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A NEW CLASS OF GENERALIZED POLYNOMIALS ASSOCIATED WITH HERMITE AND BERNOULLI POLYNOMIALS

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In this paper, we introduce a new class of generalized polynomials associated with the modified Milne-Thomson's polynomials $\Phi_n^{(\alpha)}(x,\nu)$ of degree n and order α introduced by Derre and Simsek. The concepts of Bernoulli numbers B_n, Bernoulli polynomials $B_n(x)$, Bernoulli numbers $B_n(a,b)$, generalized Bernoulli polynomials $B_n(x;a,b,c)$ of Luo et al., Hermite-Bernoulli polynomials ${}_HB_n(x,y)$ of Dattoli et al. and ${}_HB_n^{(\alpha)}(x,y)$ of Pathan are generalized to the one ${}_HB_n^{(\alpha)}(x,y,a,b,c)$ which is called the generalized polynomials depending on three positive real parameters. Numerous properties of these polynomials and some relationships between B_n, $B_n(x)$, $B_n(a,b)$, $B_n(x;a,b,c)$ and ${}_HB_n^{(\alpha)}(x,y;a,b,c)$ are established. Some implicit summation formulae and general symmetry identities are derived by using different analytical means and applying generating functions. These results extend some known summations and identities of generalized Bernoulli numbers and polynomials.

1. Introduction

Derre and Simsek [7] modified the Milne-Thomson's polynomials $\Phi_n^{(\alpha)}(x)$ (see for detail [12]) as $\Phi_n^{(\alpha)}(x,\nu)$ of degree n and order α by the means of the fol-

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lowing generating function:

$$g_1(t,x;\alpha,\nu) = f(t,\alpha)e^{xt+h(t,\nu)} = \sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x,\nu)\frac{t^n}{n!} \tag{1}$$

where $f(t,\alpha)$ is a function of t and integer α.
Observe that $\Phi_n^{(\alpha)}(x,0) = \Phi_n^{(\alpha)}(x)$ (cf. [12]).

Setting $f(t,\alpha) = \left(\frac{t}{e^t-1}\right)^{\alpha}$ in (1), we obtain the following polynomials given by the generating function

$$g_2(t,x;\alpha,\nu) = \left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt+h(t,\nu)} = \sum_{n=0}^{\infty} \frac{B_n^{(\alpha)}(x,\nu)t^n}{n!}. \tag{2}$$

Observe that the polynomials $B_n^{(\alpha)}(x,\nu)$ are related to not only Bernoulli polynomials but also the Hermite polynomials. For example, if $h(t,0)=0$ in (2), we have

$$B_n^{(\alpha)}(x,0) = B_n^{(\alpha)}(x)$$

where $B_n^{(\alpha)}(x,\nu)$ denotes the Bernoulli polynomials of higher order which is defined by means of the following generating function

$$F_B(t,x;\alpha) = \left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}. \tag{3}$$

One can easily see that

$$B_n^{(\alpha)}(0,0) = B_n^{(\alpha)}$$

that is

$$F_B(t;\alpha) = \left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!} \tag{4}$$

where $B_n^{(\alpha)}$ are generalized Bernoulli numbers. For more information about Bernoulli numbers and Bernoulli polynomials, we refer to [6, 14, 16]. If we take $h(t,\nu) = h(t,y) = yt^2$ in (1), we get generalized Hermite-Bernoulli polynomials of two variables ${}_HB_n^{(\alpha)}(x,y)$ introduced by Pathan [14] in the form

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{(\alpha)}(x,y)\frac{t^n}{n!} \tag{5}$$

which is essentially a generalization of Bernoulli numbers, Bernoulli polynomials, Hermite polynomials and Hermite-Bernoulli polynomials ${}_HB_n^{(\alpha)}(x,y)$ introduced by Dattoli et al. ([5], p.386 (1.6)) in the form

$$\left(\frac{t}{e^t-1}\right) e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n(x,y)\frac{t^n}{n!}. \tag{6}$$

