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RATIONAL CUSPIDAL CURVES WITH FOUR CUSPS ON HIRZEBRUCH SURFACES

TORGUNN KAROLINE MOE

The purpose of this article is to shed light on the question of how many and what kind of cusps a rational cuspidal curve on a Hirzebruch surface can have. Our main result is a list of rational cuspidal curves with four cusps, their type, cuspidal configurations and the surfaces they lie on. We use birational transformations to construct these curves. Moreover, we find a general expression for and compute the Euler characteristic of the logarithmic tangent sheaf in these cases. Additionally, we show that there exist real rational cuspidal curves with four real cusps. Last, we show that for rational cuspidal curves with two or more cusps on a Hirzebruch surface, there is a lower bound on one of the multiplicities.

1. Introduction

Let C be a reduced and irreducible curve on a smooth complex surface X. Let g denote the geometric genus of the curve; *rational* if $g = 0$. A point p on C is called a *cusp* if it is singular and if the germ (C, p) of C at p is irreducible. A curve C is called *cuspidal* if all of its singularities are cusps, and j-cuspidal if it has j cusps $p_1, \ldots, p_j$.

Plane rational cuspidal curves have been studied quite intensively both classically and the last 20 years. Classically, the study was part of the process of

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classifying plane curves of given degree d, and additionally, bounds on the number of cusps were produced (see [3, 16, 27, 28, 30, 32]). In the modern context, rational cuspidal curves with many cusps play an important role in the study of open surfaces (see [5, 8, 31]). The study of these curves was further motivated by the discoveries of both a particular kind of affine plane rational cuspidal curves by Lin and Zaidenberg in [17], and a bound on the multiplicities of the cusps by Matsuoka and Sakai in [18]. In the mid 1990s, Sakai in [11] suggested the following two tasks: first to classify all rational and elliptic cuspidal plane curves, and second to find the maximal number of cusps on a rational cuspidal plane curve.

Much has been done to answer the two questions, and rational cuspidal plane curves have been classified up to equisingular equivalence up to degree 6 [4, 5, 23]. But one of the interesting points for the authors chasing Sakai's challenges is that, while there is *one* known rational cuspidal curve with four cusps (up to topological type of the singularities) [23], there are apparently no rational cuspidal curves with more cusps. It has not yet been possible to prove that there does not exist rational cuspidal plane curves with more than four cusps, and this was conjectured to hold by Orevkov (Zaidenberg, personal communication). An upper bound of eight as the maximal number of cusps was found by Tono in [29]. Note that this result was recently improved by Palka in [25]. Moreover, there seems to be astounishingly few rational cuspidal plane curves with three cusps, and, except one curve of degree 5, all known curves can be found in three series of curves (infinite in degree) (see [5, 9, 10, 26]).

Inspired by this hunt for rational cuspidal curves on the projective plane, we move the quest to cuspidal curves on Hirzebruch surfaces. In this article we show that there exist rational cuspidal curves with four cusps on these surfaces, and we construct series of such curves by applying birational transformations to plane curves. The fact that we have several curves with four cusps is a perhaps surprising contrast to the single plane curve with four cusps. Note that we in [22] show that on the Hirzebruch surfaces an upper bound for the number of cusps on a rational cuspidal curve is fourteen.

Now, let C be a curve of type (a,b) on a Hirzebruch surface $\mathbb{P}(\mathscr{O}\oplus\mathscr{O}(-e))$, denoted by $\mathbb{F}_e$ for $e \geq 0$. For p a cusp on C, let m denote the multiplicity of p, and let $\overline{m}$ denote its multiplicity sequence consisting of multiplicities of subsequent infinitely near points to p (see Section 2 for a precise definition). The collection of multiplicity sequences of all the cusps on a cuspidal curve will be referred to as the *cuspidal configuration* of the curve.

Our main results can be summarized in the following theorem. To distinguish the Hirzebruch surfaces $\mathbb{F}_0$ and $\mathbb{F}_1$, playing a special role, we employ a separate index $h \in \{0,1\}$ in these cases.

Theorem 1.1. *The following rational cuspidal curves with four cusps exist on* $\mathbb{F}_e$ *and* $\mathbb{F}_h$ *for all* $e \geq 0$ *and* $h \in \{0,1\}$, *except the pairs* $(e,k) = (h,k) = (0,0)$.

Type	Cuspidal configuration	For	Surface
$(2k+1,4)$	$[4_{k-1+e}, 2_3], [2], [2], [2]$	$k \geq 0$	$\mathbb{F}_e$
$(3k+1-h,5)$	$[4_{2k-1+h}, 2_3], [2], [2], [2]$	$k \geq 0$	$\mathbb{F}_h$
$(2k+2-h,4)$	$[3_{2k-1+h}, 2], [2_3], [2], [2]$	$k \geq 0$	$\mathbb{F}_h$
$(k+1-h,3)$	$[2_{n_1}], [2_{n_2}], [2_{n_3}], [2_{n_4}]$	$k \geq 2$ and $\sum n_i = 2k+h$	$\mathbb{F}_h$
$(0,3)$	$[2], [2], [2], [2]$		$\mathbb{F}_2$

Table 1: Rational cuspidal curves with four cusps on Hirzebruch surfaces.

We have explored many possibilities in the search for curves with more cusps, but the search has so far been unsuccessful. The lack of examples of curves with more than four cusps leads us to the following conjecture, extending Orevkov's conjecture in the plane case.

Conjecture 1.2. A rational cuspidal curve on a Hirzebruch surface has at most four cusps.

Associated to the curves in Table 1, in this article we additionally find other results that resemble results for plane cuspidal curves. First, we find a general expression for and compute the Euler characteristic of the logarithmic tangent sheaf, $\chi(\Theta_V \langle D \rangle)$ (where (V,D) is the minimal embedded resolution of the curve), for the curves in Table 1, and we find that the value is not necessarily bounded below. This differs from the result for plane curves, where in this setting, $\chi(\Theta_V \langle D \rangle)$ is conjectured to be equal to zero [7, 8].

Second, we give an example of a *real* rational cuspidal curve with four *real* cusps on a Hirzebruch surface. The example is interesting since this is impossible for the only known rational cuspidal plane curve with four cusps (see [20, Section 9.1]).

Third, for rational cuspidal plane curves, Matsuoka and Sakai showed in [18] that $m > \frac{d}{3}$, where m is the maximal multiplicity of the cusps. Here we prove a similar result for rational cuspidal curves on Hirzebruch surfaces.

Theorem 1.3. *A rational cuspidal curve C of type* (a,b) *on* $\mathbb{F}_e$ *with two or more cusps, must have at least one cusp* p_1 *with multiplicity* $m := m_{p_1}$ *that satisfies the lower bound,*

$$m > 1 + a + b + \frac{1}{2}be - \frac{1}{2}\sqrt{4(a^2+b^2-1) + 16(a+b) + be(be+4a+8)}.$$

1.1. Structure

In Section 2 we recall precise definitions and frequently used notation, and we give the basic definitions and preliminary results for rational cuspidal curves on Hirzebruch surfaces. Section 3 contains the main results of this article. Here we construct several series of rational cuspidal curves with four cusps on Hirzebruch surfaces. In Section 4 we present associated results. First, we compute $\chi(\Theta_V \langle D \rangle)$ for the curves we have found in Section 3. Then we explicitly construct a real rational cuspidal curve with four real cusps, and show that this is possible for an entire series. Last, we give a lower bound for the highest multiplicity of a cusp on a rational cuspidal curve with two or more cusps.

2. Notation and preliminary results

In this section we clarify the definitions and notation used in this article. First we remind the reader of the basic objects. Then we move to the Hirzebruch surfaces, and curves on these surfaces. Last we state preliminary results that give restrictions on the cuspidal configurations of rational cuspidal curves in this case.

