\frac{f(z)}{z^p}\right)^{\frac{1}{b}} + \arg\left\{\frac{1}{b}\left(\frac{zf'(z)}{f(z)} - p + b\right)\right\}$$

In view of Lemma 2.2, we see that if

$$\left|\arg\left(\frac{f(z)}{z^p}\right)^{\frac{1}{b}} + \arg\left\{\frac{1}{b}\left(\frac{zf'(z)}{f(z)} - p + b\right)\right\}\right| < \frac{\pi}{2}\delta_n$$

then $h(z) \in \mathcal{S}^*(n;1)$. This implies that $f(z) \in \mathcal{S}_p^*(n;b)$. □

Setting $n = p = 1$ in Theorem 3.1, we get the following

**Corollary 3.4.** *If $f(z) \in \mathcal{A}$ satisfies*

$$\left|\left(\frac{f(z)}{z}\right)^{\frac{1}{b}}\left[\frac{zf'(z)}{f(z)} - 1\right]\right| < \frac{|b|}{2}$$

*then $f(z) \in \mathcal{S}^*(b)$.*

Setting $b = 1$ in above Corollary, we get

**Corollary 3.5.** *If $f(z) \in \mathcal{A}$ satisfies*

$$\left|f'(z) - \frac{f(z)}{z}\right| < \frac{1}{2}$$

*then $f(z) \in \mathcal{S}^*$.*

## 4. Conditions for Convexity of Complex Order $b$

**Theorem 4.1.** *If $f(z) \in \mathcal{A}_p(n)$ satisfies*

$$\left|\left\{\frac{(f'(z))^{1-b}}{pz^{p-1}}\right\}^{\frac{1}{b}}\left[zf''(z) + (1-p)f'(z)\right]\right| < \frac{n}{n+1}|b|$$

*then $f(z) \in \mathcal{C}_p(n;b)$.*

*Proof.* Let us define a function $h(z)$ by

$$h(z) = \int_0^z \left(\frac{f'(t)}{pt^{p-1}}\right)^{\frac{1}{b}} dt = z + \frac{(p+n)\,a_{p+n}z^{n+1}}{(n+1)\,pb} + \dots$$

Further, let

$$\begin{aligned} g(z) = zh'(z) = z\left(\frac{f'(z)}{pz^{p-1}}\right)^{\frac{1}{b}} &= z\left(1 + \frac{p+n}{pb}a_{p+n}z^n + \dots\right) \\ &= z + \frac{(p+n)\,a_{p+n}z^{n+1}}{(n+1)\,pb} + \dots \end{aligned}$$

Obviously $h(z) \in \mathcal{A}(n)$ and $g(z) \in \mathcal{A}(n)$. Now

$$g(z) = z\left(\frac{f'(z)}{pz^{p-1}}\right)^{\frac{1}{b}}$$

Differentiating logarithmically, we find after some computation that

$$g'(z) = \left(\frac{(f'(z))^{1-b}}{pz^{p-1}}\right)^{\frac{1}{b}} \left[\frac{1}{b}\{zf''(z) + (1-p+b)f'(z)\}\right]$$

we see that

$$\begin{aligned} &\left|g'(z) - \frac{g(z)}{z}\right| \\ &\qquad = \left|\left(\frac{(f'(z))^{1-b}}{pz^{p-1}}\right)^{\frac{1}{b}} \left[\frac{1}{b}\{zf''(z) + (1-p)f'(z)\}\right]\right| \\ &\qquad\qquad < \frac{n}{n+1} \quad (z \in \mathcal{U}, b \in \mathbb{C}\backslash\{0\}) \end{aligned}$$

Thus application of Lemma 2.1 gives

$$g(z) = zh'(z) \in \mathcal{S}^*(n;1) \Rightarrow h(z) \in \mathcal{C}(n;1)$$

Since

$$\frac{zh''(z)}{h(z)} = \frac{1}{b}\left[\frac{zf''(z)}{f'(z)} - (p-1)\right]$$

therefore

$$\mathrm{Re}\left(1 + \frac{zh''(z)}{h(z)}\right) = \mathrm{Re}\left[1 + \frac{1}{b}\left(\frac{zf''(z)}{f'(z)} - (p-1)\right)\right] > 0 \quad (as\, h(z) \in \mathcal{C}(n;1))$$

it follows that $f(z) \in \mathcal{C}_p(n;b)$. This completes the proof of the theorem. □

**Theorem 4.2.** *If $f(z) \in \mathcal{A}_p(n)$ satisfies*

$$\left| f''(z) \left( \frac{(f'(z))^{1-b}}{pz^{p-1}} \right)^{\frac{1}{b}} - \frac{(p-1)}{z} \left( \frac{f'(z)}{pz^{p-1}} \right)^{\frac{1}{b}} \right| < \frac{n}{n+1} |b|$$