For each integer $k \geq 0$, $S_k(n) = \sum_{i=0}^{n} i^k$ is called sum of integer powers or simply power sum. The exponential generating function for $S_k(n)$ is

$$\sum_{k=0}^{\infty} S_k(n)\frac{t^k}{k!} = 1 + e^t + e^{2t} + \cdots + e^{nt} = \frac{e^{(n+1)t} - 1}{e^t - 1}.$$

In [9, 16] Qi and Guo generalized the concept of Bernoulli numbers as follows. Let a, b and $a \neq b$. The generalized Bernoulli numbers $B_n(a,b)$ for nonnegative integer n are defined by

$$\Phi(t;a,b) = \frac{t}{a^t - b^t} = \sum_{n=0}^{\infty} B_n(a,b)\frac{t^n}{n!}, \qquad |\, t \,| < 2\pi. \tag{7}$$

In [11] Luo et al. gave the following definition of the generalized Bernoulli polynomials which generalize the concepts stated above. Let $a, b > 0$ and $a \neq b$. The generalized Bernoulli polynomials $B_n(x;a,b,c)$ for nonnegative integer n are defined by

$$\Phi(x,t;a,b,c) = \frac{tc^{xt}}{a^t - b^t} = \sum_{n=0}^{\infty} B_n(x;a,b,c)\frac{t^n}{n!}, \qquad |\, t \,| < 2\pi.$$

It is easy to see that the above definition given by Luo et al. [11] is a natural and essential generalization of the concepts of Bernoulli numbers B_n, Bernoulli polynomials $B_n(x)$ and the generalized Bernoulli numbers $B_n(a,b)$.

Definition 1.1. Let $c > 0$. The generalized 2-variable 1-parameter Hermite-Kampé de Fériet polynomials $H_n(x,y,c)$ polynomials for nonnegative integer n are defined by

$$c^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y,c)\frac{t^n}{n!}. \tag{8}$$

This is an extended 2-variable Hermite-Kampé de Fériet polynomials $H_n(x,y)$ (see [3]) defined by

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}.$$

Note that

$$H_n(x,y,e) = H_n(x,y).$$

In order to collect the powers of t we expand the left hand side of (8) to get

$$\left(\sum_{n=0}^{\infty} \frac{x^n(\ln c)^n t^n}{n!}\right)\left(\sum_{j=0}^{\infty} \frac{y^j(\ln c)^j t^{2j}}{j!}\right) = \sum_{n=0}^{\infty} H_n(x,y,c)\frac{t^n}{n!}.$$

Thus we led to the representation

$$H_n(x,y,c) = n!\sum_{j=0}^{[\frac{n}{2}]} \frac{(\ln c)^{n-j}x^{n-2j}y^j}{j!(n-2j)!}.$$

In this note we first give definitions of the generalized Bernoulli polynomials $B_n^{(\alpha)}(x,y;a,b.c)$ which generalize the concepts stated above and then research their basic properties and relationships with Bernoulli numbers B_n, Bernoulli polynomials $B_n(x)$ and the generalized Bernoulli numbers $B_n(a,b)$ generalized Bernoulli polynomials $B_n(x;a,b,c)$ of Luo et al Hermite-Bernoulli polynomials ${}_HB_n(x,y)$ of Dattoli et al. and ${}_HB_n^{(\alpha)}(x,y)$ of Pathan. The remainder of this paper is organized as follows. We modify generating functions for the Milne-Thomson's polynomials [12] and derive some identities related to Hermite polynomials, Bernoulli polynomials and power sums. Some implicit summation formulae and general symmetry identities are derived by using different analytical means and applying generating functions. These results extend some known summations and identities of generalized Hermite-Bernoulli polynomials studied by Dattoli et al., Natalini et al., Zhang et al., Yang and Pathan.