Let C be a reduced and irreducible curve on a smooth complex surface X. There exists a sequence of t monoidal transformations,

$$V = V_t \xrightarrow{\sigma_t} V_{t-1} \to \cdots \to V_1 \xrightarrow{\sigma_1} V_0 = X,$$

such that the reduced total inverse image of C under the composition $\sigma : V \to X$,

$$D := \sigma^{-1}(C)_{\text{red}},$$

is a *simple normal crossing divisor* (SNC-divisor) on the smooth complete surface V (see [15]). The pair (V, D) and the transformation σ are referred to as an *embedded resolution* of C, and it is called a *minimal embedded resolution* of C when t is the smallest integer such that D is an SNC-divisor.

Let p be a cusp on C, let m denote the multiplicity of p, and let m_i denote the multiplicity of the infinitely near points p_i of p. Then the *multiplicity sequence* $\overline{m}$ of the cusp p is defined to be the sequence of integers

$$\overline{m} = [m, m_1, \ldots, m_{t-1}],$$

where t is the number of monoidal transformations in the local minimal embedded resolution of the cusp, and we have $m_{t-1} = 1$ (see [2]). We follow the convention of compacting the notation by omitting the number of ending 1s in the sequence and indexing the number of repeated elements, for example, we write

$$[6,6,3,3,3,2,1,1] = [6_2, 3_3, 2].$$

Note that there are strong relations between the elements in the sequence (see [9]). The collection of multiplicity sequences of a cuspidal curve will be referred to as its *cuspidal configuration*.

Following standard notation, we write $(C\cdot C')_p$ for the local intersection multiplicity of two curves C and C' (without common components) at a common point p. Viewing the intersection of C and C' as a 0-cycle, we express this by the notation $C\cdot C'$, where

$$C\cdot C' = \sum_{p\in C\cap C'} (C\cdot C')_p p.$$

The intersection number of two divisors C and C' on X is written $C\,.\,C'$.

For the sake of completeness, we recall the basic notation for the projective plane and the Hirzebruch surfaces.

We let $\mathbb{P}^2$ denote the projective plane with coordinates $(x:y:z)$ and coordinate ring $\mathbb{C}[x,y,z]$. A reduced and irreducible curve C of degree d on $\mathbb{P}^2$ is given as the zero set $\mathscr{V}(F)$ of a homogeneous, reduced and irreducible polynomial $F(x,y,z)\in\mathbb{C}[x,y,z]_d$.

We let $\mathbb{F}_e=\mathbb{P}(\mathscr{O}\oplus\mathscr{O}(-e))$ denote the Hirzebruch surface of type e for any $e\geq 0$. The coordinate ring of $\mathbb{F}_e$ is denoted by $S_e:=\mathbb{C}[x_0,x_1,y_0,y_1]$, where the variables are given a bigrading,

$$\deg x_0=(1,0),\quad \deg x_1=(1,0),\quad \deg y_0=(0,1),\quad \deg y_1=(-e,1).$$

Let $S_e(a,b)$ denote the (a,b)-graded part of S_e,

$$S_e(a,b):=\mathrm{H}^0(\mathbb{F}_e,\mathscr{O}_{\mathbb{F}_e}(a,b))=\bigoplus_{\substack{\alpha_0+\alpha_1-e\beta_1=a\\ \beta_0+\beta_1=b}}\mathbb{C}x_0^{\alpha_0}x_1^{\alpha_1}y_0^{\beta_0}y_1^{\beta_1}.$$

A reduced and irreducible curve C of type (a,b) on $\mathbb{F}_e$ is given as the zero set $\mathscr{V}(F)$ of a reduced and irreducible polynomial $F(x_0,x_1,y_0,y_1)\in S_e(a,b)$.

In the language of divisors, let L be a *fiber* of $\pi:\mathbb{F}_e\longrightarrow\mathbb{P}^1$ and M_0 the *special section* (in the literature also referred to as *exceptional section*) of π. The Picard group of $\mathbb{F}_e$, $\mathrm{Pic}(\mathbb{F}_e)$, is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$. We choose L and $M\sim eL+M_0$ as a generating set of $\mathrm{Pic}(\mathbb{F}_e)$, and then

$$L^2=0,\quad L\,.\,M=1,\quad M^2=e.$$

Note that $M\,.\,M_0=0$. The canonical divisor K on $\mathbb{F}_e$ can be expressed as

$$K\sim(e-2)L-2M \text{ and } K^2=8.$$

Any irreducible curve $C\neq L,M_0$ corresponds to a divisor on $\mathbb{F}_e$ given by

$$C\sim aL+bM,\quad b>0, a\geq 0.$$

Recall that there are *birational transformations* between these surfaces and the projective plane, and these can be given in a quite explicit way (see [21]). Using the birational transformations, we are able to construct curves on one surface from a curve on another surface by taking the strict transform.

Last in this section, we recall two important facts that give restrictions for the singularities on a curve. The genus formula for curves on Hirzebruch surfaces is fundamental.

Corollary 2.1 (Genus formula)**.** *A cuspidal curve C of type (a,b) with cusps p_j, for $j = 1,\dots,s$, and multiplicity sequences $\overline{m}_j = [m_0, m_1, \dots, m_{t_j-1}]$ on the Hirzebruch surface $\mathbb{F}_e$ has genus g, where*

$$g = \frac{(b-1)(2a-2+be)}{2} - \sum_{j=1}^{s}\sum_{i=0}^{t_j-1} \frac{m_{i,j}(m_{i,j}-1)}{2}.$$

Proof. The proof follows from the above discussion and [12, Example V 3.9.2, p.393]. □

Additionally, the structure of the Hirzebruch surfaces gives restrictions on the multiplicity sequence of a cusp on a curve on such a surface. Here we only give the most important of these restrictions, and we refer to [21] for further results.

Proposition 2.2. *Let p be a cusp on a reduced and irreducible curve C of type (a,b) with multiplicity m on a Hirzebruch surface $\mathbb{F}_e$. Then $m \leq b$.*

Proof. Let L be the fiber through p. By intersection theory [12]

$$m \leq (L \cdot C)_p \leq L.C = b.$$

□

3. Rational cuspidal curves with four cusps

In this section we give examples of rational cuspidal curves with four cusps on Hirzebruch surfaces. Our aim is to shed light on the question of how many and what kind of cusps a rational cuspidal curve on a Hirzebruch surface can have.

On each $\mathbb{F}_e$, with $e \geq 0$, we construct one infinite series of rational cuspidal curves with four cusps. On $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$ we construct another three infinite series of rational cuspidal curves with four cusps, and on $\mathbb{F}_2$ we construct a single additional rational cuspidal curve with four cusps. Note that all these curves can be constructed explicitly using known plane curves, appropriate change of coordinates and the birational transformations indicated in the

Curve	Degree	Cuspidal conf.	Parametrization
C_1	4	$[2],[2],[2]$	$(s^3t-\frac{1}{2}s^4 : s^2t^2 : t^4-2st^3)$
C_2	5	$[2_3],[2],[2],[2]$	$(s^4t : s^2t^3-s^5 : t^5+2s^3t^2)$

Table 2: Two plane rational cuspidal curves used in the construction of cuspidal curves on Hirzebruch surfaces.

proofs (see [21]). In particular, the two plane cuspidal curves in Table 2 are the starting points of our constructions [23].

The following theorem presents the series of rational cuspidal curves with four cusps that consists of curves on *all* the Hirzebruch surfaces.