*then $f(z) \in \mathcal{C}_p(n;b)$.*

*Proof.* Let

$$h(z) = \int_0^z \left( \frac{f'(t)}{pt^{p-1}} \right)^{\frac{1}{b}} dt,$$

then

$$zh'(z) = z \left( \frac{f'(z)}{pz^{p-1}} \right)^{\frac{1}{b}}.$$

Further, suppose that $g(z) = zh'(z)$. Then we obtain

$$g(z) = z + \frac{(p+n)\, a_{p+n} z^{n+1}}{pb} + \cdots \in \mathcal{A}(n)$$

and

$$\left| g'(z) - \frac{g(z)}{z} \right| = \left| zh''(z) \right| = \left| \frac{z}{b} \left[ f''(z) \left( \frac{(f'(z))^{1-b}}{pz^{p-1}} \right)^{\frac{1}{b}} - \frac{(p-1)}{z} \left( \frac{f'(z)}{pz^{p-1}} \right)^{\frac{1}{b}} \right] \right|$$
$$\leq \frac{n}{n+1} |z| < \frac{n}{n+1}.$$

Thus using the Lemma 2.1, we obtain $g(z) \in \mathcal{S}^*(n;1)$, that is $zh'(z) \in \mathcal{S}^*(n;1)$ which means that $h(z) \in \mathcal{C}(n;1)$. Thus we conclude that $f(z) \in \mathcal{C}_p(n;b)$. □

**Theorem 4.3.** *If $f(z) \in \mathcal{A}_p(n)$ satisfies*

$$\left| \arg\left\{ \frac{1}{b} \left( \frac{f'(z)}{pz^{p-1}} \right)^{\frac{1}{b}} \right\} + \arg\left( 1 + \frac{zf''(z)}{f'(z)} - p + b \right) \right| < \frac{\pi}{2} \delta_n$$

*then $f(z) \in \mathcal{C}_p(n;b)$.*

*Proof.* If we define the function $h(z)$ by

$$h(z) = \int_0^z \left( \frac{f'(t)}{pt^{p-1}} \right)^{\frac{1}{b}} dt = z + \frac{(p+n)\, a_{p+n} z^{n+1}}{(n+1)\, pb} + \ldots$$

and the function $g(z)$ by $g(z) = zh'(z)$, then we have

$$g'(z) = \frac{1}{b}\left(\frac{f'(z)}{pz^{p-1}}\right)^{\frac{1}{b}}\left[1 + \frac{zf''(z)}{f'(z)} + b - p\right]$$

Thus by applying Lemma 2.2, we obtain

$$\left|\arg g'(z)\right| = \left|\arg\left\{\frac{1}{b}\left(\frac{f'(z)}{pz^{p-1}}\right)^{\frac{1}{b}}\right\} + \arg\left(1 + \frac{zf''(z)}{f'(z)} + b - p\right)\right| < \frac{\pi}{2}\delta_n$$

which shows that $g(z) \in \mathcal{S}^*(n;1)$. This gives us that $h(z) \in \mathcal{C}(n;1)$, that is $f(z) \in \mathcal{C}_p(n;b)$. □

Setting $p = b = 1$, the Theorem (4.2) reduces to

**Corollary 4.4.** *If* $f(z) \in \mathcal{A}(n)$ *satisfies*

$$\left|f''(z)\right| < \frac{n}{n+1}$$

*then* $f(z) \in \mathcal{C}(n;1)$

Setting $p = n = 1$ in Theorem 4.1, we get the following corollary

**Corollary 4.5.** *If* $f(z) \in \mathcal{A}$ *satisfies*

$$\left|zf''(z)\left(f'(z)\right)^{\frac{1-b}{b}}\right| < \frac{|b|}{2}$$

*then* $f(z) \in \mathcal{C}(1;b)$

## 5. An application of generalized Alexander integral operator

For $f(z) \in \mathcal{A}_p(n)$, define

$$g(z) = \int_0^z \left(\frac{f(t)}{t^p}\right)^{\frac{1}{b}} dt = z + \frac{a_{p+n}}{(n+1)b} z^{n+1} + \ldots \tag{12}$$

Here note that $g(z) \in \mathcal{A}(n)$ and for $p = 1$ and $b = 1$ we obtain the well known Alexander integral operator [1].

Considering the above generalized Alexander integral operator, we derive

**Theorem 5.1.** *If* $f(z) \in \mathcal{A}_p(n)$ *satisfies*

$$\left|\frac{1}{z}\left(\frac{f(z)}{z^p}\right)^{\frac{1}{b}}\left(\frac{zf'(z)}{f(z)} - p\right)\right| < \frac{n}{n+1}|b| \tag{13}$$

*then* $f(z) \in \mathcal{S}_p^*(n;b)$

*Proof.* From (12), we get

$$g'(z) = \left(\frac{f(z)}{z^p}\right)^{\frac{1}{b}}. \tag{14}$$

Now differentiating (14) logarithmically and multiplying by $z$, we get

$$\frac{zg''(z)}{g'(z)} = \frac{1}{b}\left[\frac{zf'(z)}{f(z)} - p\right]. \tag{15}$$

Therefore

$$\left|g''(z)\right| = \left|\frac{1}{bz}\left(\frac{f(z)}{z^p}\right)^{\frac{1}{b}}\left(\frac{zf'(z)}{f(z)} - p\right)\right| < \frac{n}{n+1}\,(z \in \mathcal{U}).$$

Since $g(z) \in \mathcal{A}(n)$, therefore by Corollary 4.4, we get $g(z) \in \mathcal{C}\,(n;1)$
From (15), we obtain

$$\operatorname{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) = \operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - p\right)\right\} > 0 \qquad (as\; g(z) \in \mathcal{C}\,(n;1))$$

which proves that $f(z) \in \mathcal{S}_p^*\,(n;b)$. □

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