2. Definitions and Properties of the Generalized Bernoulli polynomials $B_n^{(\alpha)}(x,y;a,b,c)$

In the modified Milne Thomson's polynomials due to Derre and Simsek [7, 12] defined by (1) if we set $f(t,\alpha) = \left(\frac{t}{a^t-b^t}\right)^\alpha$, we obtain the following generalized polynomials $B_n^{(\alpha)}(x,\nu;a,b,c)$

Definition 2.1. Let $a,b,c>0$ and $a\neq b$. The generalized Bernoulli polynomials $B_n^{(\alpha)}(x,\nu;a,b,c)$ for nonnegative integer n are defined by

$$G_1(t,x;\alpha,a,b,\nu) = \left(\frac{t}{a^t-b^t}\right)^\alpha c^{xt+h(t,\nu)} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x,\nu;a,b,c)\frac{t^n}{n!} \tag{9}$$

$$|\,t\,| < 2\pi/(|\ln a - \ln b\,|),\ x\in\mathbb{R}.$$

Setting $h(t,\nu) = h(t,y) = yt^2$ in (9), we get

Definition 2.2. Let $a,b,c>0$ and $a\neq b$. The generalized Bernoulli polynomials $B_n^{(\alpha)}(x,y;a,b,c)$ for nonnegative integer n are defined by

$$G_2(t,x,y;\alpha,a,b,c) = \left(\frac{t}{a^t-b^t}\right)^\alpha c^{xt+yt^2} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x,y;a,b,c)\frac{t^n}{n!} \tag{10}$$

$$|\,t\,| < 2\pi/(|\ln a - \ln b\,|),\ x\in\mathbb{R}.$$

For $\alpha = 1$, we obtain from (10) the generating function

$$G_2(t,x,y;1,a,b,c) = \frac{t}{a^t - b^t} c^{xt+yt^2} = \sum_{n=0}^{\infty} B_n(x,y;a,b,c)\frac{t^n}{n!}$$

whereas for $x = 0$ gives

$$B_n^{(\alpha)}(0,y;a,b,c) = \sum_{k=0}^{[\frac{n}{2}]} \frac{n!}{k!(n-2k)!}(\ln c)^k B_{n-2k}^{(\alpha)}(a,b)y^k \tag{11}$$

Another special case of (10) for $x = 0, y = 0$ leads to the extension of the generalized Bernoulli numbers $B_n(a,b)$ for nonnegative integer n defined by (7) in the form

Definition 2.3. Let $a,b,c > 0$ and $a \neq b$. The generalized Bernoulli polynomials $B_n^{(\alpha)}(a,b)$ for nonnegative integer n are defined by

$$\Phi(t;\alpha,a,b) = \left(\frac{t}{a^t - b^t}\right)^{\alpha} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(a,b)\frac{t^n}{n!}, \quad |\, t \,| < 2\pi/(|\ln a - \ln b\,|), x \in \mathbb{R}. \tag{12}$$

It is easy to prove that

$$B_n^{\alpha+\beta}(a,b) = \sum_{m=0}^{n} \binom{n}{m} B_m^{(\alpha)}(a,b) B_{n-m}^{(\beta)}(a,b)$$

Further setting $c = e$ in (10), we get

Definition 2.4. Let $a,b,c > 0$ and $a \neq b$. The generalized Hermite-Bernoulli polynomials ${}_HB_n^{(\alpha)}(x,y;a,b,e)$ for nonnegative integer n are defined by

$$G_2(t,x,y;\alpha,a,b,e) = \left(\frac{t}{a^t - b^t}\right)^{\alpha} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{(\alpha)}(x,y;a,b,e)\frac{t^n}{n!} \tag{13}$$
$$|\, t \,| < 2\pi/(|\ln a - \ln b\,|),\ x \in \mathbb{R}$$

The generalized Bernoulli polynomials $B_n^{(\alpha)}(x,y;a,b,c)$ defined by (10) have the following properties which are stated as theorems below.