Theorem 3.1. *For all $e \geq 0$ and $k \geq 0$, except for the pair $(e,k)=(0,0)$, there exists on the Hirzebruch surface $\mathbb{F}_e$ a rational cuspidal curve $C_{e,k}$ of type $(2k+1,4)$ with four cusps and cuspidal configuration*

$$[4_{k-1+e},2_3],[2],[2],[2].$$

Proof. We will show that for each $e \geq 0$ there is an infinite series of curves on $\mathbb{F}_e$, and we show this by induction on k. The proof is split in two, and we treat the case of k odd and even separately. We construct the series of curves $C_{e,0}$ for $e \geq 1$, and then we construct the initial series $C_{e,1}$ and $C_{e,2}$, with $e \geq 0$. We only treat the induction to prove the existence of $C_{e,k}$ for odd values of k, as the proof for even values of k is completely parallel.

Let C be a rational cuspidal curve of degree 5 on $\mathbb{P}^2$ with cuspidal configuration $[2_3],[2],[2],[2]$. Let p be the cusp with multiplicity sequence $[2_3]$, and let T be the tangent line to C at p. Then $T \cdot C = 4p + r$, with r a smooth point on C. Blowing up at r, the strict transform of C is a curve $C_{1,0}$ of type $(1,4)$ on $\mathbb{F}_1$ with cuspidal configuration $[2_3],[2],[2],[2]$. Letting $T_{1,0}$ denote the strict transform of T and $p_{1,0}$ the strict transform of p, we have $T_{1,0} \cdot C_{1,0} = 4p_{1,0}$. We observe that $p_{1,0}$ is tangent to the fiber. Let E_1 denote the special section on $\mathbb{F}_1$, and let $s_{0,1} = E_1 \cap T_{1,0}$.

From $C_{1,0}$ we can proceed with the construction of curves on Hirzebruch surfaces in three ways.

First, we show by induction on e that the curves $C_{e,0}$ exist on the Hirzebruch surfaces $\mathbb{F}_e$, for all $e \geq 1$. We have already seen that $C_{1,0}$ exists on $\mathbb{F}_1$, and that there exists a fiber $T_{1,0}$ with the property that $T_{1,0} \cdot C_{1,0} = 4p_{1,0}$ for the first cusp $p_{1,0}$. Now assume $e \geq 2$ and that the curve $C_{e-1,0}$ of type $(1,4)$ exists on $\mathbb{F}_{e-1}$ with cuspidal configuration $[4_{e-2},2_3],[2],[2],[2]$, where $p_{e-1,0}$ denotes the first cusp and $T_{e-1,0}$ has the property that $T_{e-1,0} \cdot C_{e-1,0} = 4p_{e-1,0}$. Then, with

E_{e-1} the special section of $\mathbb{F}_{e-1}$, blowing up at the intersection $s_{e-1,0} \in E_{e-1} \cap T_{e-1,0}$ and contracting $T_{e-1,0}$, we get $C_{e,0}$ on $\mathbb{F}_e$ of type $(1,4)$ with cuspidal configuration $[4_{e-1}, 2_3], [2], [2], [2]$. Moreover, we note that there exists a fiber $T_{e,0}$ with $T_{e,0} \cdot C_{e,0} = 4p_{e,0}$. So the series $C_{e,0}$ exists for all $e \geq 1$ ($k = 0$).

Second, we construct the series $C_{e,1}$. Note that from the curve $C_{1,0}$ on $\mathbb{F}_1$ it is possible to construct the curve $C_{0,1}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up at $p_{1,0}$ before contracting $T_{1,0}$. The curve $C_{0,1}$ is a curve of type $(3,4)$ with cuspidal configuration $[2_3], [2], [2], [2]$, and there is a fiber $T_{0,1}$ such that $T_{0,1} \,.\, C_{0,1} = 4p_{0,1}$. Blowing up at a point $s_{0,1} \in T_{0,1} \setminus \{p_{0,1}\}$ and contracting $T_{0,1}$ results in the curve $C_{1,1}$ on $\mathbb{F}_1$ of type $(3,4)$ with cuspidal configuration $[4, 2_3], [2], [2], [2]$. Moreover, there exists a fiber $T_{1,1}$ with $T_{1,1} \cdot C_{1,1} = 4p_{1,1}$ and $p_{1,1} \notin E_1$. The same induction on e as above proves that the series $C_{e,1}$ exists for $e \geq 0$ ($k = 1$).

Third, we construct the series $C_{e,2}$. Note that from the curve $C_{1,0}$ on $\mathbb{F}_1$ it is possible to construct the curve $C_{0,2}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up at a point $t_{1,0} \in T_{1,0} \setminus \{p_{1,0}, s_{1,0}\}$ before contracting $T_{1,0}$. The curve $C_{0,2}$ is a curve of type $(5,4)$ with cuspidal configuration $[4, 2_3], [2], [2], [2]$, and there is a fiber $T_{0,2}$ such that $T_{0,2} \cdot C_{0,2} = 4p_{0,2}$. Blowing up at a point $s_{0,2} \in T_{0,2} \setminus \{p_{0,2}\}$ and contracting $T_{0,2}$ gives the curve $C_{1,2}$ on $\mathbb{F}_1$ of type $(5,4)$ with cuspidal configuration $[4_2, 2_3], [2], [2], [2]$. Moreover, there exists a fiber $T_{1,2}$ with $T_{1,2} \cdot C_{1,2} = 4p_{1,2}$ and $p_{1,2} \notin E_1$. The same induction on e as above proves that the series $C_{e,2}$ exists for $e \geq 0$ ($k = 2$).

Next assume $k \geq 3$, with k odd, and that there exists a series of curves $C_{e,k-2}$ of type $(2k-3, 4)$ on $\mathbb{F}_e$ for all $e \geq 0$ with cuspidal configuration $[4_{e+k-3}, 2_3], [2], [2], [2]$. Then, in particular, the curve $C_{1,k-2}$ on $\mathbb{F}_1$ with cuspidal configuration $[4_{k-2}, 2_3], [2], [2], [2]$ exists. Moreover, there exists a fiber $T_{1,k-2}$ on $\mathbb{F}_1$ such that $T_{1,k-2} \cdot C_{1,k-2} = 4p_{1,k-2}$, where $p_{1,k-2}$ denotes the cusp with multiplicity sequence $[4_{k-2}, 2_3]$. With E_1 the special section on $\mathbb{F}_1$, let $s_{1,k-2} \in E_1 \cap T_{1,k-2}$. We now blow up at a point $t_{1,k-2} \in T_{1,k-2} \setminus \{p_{1,k-2}, s_{1,k-2}\}$ and subsequently contract $T_{1,k-2}$. This gives the curve $C_{0,k}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ of type $(2k+1, 4)$ with cuspidal configuration $[4_{k-1}, 2_3], [2], [2], [2]$. With $T_{0,k}$ the strict transform of the exceptional line of the latter blowing up, we have $T_{0,k} \cdot C_{0,k} = 4p_{0,k}$. Blowing up at a point $s_{0,k} \in T_{0,k} \setminus \{p_{0,k}\}$ and contracting $T_{0,k}$ gives the curve $C_{1,k}$ on $\mathbb{F}_1$ of type $(2k+1, 4)$ with cuspidal configuration $[4_k, 2_3], [2], [2], [2]$. Moreover, there is a fiber $T_{1,k}$ with the property that $T_{1,k} \cdot C_{1,k} = 4p_{1,k}$. With the same induction on e as above, we get the series of curves $C_{e,k}$. □

In the following we show that there are three infinite series of rational cuspidal curves with four cusps that can be found on the Hirzebruch surfaces $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$.

Before we list these three series, we consider the rational cuspidal curves with four cusps on $\mathbb{F}_1$ that we can get, taking the strict transform of a plane rational cuspidal curve of degree 5 with four cusps, by blowing up a single point on $\mathbb{P}^2$. These curves represent examples from the series.