Theorem 2.5. *Let $a,b,c > 0$ and $a \neq b$. For $x \in R$ and $n \geq 0$:*

$$B_n^{(\alpha)}(x,y,e,1,e) = H_n^{B(\alpha)}(x,y), B_n^{(\alpha)}(0,0,a,b,1) = B_n^{(\alpha)}(a,b),$$
$$B_n^{(\alpha)}(0,0,a,b,1) = B_n^{(\alpha)}, B_n^{(\alpha)}(x,y,a,b,1) = B_n^{(\alpha)}(a,b) \tag{14}$$

$$B_n^{(\alpha+\beta)}(x+y,z+u;a,b,c) = \sum_{m=0}^{\infty} \binom{n}{m} B_m^{(\beta)}(z,u;a,b,c) B_{n-m}^{(\alpha)}(x,y;a,b,c) \tag{15}$$

$$B_n^{(\alpha)}(x+z,y;a,b,c) = \sum_{n=0}^{m} \binom{m}{n} B_{n-m}^{(\alpha)}(x;a,b,c) H_m(z,y;c) \tag{16}$$

Proof. The formulas in (14) are obvious. Applying Definition 2.2, we have

$$\sum_{n=0}^{\infty} B_n^{(\alpha+\beta)}(x+y,z+u;a,b,c)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty} B_n^{(\alpha)}(x,y;a,b,c)\frac{t^n}{n!}\sum_{m=0}^{\infty} B_m^{(\beta)}(z,u;a,b,c)\frac{t^m}{m!}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n} B_m^{(\beta)}(z,u;a,b,c)\frac{t^m}{m!}B_{n-m}^{(\alpha)}(x,y;a,b,c)\frac{t^{n-m}}{(n-m)!}$$

Now equating the coefficients of the like powers of t in the above equation, we get the result (15). Again by Definition 2.2 of generalized Bernoulli polynomials, we have

$$\left(\frac{t}{a^t-b^t}\right)^{\alpha} c^{(x+z)t+yt^2}=\sum_{n=0}^{\infty} B_n^{(\alpha)}(x+z,y;a,b,c)\frac{t^n}{n!} \tag{17}$$

which can be written as

$$\left(\frac{t}{a^t-b^t}\right)^{\alpha} c^{xt}c^{zt+yt^2}=\sum_{n=0}^{\infty} B_n^{(\alpha)}(x,y;a,b,c)\frac{t^n}{n!}\sum_{n=0}^{\infty} H_n(z,y,c)\frac{t^n}{n!} \tag{18}$$

Replacing n by $n-m$ in (18), comparing with (17) and equating their coefficients of t^n leads to formula (16). □

3. Implicit Summation formulae involving generalized Bernoulli and generalized Hermite-Bernoulli polynomials

For the derivation of implicit formulae involving generalized Bernoulli polynomials $B_n^{\alpha}(x,y;a,b,c)$ and generalized Hermite-Bernoulli polynomials ${}_HB_n^{(\alpha)}(x,y;a,b,e)$ the same considerations as developed for the ordinary Hermite and related polynomials in Khan et al. [10] and Hermite-Bernoulli polynomials in Pathan [14] holds as well. First we prove the following results involving generalized Bernoulli polynomials $B_n^{\alpha}(x,y;a,b,c)$.

Theorem 3.1. *Let $a,b,c>0$ and $a\neq b$. Then for $x,y\in R$ and $n\geq 0$, the following implicit summation formulae for generalized Bernoulli polynomials $B_n^{(\alpha)}(x,y;a,b,c)$ holds true:*

$$B_{k+l}^{(\alpha)}(z,y;a,b,c) \;=\; \sum_{n,m=0}^{k,l}\binom{l}{m}\binom{k}{n}(z-x)^{n+m}H_{k+l-n-m}^{(\alpha)}(x,y;a,b,c) \tag{19}$$

Proof. We replace t by $t+u$ and rewrite the generating function (10) as

$$\left(\frac{t+u}{a^{t+u}-b^{t+u}}\right)^{\alpha} c^{y(t+u)^2} = c^{-x(t+u)} \sum_{k,l=0}^{\infty} B_{k+l}^{(\alpha)}(x,y;a,b,c)\frac{t^k}{k!}\frac{u^l}{l!} \tag{20}$$