Theorem 3.2. *Let C be a rational cuspidal curve with four cusps of degree 5 on $\mathbb{P}^2$. The following rational cuspidal curves with four cusps on $\mathbb{F}_1$ can be constructed from C by blowing up a single point on $\mathbb{P}^2$.*

Curve	**Type**	**Cuspidal configuration**
C_1	$(0,5)$	$[2_3],[2],[2],[2]$
C_2	$(1,4)$	$[2_3],[2],[2],[2]$
C_3	$(2,3)$	$[2_2],[2],[2],[2]$

Table 3: Rational cuspidal curves on $\mathbb{F}_1$ with four cusps.

Proof. The curve C_1 is constructed by blowing up any point r on $\mathbb{P}^2 \setminus C$. Note that if r is on the tangent line to a cusp on C, then C_1 has a cusp that is tangent to a fiber. If r is only on tangent lines to smooth points on C, then C_1 has smooth fiber tangential points.

The curve C_2 is constructed by blowing up any smooth point r on C. Again, if r is on a tangent line to any other point of C, C_2 will have points that are fiber tangential.

The curve C_3 is constructed by blowing up the cusp with multiplicity sequence $[2_3]$. □

We now give the three series of rational cuspidal curves with four cusps on $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$.

Theorem 3.3. *For $h \in \{0,1\}$ and all integers $k \geq 0$, except the pair $(h,k) = (0,0)$, there exists on the Hirzebruch surface $\mathbb{F}_h$ a rational cuspidal curve $C_{h,k}$ of type $(3k+1-h,5)$ with four cusps and cuspidal configuration*

$$[4_{2k-1+h}, 2_3], [2], [2], [2].$$

Proof. The proof is by construction and induction on k. Let C be a rational cuspidal curve of degree 5 on $\mathbb{P}^2$ with cuspidal configuration $[2_3],[2],[2],[2]$. Let p be the cusp with multiplicity sequence $[2_3]$, and let T be the tangent line to C at p. There is a smooth point $r \in C$, such that $T \cdot C = 4p + r$. Blowing up at any point $t \in T \setminus \{p,r\}$, we get the curve $C_{1,0}$ of type $(0,5)$ and cuspidal configuration $[2_3],[2],[2],[2]$ on $\mathbb{F}_1$. Moreover, with $T_{1,0}$ the strict transform of T and $p_{1,0}$, $r_{1,0}$ the strict transforms of the points p and r, we have $T_{1,0} \cdot C_{1,0} = 4p_{1,0} + r_{1,0}$.

Now assume that the curve $C_{1,k-1}$ of type $(3(k-1),5)$ exists on $\mathbb{F}_1$ with cuspidal configuration $[4_{2(k-1)},2_3],[2],[2],[2]$, and the intersection $T_{1,k-1}\cdot C_{1,k-1} = 4p_{1,k-1}+r_{1,k-1}$ for a fiber $T_{1,k-1}$ and points as above. Then blowing up at $r_{1,k-1}$ and contracting $T_{1,k-1}$, we get a curve $C_{0,k}$ on $\mathbb{P}^1\times\mathbb{P}^1$ of type $(3k+1,5)$ and cuspidal configuration $[4_{2k-1},2_3],[2],[2],[2]$. Moreover, there is a fiber $T_{0,k}$ with the property that $T_{0,k}\cdot C_{0,k} = 4p_{0,k}+r_{0,k}$. Blowing up at $r_{0,k}$ and contracting $T_{0,k}$, we get a rational cuspidal curve $C_{1,k}$ of type $(3k,5)$ on $\mathbb{F}_1$ with cuspidal configuration $[4_{2k},2_3],[2],[2],[2]$. □

Theorem 3.4. *For $h\in\{0,1\}$ and all integers $k\geq 0$, except the pair $(h,k)=(0,0)$, there exists on the Hirzebruch surface $\mathbb{F}_h$ a rational cuspidal curve of type $(2k+2-h,4)$ with four cusps and cuspidal configuration*

$$[3_{2k-1+h},2],[2_3],[2],[2].$$

Proof. The proof is by construction and induction on k. Let C be the rational cuspidal curve of degree 5 on $\mathbb{P}^2$ with cuspidal configuration $[2_3],[2],[2],[2]$. Let q be one of the cusps with multiplicity sequence $[2]$, and let T be the tangent line to C at q. Then there are smooth points $r,s\in C$, such that $T\cdot C=3q+r+s$. Blowing up at s, we get the curve $C_{1,0}$ of type $(1,4)$ and cuspidal configuration $[2_3],[2],[2],[2]$ on $\mathbb{F}_1$. Moreover, with $T_{1,0}$ the strict transform of T and $p_{1,0}$, $r_{1,0}$ the strict transforms of the points p and r, we have $T_{1,0}\cdot C_{1,0}=3p_{1,0}+r_{1,0}$.

Now assume that the curve $C_{1,k-1}$ of type $(2k-1,4)$ exists on $\mathbb{F}_1$ with cuspidal configuration $[3_{2k-2},2],[2_3],[2],[2]$, and the intersection $T_{1,k-1}\cdot C_{1,k-1} = 3p_{1,k-1}+r_{1,k-1}$ for a fiber $T_{1,k-1}$ and points as above. Then blowing up at $r_{1,k-1}$ and contracting $T_{1,k-1}$, we get a curve $C_{0,k}$ on $\mathbb{P}^1\times\mathbb{P}^1$ of type $(2k+2,4)$ and cuspidal configuration $[3_{2k-1},2],[2_3],[2],[2]$. Moreover, there is a fiber $T_{0,k}$ with the property that $T_{0,k}\cdot C_{0,k}=3p_{0,k}+r_{0,k}$. Blowing up at $r_{0,k}$ and contracting $T_{0,k}$, we get a rational cuspidal curve $C_{1,k}$ of type $(2k+1,4)$ on $\mathbb{F}_1$ with cuspidal configuration $[3_{2k},2],[2_3],[2],[2]$. □

Theorem 3.5. *For $h\in\{0,1\}$, all integers $k\geq 2$, and every choice of $n_j\in\mathbb{N}$, with $j=1,\dots,4$, such that $\sum_{j=1}^4 n_j = 2k+h$, there exists on the Hirzebruch surface $\mathbb{F}_h$ a rational cuspidal curve of type $(k+1-h,3)$ with four cusps and cuspidal configuration*

$$[2_{n_1}],[2_{n_2}],[2_{n_3}],[2_{n_4}].$$

Proof. We prove the existence of the curves on $\mathbb{P}^1\times\mathbb{P}^1$ by induction on k. In the proof we show that any curve on $\mathbb{F}_1$ can be reached from a curve on $\mathbb{P}^1\times\mathbb{P}^1$ by an elementary transformation, hence we prove the theorem for $h\in\{0,1\}$.

First we observe that a choice of n_j such that the condition $\sum_{j=1}^4 n_j = 2k$ means that either all four n_j are odd, two are odd and two are even, or all four are even. We split the proof into these three cases, and prove only the first case completely. The other two can be dealt with in the same way once we have proved the existence of a first curve.

We now prove the theorem when all n_j are odd. Let C be a rational cuspidal curve on $\mathbb{P}^2$ of degree 4 with three cusps and cuspidal configuration $[2],[2],[2]$ for cusps p_j, $j = 1,2,3$. Let p_4 be a general smooth point on C and let T be the tangent line to C at p_4. Then $T \cdot C = 2p_4 + t_1 + t_2$, where t_1, t_2 are two smooth points on C. Blowing up at t_1 and t_2 and contracting T, we get a rational cuspidal curve on $\mathbb{P}^1 \times \mathbb{P}^1$ of type $(3,3)$ with four ordinary cusps.

Fixing notation, we say that we have a curve C_2 on $\mathbb{P}^1 \times \mathbb{P}^1$ of type $(3,3)$ and four cusps p_j^2, $j = 1,\dots,4$, all with multiplicity sequence $[2]$. Since the choice of $p_4 \in \mathbb{P}^2$ was general, there are four $(1,0)$-curves L_j^2 such that

$$L_j^2 \cdot C_2 = 2p_j^2 + r_j^2,$$

for smooth points $r_j^2 \in C_2$. Now assume that we have a curve C_{k-1} on $\mathbb{P}^1 \times \mathbb{P}^1$ of type $((k-1)+1,3)$, with cuspidal configuration $[2_{n_1-2}],[2_{n_2}],[2_{n_3}],[2_{n_4}]$ such that all n_j are odd, and such that there exist fibers L_j^{k-1} with

$$L_j^{k-1} \cdot C_{k-1} = 2p_j^{k-1} + r_j^{k-1},$$

for smooth points r_j^{k-1} on C_{k-1}.

We blow up at r_1^{k-1}, contract the corresponding L_1^{k-1} and get a curve $C_{1,k-1}$ on $\mathbb{F}_1$ of type $(k-1,3)$ with cuspidal configuration $[2_{n_1-1}],[2_{n_2}],[2_{n_3}],[2_{n_4}]$. Moreover, since r_1^{k-1} was not fiber tangential, we have that $r_1^{1,k-1} \notin E_1$, and the strict transform of the exceptional fiber of the blowing up, $L_1^{1,k-1}$, has intersection with $C_{1,k-1}$,

$$L_1^{1,k-1} \cdot C_{1,k-1} = 2p_1^{1,k-1} + r_1^{1,k-1}.$$

Blowing up at $r_1^{1,k-1}$ and contracting $L_1^{1,k-1}$ bring us back to $\mathbb{P}^1 \times \mathbb{P}^1$ and a curve C_k of type $(k+1,3)$ and cuspidal configuration $[2_{n_1}],[2_{n_2}],[2_{n_3}],[2_{n_4}]$. This takes care of the case when all n_j are odd.

To prove the theorem when two n_j are even or all n_j are even, we only show that there exist curves on $\mathbb{P}^1 \times \mathbb{P}^1$ of the right type and cuspidal configurations $[2_2],[2_2],[2],[2]$ and $[2_2],[2_2],[2_2],[2_2]$. The rest of the argument is then similar to the above. To get the first curve, we blow up C_2 in r_1^2 and r_2^2 and contract L_1^2 and L_2^2. This is a curve C_3 of type $(4,3)$ with cuspidal configuration $[2_2],[2_2],[2],[2]$. The curve is on $\mathbb{P}^1 \times \mathbb{P}^1$ since it can be shown with direct calculations in `Maple` that r_1^2 and r_2^2 are not on the same $(0,1)$-curve. To get the second curve, we blow up at the analogous r_3^3 and r_4^3 on the curve C_3, before contracting L_3^3 and L_4^3. We are again on $\mathbb{P}^1 \times \mathbb{P}^1$ by a similar argument to the above, and the curve C_4 is of type $(5,3)$ and has cuspidal configuration $[2_2],[2_2],[2_2],[2_2]$. □

The series in Theorem 3.5 can be extended to a series of rational cuspidal curves with less than four cusps in an obvious way. We state this as a corollary.

Corollary 3.6. *For $h \in \{0,1\}$, all integers $k \geq 0$, and every choice of $n_j \in \mathbb{N} \cup \{0\}$, with $j = 1,\dots,4$, such that $\sum_{j=1}^{4} n_j = 2k+h$, there exists on the Hirzebruch surface $\mathbb{F}_h$ a rational cuspidal curve of type $(k+1-h,3)$ with $s \in \{0,1,2,3,4\}$ cusps and cuspidal configuration*

$$[2_{n_1}],[2_{n_2}],[2_{n_3}],[2_{n_4}].$$

Proof. These curves can be constructed from the curves in Theorem 3.5 by a similar construction. In order to construct the curves with less than four cusps we have to blow up cusps on the curves in the series from Theorem 3.5. □

An example of a curve from Theorem 3.5 can be found in Section 4.3. Additionally, note that after contracting the special section on $\mathbb{F}_1$, the strict transform of the curves on $\mathbb{F}_1$ in this series are examples of plane curves with three cusps with multiplicity 2 and one non-cuspidal singularity with multiplicity 3.

Last in this section we provide an example of a curve not represented in any of the above series. This is the only example we have found of such a curve, and in particular the only such curve on $\mathbb{F}_2$.

Theorem 3.7. *On $\mathbb{F}_2$ there exists a rational cuspidal curve of type $(0,3)$ with four cusps and cuspidal configuration*

$$[2],[2],[2],[2].$$

Proof. Let C be a plane rational cuspidal curve of degree 5 with four cusps. Let p be the cusp $[2_3]$ and p_i, $i = 1,2,3$, the cusps with multiplicity sequence $[2]$. Let T be the tangent line to C at p. Let L_i denote the line through p and p_i, with $i = 1,2,3$. There are smooth points s and r_i, $i = 1,2,3$, on C, such that

$$T \cdot C = 4p + s, \qquad L_i \cdot C = 2p + 2p_i + r_i.$$

Blowing up at p gives a $(2,3)$-curve on $\mathbb{F}_1$ with cuspidal configuration $[2_2],[2],[2],[2]$. Let C' denote the strict transform of C, T' and L_i' the strict transforms of T and L_i, and let E' be the special section on $\mathbb{F}_1$. Let p' be the cusp $[2_2]$, p_i' the other cusps, and s' and r_i' the strict transforms of the points s and r_i. Then we have the following intersections,

$$E' \cdot C' = 2p', \qquad T' \cdot C' = 2p' + s', \qquad L_i' \cdot C' = 2p_i' + r_i'.$$

Since $p' \in E'$, blowing up at p' and contracting T', we get a cuspidal curve on $\mathbb{F}_2$ of type $(0,3)$ and cuspidal configuration $[2],[2],[2],[2]$. □

4. Associated results

The main result in this section is that $\chi(\Theta_V \langle D \rangle)$ is not necessarily bounded below for rational cuspidal curves on Hirzebruch surfaces. First in this section we state two lemmas for rational cuspidal curves on Hirzebruch surfaces, the first analogous to a lemma by Flenner and Zaidenberg [8, Lemma 1.3, p.148], and the other a lemma bounding the sum of the so-called M-numbers of the curve. We write down the proof of the second lemma. Second, we use these lemmas to give an explicit formula for $\chi(\Theta_V \langle D \rangle)$ in this case. We calculate this value for the curves constructed in Section 3, and with that we provide examples of curves for which $\chi(\Theta_V \langle D \rangle) \neq 0$. We additionally investigate real cuspidal curves on Hirzebruch surfaces. Last, we show that for rational cuspidal curves with two or more cusps on a Hirzebruch surface, there is a lower bound on one of the multiplicities.

4.1. Two lemmas

We now state two lemmas for rational cuspidal curves on Hirzebruch surfaces. The first lemma is a variant of [8, Lemma 1.3, p.148].

Lemma 4.1. *Let C be a rational cuspidal curve on $\mathbb{F}_e$. Let (V,D) be the minimal embedded resolution of C, and let K_V be the canonical divisor on V. Moreover, let $D_1,\dots,D_r$ be the irreducible components of D, Θ_V the tangent sheaf of V, $\mathscr{N}_{D/V}$ the normal sheaf of D in V, and let c_2 be the second Chern class of V. Then the following hold.*

(0) *D is a rational tree.*

(1) $\chi(\Theta_V) = 8-2r.$

(2) $K_V^2 = 9-r.$

(3) $c_2 := c_2(V) = 3+r.$

(4) $\chi\left(\bigoplus \mathscr{N}_{D_i/V}\right) = r + \sum_{i=1}^{r} D_i^2.$

(5) $\chi(\Theta_V \langle D \rangle) = K_V\,.\,(K_V+D)-1.$

Proof. The proof is very similar to the proof of [8, Lemma 1.3, p.148], and only standard details (f.ex. the value of $c_2(X)$) are changed. A full proof can be found in [21]. For later reference we include the proof of (5).