Replacing x by z in the above equation and equating the resulting equation to the above equation, we get

$$c^{(z-x)(t+u)} \sum_{k,l=0}^{\infty} B_{k+l}^{(\alpha)}(x,y;a,b,c)\frac{t^k}{k!}\frac{u^l}{l!} = \sum_{k,l=0}^{\infty} B_{k+l}^{(\alpha)}(z,y;a,b,c)\frac{t^k}{k!}\frac{u^l}{l!} \tag{21}$$

On expanding exponential function (21) gives

$$\sum_{N=0}^{\infty} \frac{[(z-x)(t+u)]^N}{N!} \sum_{k,l=0}^{\infty} B_{k+l}^{(\alpha)}(x,y;a,b,c)\frac{t^k}{k!}\frac{u^l}{l!} = \sum_{k,l=0}^{\infty} B_{k+l}^{(\alpha)}(z,y;a,b,c)\frac{t^k}{k!}\frac{u^l}{l!}$$

which on using formula [17, p. 52 (2)]

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}$$

in the left hand side becomes

$$\sum_{n,p=0}^{\infty} \frac{(z-x)^{n+p}}{n!p!} \sum_{k,l=0}^{\infty} B_{k+l}^{(\alpha)}(x,y;a,b,c)\frac{t^k}{k!}\frac{u^l}{l!} = \sum_{k,l=0}^{\infty} B_{k+l}^{(\alpha)}(z,y;a,b,c)\frac{t^k}{k!}\frac{u^l}{l!} \tag{22}$$

Now replacing k by $k-n$, l by $l-p$ and using the lemma [17, p. 100 (1)] in the left hand side of (22), we get

$$\sum_{n,p=0}^{\infty}\sum_{k,l=0}^{\infty} \frac{(z-x)^{n+p}}{n!p!} B_{k+l-n-p}^{(\alpha)}(x,y;a,b,c)\frac{t^k}{(k-n)!}\frac{u^l}{(l-p)!}$$
$$= \sum_{k,l=0}^{\infty} B_{k+l}^{(\alpha)}(z,y;a,b,c)\frac{t^k}{k!}\frac{u^l}{l!} \tag{23}$$

Finally on equating the coefficients of the like powers of t and u in the above equation, we get the required result. □

Remark 3.2. By taking $l=0$ in equation (19), we immediately deduce the following result.

Corollary 3.3. *The following implicit summation formula for Bernoulli polynomials $B_n^{(\alpha)}(z,y;a,b,c)$ holds true:*

$$B_k^{(\alpha)}(z,y;a,b,c) = \sum_{n=0}^{k} \binom{k}{n} (z-x)^n B_{k-n}^{(\alpha)}(x,y,a,b,c)$$

Remark 3.4. On replacing z by $z+x$ and setting $y=0$ in Theorem 3.1, we get the following result involving generalized Bernoulli polynomials of one variable

$$B_{k+l}^{(\alpha)}(z+x;a,b,c)=\sum_{n,m=0}^{k,l}\binom{l}{m}\binom{k}{n}(z)^{n+m}B_{k+l-m-n}^{(\alpha)}(x;a,b,c)$$

whereas by setting $z=0$ in Theorem 3.1, we get another result involving generalized Bernoulli polynomials of one and two variables

$$B_{k+l}^{(\alpha)}(y)=\sum_{n,m=0}^{k,l}\binom{l}{m}\binom{k}{n}(-x)^{n+m}B_{k+l-m-n}^{(\alpha)}(x,y;a,b,c)$$

Remark 3.5. Along with the above results we will exploit extended forms of generalized Bernoulli polynomials $B_{k+l}^{(\alpha)}(z;a,b,c)$ by setting $y=0$ in the Theorem 3.1 to get

$$B_{k+l}^{(\alpha)}(z;a,b,c)=\sum_{n,m=0}^{k,l}\binom{l}{m}\binom{k}{n}(z-x)^{n+m}B_{k+l-m-n}^{(\alpha)}(x;a,b,c)$$

Remark 3.6. A straightforward expression of the $B_{k+l}(z,y;a,b,c)$ is suggested by a special case of the Theorem 3.1 for $\alpha=1$ in the following form

$$B_{k+l}(z,y;a,b,c)=\sum_{n,m=0}^{k,l}\binom{k}{n}\binom{l}{m}(z-x)^{n+m}B_{k+l-m-n}(x,y;a,b,c)$$

where $B_{k+l}(z,y;a,b,c)$ denotes the generalized Bernoulli polynomials [18] defined by Luo et al. [11].