Proof of (5) Observe first that since D is an SNC-divisor, we have by direct calculation

$$
\begin{aligned}
D^2 &= \sum_{i=1}^{r} D_i^2 + \sum_{i\neq j} D_i D_j \\
&= \sum_{i=1}^{r} D_i^2 + (1 + 2(r-2) + 1) \\
&= \sum_{i=1}^{r} D_i^2 + 2r - 2.
\end{aligned}
$$

Since D is an effective divisor, we have by definition that $p_a(D) = 1 - \chi(\mathscr{O}_D)$. Since D is a rational tree, by [8, Lemma 1.2, p.148], $p_a(D) = 0$. So by the adjunction formula [12, Exericise V 1.3, p.366],

$$K_V \,.\, D = -D^2 - 2.$$

Since Γ is a rational tree, we have

$$1 = e(\Gamma) = r - \frac{1}{2}\sum_{i\neq j} D_i D_j.$$

Thus,

$$\sum_{i=1}^{r} D_i^2 = D^2 - 2r + 2$$

Using the additivity of χ on the short exact sequence (see [8, pp.147,162]),

$$0 \longrightarrow \Theta_V \langle D\rangle \longrightarrow \Theta_V \longrightarrow \bigoplus \mathscr{N}_{D_i/V} \longrightarrow 0,$$

and the previous results and remarks, we get

$$
\begin{aligned}
\chi(\Theta_V \langle D\rangle) &= \chi(\Theta_V) - \chi\left(\bigoplus \mathscr{N}_{D_i/V}\right) \\
&= (8-2r) - \left(r + \sum_{i=1}^{r} D_i^2\right) \\
&= 8 - 2r - (r + D^2 - 2r + 2) \\
&= 6 - r - D^2 \\
&= K_V^2 - D^2 - 3 \\
&= K_V^2 + 2K_V \,.\, D - 2(-D^2 - 2) - D^2 - 3 \\
&= (K_V + D)^2 + 1 \\
&= K_V \,.\, (K_V + D) - 1.
\end{aligned}
$$

□

The second lemma bounds the sum of the so-called M-numbers by the type of the curve. This result is inspired by Orevkov (see [24]).

The associated *M-number* to a cusp p can be calculated directly from the multiplicity sequence $\overline{m}$,

$$M = \sum_{i=0}^{t-1}(m_i - 1) + \sum_{i=1}^{t-1}\left(\left\lceil \frac{m_{i-1}}{m_i} \right\rceil - 1\right) - 1,$$

where $\lceil a \rceil$ is the smallest integer $\geq a$, [5, Definition 1.5.23, p.44] and [24, p.659].

Before we state the lemma, recall that $\overline{\kappa}(\mathbb{F}_e \setminus C)$ denotes the logarithmic Kodaira dimension of the complement to C in $\mathbb{F}_e$ (see [15]).

Lemma 4.2. *For a rational cuspidal curve C of type (a,b) on $\mathbb{F}_e$ with s cusps and $\overline{\kappa}(\mathbb{F}_e \setminus C) \geq 0$, we have*

$$\sum_{j=1}^{s} M_j \leq 2(a+b) + be.$$

Proof. Let (V,D) and $\sigma = \sigma_1 \circ \ldots \circ \sigma_t$ be a minimal embedded resolution of C. Write $\sigma^*(C) = \tilde{C} + \sum_{i=1}^{t} m_{i-1}E_i$, with $\tilde{C}$ the strict transform of C under σ, m_{i-1} the multiplicity of the center of σ_i and E_i the exceptional curve of σ_i. Then by induction, [12, Proposition V 3.2, p.387] and the genus formula we find that

$$\begin{aligned}
\tilde{C}^2 &= \left(\sigma^*(C) - \sum_{i=1}^{t} m_{i-1}E_i\right)^2 \\
&= C^2 - \sum_{i=0}^{t-1} m_i^2 \\
&= b^2e + 2ab - \sum_{i=0}^{t-1} m_i^2 \\
&= be + 2(a+b) - 2 - \sum_{i=0}^{t-1} m_i.
\end{aligned}$$

Moreover, for $D = \tilde{C} + \sum_{i=1}^{t} E_i'$, with E_i' the strict transform of E_i under the composition $\sigma_{i+1} \circ \cdots \circ \sigma_t$, we have the formula

$$\begin{aligned}
D^2 &= \tilde{C}^2 + 2\tilde{C}\,.\left(\sum_{i=1}^{t} E_i'\right) + \left(\sum_{i=1}^{t} E_i'\right)^2 \\
&= \tilde{C}^2 + 2s + \left(\sum_{i=1}^{t} E_i'\right)^2.
\end{aligned}$$

Now we split the latter term in this sum into the sum of the strict transforms of the exceptional divisors for each cusp,

$$\sum_{i=1}^{t} E_i' = \sum_{j=1}^{s} E_{p_j},$$

where s denotes the number of cusps. By [18, Lemma 2, p.235], we have

$$\omega_j = -E_{p_j}^2 - 1,$$

where ω_j is the number of blowing ups in the minimal embedded resolution which center is an intersection point of the strict transforms of two exceptional curves of the resolution, i.e., an *inner* blowing up.

Combining the above results, we get

$$\begin{aligned} D^2 &= be + 2(a+b) - 2 - \sum_{i=0}^{t-1} m_i + 2s - \sum_{j=1}^{s} (\omega_j + 1) \\ &= be + 2(a+b) - 2 - \sum_{i=0}^{t-1} m_i - \sum_{j=1}^{s} (\omega_j - 1). \end{aligned}$$

By the proof of Lemma 4.1, we have

$$6 - r - D^2 = (K_V + D)^2 + 1,$$

where r denotes the number of components of the divisor D. This number is equal to the total number of blowing ups needed to resolve the singularities, plus one component from the strict transform of the curve itself. Following the notation established, we have $r = t+1$. Moreover, by assumption, $\overline{\kappa}(\mathbb{F}_e \setminus C) \geq 0$. By the logarithmic Bogomolov–Miyaoka–Yau-inequality (B–M–Y-inequality) in the form given by Orevkov [24, Theorem 2.1, p.660] and the topological Euler characteristic of the complement to the curve (see [21]), we obtain

$$(K_V + D)^2 \leq 6.$$

So we get

$$\begin{aligned} 0 &\leq 1 + r + D^2 \\ &\leq 1 + r + be + 2(a+b) - 2 - \sum_{i=0}^{t-1} m_i - \sum_{j=1}^{s} (\omega_j - 1) \\ &\leq -1 + 1 + be + 2(a+b) - \sum_{i=0}^{t-1} (m_i - 1) - \sum_{j=1}^{s} (\omega_j - 1) \\ &\leq be + 2(a+b) - \sum_{j=1}^{s} M_j. \end{aligned}$$