Theorem 3.7. *Let $a,b,c>0$ and $a\neq b$. Then for $x\in\mathbb{R}$ and $n\geq 0$:*

$$B_n^{(\alpha)}(x;a,b,c)=\sum_{m=0}^{n}\binom{n}{m}B_m^{(\alpha-1)}(a,b)B_{n-m}^{(\alpha)}(x;\frac{a}{c},\frac{b}{c},c)$$

Proof. We start with the definition

$$\sum_{n=0}^{\infty}B_n^{(\alpha)}(x+1;a,b,c)\frac{t^n}{n!}$$
$$=\left(\frac{t}{a^t-b^t}\right)^{\alpha}c^{(x+1)t}=\left(\frac{t}{a^t-b^t}\right)^{\alpha-1}\left(\frac{t}{a^t-b^t}\right)c^{(x+1)t} \tag{24}$$

and the result of Luo et al. [11, p. 3771 (2.12)] to get

$$\sum_{n=0}^{\infty}B_n^{(\alpha)}(x;a,b,c)\frac{t^n}{n!}=\sum_{m=0}^{\infty}B_m^{(\alpha-1)}(a,b)\frac{t^m}{m!}=\sum_{n=0}^{\infty}B_n^{(\alpha)}(x;\frac{a}{c},\frac{b}{c},c)\frac{t^n}{n!}$$

Now replacing n by $n-m$ and equating the coefficients of t^n leads to formula (11). ∎

Theorem 3.8. *Let $a,b,c>0$ and $a\neq b$. Then for $x,y\in\mathbb{R}$ and $n\geq 0$:*

$$B_n^{(\alpha)}(x+\alpha,y;a,b,c)=\sum_{j=0}^{[\frac{n}{2}]}\frac{n!}{j!(n-2j)!}(\ln c)^j B_{n-2j}^{(\alpha)}(x;\frac{a}{c},\frac{b}{c},c) \tag{25}$$

Proof. Since

$$B_n^{(\alpha)}(x+\alpha,y;a,b,c)\frac{t^n}{n!}=\left(\frac{t}{a^t-b^t}\right)^{\alpha}c^{(x+\alpha)t+yt^2}=\left(\frac{t}{(\frac{a}{c})^t-(\frac{b}{c})^t}\right)^{\alpha}c^{xt}c^{yt^2}$$
$$=\left(\sum_{n=0}^{\infty}B_n^{(\alpha)}(x;\frac{a}{c},\frac{b}{c},c)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}y^j(\ln c)^j\frac{t^2 j}{j!}\right)$$

Now replacing n by $n-2j$ and comparing the coefficients of t^n, we get the result (25). □

Remark 3.9. For $\alpha=1$, the above theorem reduces to

$$B_n(x+1,y;a,b,c)=\sum_{j=0}^{[\frac{n}{2}]}\frac{n!}{j!(n-2j)!}y^j(\ln c)^j B_{n-2j}(x;\frac{a}{c},\frac{b}{c},c)$$

whereas for $y=0$, it reduces to the known result of Luo et al. [11, p. 3771 (2.12)]

$$B_n(x+1,y;a,b,c)=B_n(x;\frac{a}{c},\frac{b}{c},c)$$

It is possible to find the explicit form of the generalized Bernoulli polynomials in terms of generalized Hermite polynomials which is a generalization of the result of Betti and Ricci [4].