Hence,

$$\sum_{j=1}^{s} M_j \leq 2(a+b)+be.$$

□

4.2. An expression for $\chi\left(\Theta_V\left\langle D\right\rangle\right)$

In this section we give a formula for $\chi(\Theta_V\langle D\rangle)$ for curves C on $\mathbb{F}_e$. By Flenner and Zaidenberg [8] it is conjectured that $\mathbb{Q}$-acyclic affine surfaces Y with logarithmic Kodaira dimension $\overline{\kappa}(Y)=2$ are *rigid* and *unobstructed*, that is $\mathrm{h}^1(V,\Theta_V\langle D\rangle)=\mathrm{h}^2(V,\Theta_V\langle D\rangle)=0$, for (V,D) a minimal embedded resolution of the curve C on Y. The complements to all known rational cuspidal curves with three or more cusps on the projective plane are examples of such surfaces. In that case,

$$\chi(\Theta_V\langle D\rangle)=\mathrm{h}^2(V,\Theta_V\langle D\rangle)-\mathrm{h}^1(V,\Theta_V\langle D\rangle)+\mathrm{h}^0(V,\Theta_V\langle D\rangle)=0,$$

since it is known by Iitaka that $\mathrm{h}^0(V,\Theta_V\langle D\rangle)=0$ [14, Theorem 6]. This conjecture is referred to as the *rigidity conjecture*. Complements to rational cuspidal curves with three or more cusps on Hirzebruch surfaces can be shown to have $\overline{\kappa}(\mathbb{F}_e\setminus C)=2$ (see [22]), however, these open surfaces are no longer $\mathbb{Q}$-acyclic (see [21]), so we do not expect the rigidity conjecture of Flenner and Zaidenberg to hold in this case. Indeed, we calculate the value of $\chi(\Theta_V\langle D\rangle)$ for the curves provided in Section 3, and observe that for these curves we do not necessarily have $\chi(\Theta_V\langle D\rangle)=0$.

Theorem 4.3. *For an irreducible rational cuspidal curve C of type (a,b) on $\mathbb{F}_e$ with s cusps p_j with respective M-numbers M_j, we have*

$$\chi(\Theta_V\langle D\rangle)=7-2a-2b-be+\sum_{j=1}^{s} M_j.$$

Proof. By [8, Proposition 2.4, p.445],

$$K_V\,.\,(K_V+D)=K_{\mathbb{F}_e}\,.\,(K_{\mathbb{F}_e}+C)+\sum_{j=1}^{s} M_j.$$

By Lemma 4.1, we then get

$$\begin{aligned}\chi(\Theta_V\langle D\rangle) &= K_V\,.\,(K_V+D)-1\\ &= K_{\mathbb{F}_e}\,.\,(K_{\mathbb{F}_e}+C)+\sum_{j=1}^{s}M_j-1\\ &= ((e-2)L-2M)\,.\,\Big((a+e-2)L+(b-2)M\Big)-1+\sum_{j=1}^{s}M_j\\ &= 7-2a-2b-be+\sum_{j=1}^{s}M_j.\end{aligned}$$

□

With the above result in mind, we investigate $\chi(\Theta_V\langle D\rangle)$ further. Let C be a rational cuspidal curve of type (a,b) on $\mathbb{F}_e$, and let (V,D) be as before. By the above, we have that

$$\begin{aligned}\chi(\Theta_V\langle D\rangle) &:= \mathrm{h}^0(V,\Theta_V\langle D\rangle)-\mathrm{h}^1(V,\Theta_V\langle D\rangle)+\mathrm{h}^2(V,\Theta_V\langle D\rangle)\\ &= K_V\,.\,(K_V+D)-1\\ &= 7-2(a+b)-be+\sum_{j=1}^{s}M_j.\end{aligned}$$

Moreover, when $\overline{\kappa}(\mathbb{F}_e\setminus C)\geq 0$, we see from Lemma 4.2 that

$$\chi(\Theta_V\langle D\rangle)\leq 7.$$

If $\overline{\kappa}(\mathbb{F}_e\setminus C)=2$, then Iitaka's result $\mathrm{h}^0(V,\Theta_V\langle D\rangle)=0$ still holds [14, Theorem 6]. So we have

$$\chi(\Theta_V\langle D\rangle)=\mathrm{h}^2(V,\Theta_V\langle D\rangle)-\mathrm{h}^1(V,\Theta_V\langle D\rangle).$$

In [29, Lemma 4.1, p.219] Tono shows that if first, $\overline{\kappa}(\mathbb{F}_e\setminus C)=2$, and second, the pair (V,D) is *almost minimal* (see [19]), then $K_V\,.\,(K_V+D)\geq 0$. For plane curves, this result by Tono and a result by Wakabayashi in [31] implies that for rational cuspidal curves with three or more cusps we have that $\chi(\Theta_V\langle D\rangle)\geq 0$. Similarly, a rational cuspidal curve on a Hirzebruch surface that fulfills the two prerequisites has $\chi(\Theta_V\langle D\rangle)\geq -1$. While a smiliar result to Wakabayashi's result ensures that three or more cusps implies $\overline{\kappa}(\mathbb{F}_e\setminus C)=2$ [22], rationality, however, is no longer a guarantee for almost minimality (see [21, p.98]). Therefore, for rational cuspidal curves with three or more cusps on a Hirzebruch surface, $\chi(\Theta_V\langle D\rangle)$ is not necessarily bounded below.

For rational cuspidal curves with four cusps on Hirzebruch surfaces $\mathbb{F}_e$ and $\mathbb{F}_h$, where $e \geq 0$ and $h \in \{0,1\}$, the values of $\chi(\Theta_V \langle D \rangle)$ is given in Table 4.

Type	**Cuspidal configuration**	$\chi(\Theta_V \langle D \rangle)$	**Surface**
$(2k+1,4)$	$[4_{k-1+e}, 2_3], [2], [2], [2]$	$1-k-e$	$\mathbb{F}_e$
$(3k+1-h,5)$	$[4_{2k-1+h}, 2_3], [2], [2], [2]$	-1	$\mathbb{F}_h$
$(2k+2-h,4)$	$[3_{2k-1+h}, 2], [2_3], [2], [2]$	0	$\mathbb{F}_h$
$(k+1-h,3)$	$[2_{n_1}], [2_{n_2}], [2_{n_3}], [2_{n_4}]$	-1	$\mathbb{F}_h$
$(0,3)$	$[2], [2], [2], [2]$	-1	$\mathbb{F}_2$

Table 4: $\chi(\Theta_V \langle D \rangle)$ for rational cuspidal curves with four cusps on $\mathbb{F}_e$ and $\mathbb{F}_h$. For the three first series, $k \geq 0$. For the fourth series, $k \geq 2$ and $\sum_{j=1}^{4} n_j = 2k+h$.

An important observation from this list is the fact that $\chi(\Theta_V \langle D \rangle) \leq 0$ for all these curves. We reformulate this observation in a conjecture (cf. [7, 8]).

Conjecture 4.4. Let C be a rational cuspidal curve with four or more cusps on a Hirzebruch surface $\mathbb{F}_e$. Then $\chi(\Theta_V \langle D \rangle) \leq 0$.

4.3. Real cuspidal curves

A curve $C = \mathscr{V}(F)$ is called a real curve if the polynomial F has real coefficients. The known plane rational cuspidal curves with three cusps can be defined over $\mathbb{R}$ by construction [6, 9, 10]. Due to the Klein–Schuh Theorem, that is not the case for the plane rational cuspidal quintic curve with cuspidal configuration $[2_3], [2], [2], [2]$ (see [20, Section 9.1]).

On the Hirzebruch surfaces, the question whether all cusps on real cuspidal curves can have real coordinates is still hard to answer. However, all known curves on $\mathbb{F}_e$ can be constructed from curves on $\mathbb{P}^2$ and the birational transformations can be given as real transformations. Hence, if it is possible to arrange the curve on $\mathbb{P}^2$ such that the preimages of the cusps have real coordinates, then the cusps will have real coordinates on the curve on the Hirzebruch surface as well. This is, of course, not always possible. For example, considering the rational cuspidal curves on $\mathbb{F}_e$ with four cusps, most of them are constructed from the plane rational cuspidal quintic with cuspidal configuration $[2_3], [2], [2], [2]$. Since the latter curve can not be defined over $\mathbb{R}$, we expect the same for the curves on the Hirzebruch surfaces, that is, the cusps on these curves can not all have real coordinates when the curve is real.

Contrary to this intuition, however, there are indeed examples of real rational cuspidal curves with four real cusps on the Hirzebruch surfaces.

Proposition 4.5. *The series of rational cuspidal curves of type* $(k+1-h,3)$, $k \geq 2$, *with four cusps and cuspidal configuration* $[2_{n_1}],[2_{n_2}],[2_{n_3}],[2_{n_4}]$, *where the indices satisfy* $\sum_{j=1}^{4} n_j = 2k+h$, *on the Hirzebruch surfaces* $\mathbb{F}_h$, $h \in \{0,1\}$, *has the property that all cusps can be given real coordinates on a real curve.*

Proof. We have seen that the series of curves can be constructed using the plane rational cuspidal quartic C with three cusps. Let

$$y^2z^2 + x^2z^2 + x^2y^2 - 2xyz(x+y+z)$$

be a real defining polynomial of C. Then it is possible to find a tangent line to C that intersects C in three real points. For example, choose the line T defined by

$$\frac{2048}{125}x + \frac{2048}{27}y - \frac{1048576}{3375}z = 0.$$

This line is tangent to C at the point $(\frac{64}{9} : \frac{64}{25} : 1)$, and it intersects C transversally at the points $(16 : \frac{16}{25} : 1)$ and $(\frac{4}{9} : 4 : 1)$. With this configuration, there exists a birational transformation from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1$ that preserves the real coordinates of the cusps on C and constructs a fourth cusp with real coordinates on the strict transform of C on $\mathbb{P}^1 \times \mathbb{P}^1$. We blow up the two real points at the transversal intersections and contract the tangent line T, using the birational map from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1$ (see [21]). In coordinates, the map is given by a composition of a (real) change of coordinates on $\mathbb{P}^2$ and the map ϕ,

$$\begin{array}{cccc} \phi : & \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^1 \times \mathbb{P}^1 \\ & (x:y:z) & \mapsto & (x:y;z:x), \end{array}$$

with inverse

$$\begin{array}{cccc} \phi^{-1} : & \mathbb{P}^1 \times \mathbb{P}^1 & \dashrightarrow & \mathbb{P}^2 \\ & (x_0:x_1;y_0:y_1) & \mapsto & (x_0y_1 : x_1y_1 : x_0y_0). \end{array}$$

The strict transform of C is a real curve C' on $\mathbb{P}^1 \times \mathbb{P}^1$ of type $(3,3)$ and cuspidal configuration $[2],[2],[2],[2]$, and all the cusps have real coordinates.

On $\mathbb{P}^1 \times \mathbb{P}^1$, since the cusps p_j have real coordinates, a fiber, say L_j, intersecting C' at a cusp is real. Using the defining polynomial of L_j to substitute one of the variables x_0 or x_1 in the defining polynomial of C and removing the factor of x_i^3, we are left with a polynomial with real coefficients in y_0 and y_1 of degree 3. This polynomial has a double real root, and one simple, hence real, root. The double root corresponds to the y-coordinates of the cusp p_j, and the simple root to the y-coordinates of a smooth intersection point r_j of C and L_j. Successively blowing up at any r_j and contracting the corresponding L_j lead to the desired series of curves. Since the points we blow up have real coordinates, the transformations preserve the real coordinates of the cusps. Hence, all the curves in the series can have four cusps with real coordinates. $\square$

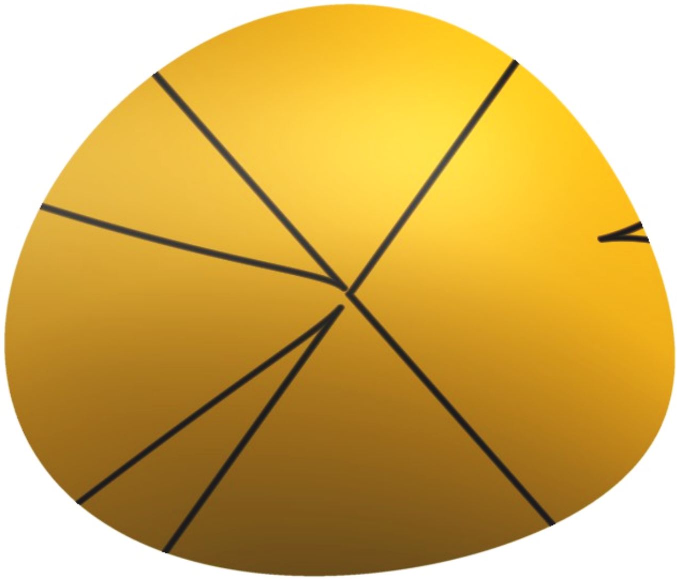

Figure 1: A real rational cuspidal curve of type $(3,3)$ with four ordinary cusps on $\mathbb{P}^1 \times \mathbb{P}^1$.

An image of a real rational cuspidal curve of type $(3,3)$ with four ordinary cusps on $\mathbb{P}^1 \times \mathbb{P}^1$ is given in Figure 1. In the figure, the surface $\mathbb{P}^1 \times \mathbb{P}^1$ is embedded in $\mathbb{P}^3$ using the Segre embedding, and we have chosen a suitable affine chart of $\mathbb{P}^3$. The image is created in cooperation with Georg Muntingh using `surfex` [13].

4.4. On the multiplicity

In the following we establish a result on the multiplicities of the cusps on a rational cuspidal curve on a Hirzebruch surface.

Assume that C is a rational cuspidal curve on a Hirzebruch surface $\mathbb{F}_e$. Let $p_1, \dots, p_s$ denote the cusps of C, and $m_{p_1}, \dots, m_{p_s}$ their multiplicities. Renumber the cusps such that $m_{p_1} \geq m_{p_2} \geq \dots \geq m_{p_s}$. Then for curves with $\overline{\kappa}(\mathbb{F}_e \setminus C) \geq 0$ we are able to establish a lower bound for m_{p_1}.

Theorem 4.6. *A rational cuspidal curve C of type (a,b) on $\mathbb{F}_e$, with two or more cusps, must have at least one cusp p_1 with multiplicity $m := m_{p_1}$ that satisfies* *the lower bound,*

$$m > 1 + a + b + \frac{1}{2}be - \frac{1}{2}\sqrt{4(a^2 + b^2 - 1) + 16(a + b) + be(be + 4a + 8)}.$$

Proof. An inequality by Borodzik et al. [1, Proposition 2.9, p.313] says that locally

$$\mu \leq m(M - m + 2).$$

The assumption that we have two or more cusps implies that $\overline{\kappa}(\mathbb{F}_e \setminus C) \geq 0$ (see [22]). Hence the M-numbers are bounded by Lemma 4.2. Summing over all cusps, we have

$$\begin{aligned}
(b-1)(2a-2+be) &\leq \sum_j m_j(M_j - m_j + 2) \\
&\leq m \sum_j (M_j - m_j + 2) \\
&\leq m \sum_j M_j + m \sum (2 - m_j) \\
&\leq m \sum_j M_j + m(2 - \sum m_j) \\
&\leq m \sum_j M_j + m(2 - m) \\
&\leq m\big(2(a+b) + be - m + 2\big).
\end{aligned}$$

Solving this inequality for m, we get that

$$m \geq 1 + a + b + \frac{1}{2}be - \frac{1}{2}\sqrt{4(a^2 + b^2 - 1) + 16(a + b) + be(be + 4a + 8)}. \qquad \square$$

Note that this bound is stronger than the one found in [21, Theorem 3.5.5, p.99], and that it does indeed exclude several eventual possibilities for cuspidal configurations on rational cuspidal curves.

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TORGUNN KAROLINE MOE
Department of Mathematics
University of Oslo
P.O. Box 1053 Blindern
NO-0316 Oslo, NORWAY
e-mail: `torgunnk@math.uio.no`