Theorem 3.10. *Let $a,b,c>0$ and $a\neq b$. Then for $x,y\in\mathbb{R}$ and $n\geq 0$*

$$B_n^{(\alpha)}(x,y;a,b,c)=\sum_{m=0}^{n}\binom{n}{m}B_{n-m}^{(\alpha)}(a,b)H_m(x,y,c) \tag{26}$$

Proof. By the definition of generalized Bernoulli polynomials and the definition (1), we have

$$\left(\frac{t}{a^t-b^t}\right)^{\alpha}c^{xt+yt^2}$$
$$=\sum_{n=0}^{\infty}B_n^{(\alpha)}(x,y;a,b,c)\frac{t^n}{n!}=\left(\sum_{n=0}^{\infty}B_n^{(\alpha)}(a,b)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}H_m(x,y;c)\frac{t^m}{m!}\right)$$

Now replacing n by $n-m$ and comparing the coefficients of t^n, we get the result (26). □

Remark 3.11. For $c = e$, (26) yields

$$ {}_HB_n^{(\alpha)}(x,y;a,b,e) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m}^{\alpha}(a,b)H_m(x,y) $$

Theorem 3.12. *Let $a,b,c > 0$ and $a \neq b$. Then for $x,y \in \mathbb{R}$ and $n \geq 0$:*

$$ B_n^{(\alpha)}(x,y;a,b,c) = \sum_{k=0}^{n-2j} \sum_{j=0}^{[\frac{n}{2}]} (\ln c)^{n-k-j} B_k^{(\alpha)}(a,b) \frac{n!}{k!j!(n-2j-k)!} \tag{27} $$

Proof. Applying the definition 2.2 to the term $\left(\frac{t}{a^t-b^t}\right)^{\alpha}$ and expanding the exponential function c^{xt+yt^2} at $t = 0$ yields

$$ \left(\frac{t}{a^t-b^t}\right)^{\alpha} c^{xt+yt^2} = \left(\sum_{k=0}^{\infty} B_k^{(\alpha)}(a,b)\frac{t^k}{k!}\right)\left(\sum_{n=0}^{\infty} x^n(\ln c)^n \frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty} y^j(\ln c)^j \frac{t^{2j}}{j!}\right) $$

$$ = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}(\ln c)^{n-k}B_k^{(\alpha)}(a,b)x^{n-k}\right)\frac{t^n}{n!}\left(\sum_{j=0}^{\infty} y^j(\ln c)^j\frac{t^{2j}}{j!}\right) $$

Replacing n by $n-2j$, we have

$$ \sum_{n=0}^{\infty} B_n^{(\alpha)}(x,y;a,b)\frac{t^n}{n!} $$

$$ = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n-2j}\sum_{j=0}^{[\frac{n}{2}]}\binom{n-2j}{k}(\ln c)^{n-k-j}B_k^{(\alpha)}(a,b)x^{n-k-2j}y^j\right)\frac{t^n}{(n-2j)!j!} \tag{28} $$

Combining (28) and (10) and equating their coefficients of t^n produce the formula (27). □

For $y = 0$, the above theorem reduces to the following result of Luo et al. [11, p. 3770 (2.3)].

Corollary 3.13. *Let $a,b,c > 0$ and $a \neq b$. Then for $x,y \in \mathbb{R}$ and $n \geq 0$:*

$$ B_n^{(\alpha)}(x;a,b,c) = \sum_{k=0}^{n}(\ln c)^{n-k}B_k^{(\alpha)}(a,b)x^{n-k}\frac{n!}{k!(n-k)!} $$

On the other hand if we set $x = 0$, the above theorem reduces to the result (11).

Theorem 3.14. *Let $a,b,c > 0$ and $a \neq b$. Then for $x,y \in \mathbb{R}$ and $n \geq 0$:*

$$ B_n^{(\alpha)}(x+1,y;a,b,c) = \sum_{j=0}^{[\frac{n}{2}]}\sum_{k=0}^{n-2j}\binom{n-2j}{k}(\ln c)^{n-k-j}B_k^{(\alpha)}(x;a,b,c) \tag{29} $$

Proof. By the definition of generalized Bernoulli polynomials, we have

$$\left(\frac{t}{a^t-b^t}\right)^{\alpha} c^{(x+1)t+yt^2} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x+1,y;a,b,c)\frac{t^n}{n!} \tag{30}$$

$$= \left(\sum_{k=0}^{\infty} B_k^{(\alpha)}(x;a,b,c)\frac{t^k}{k!}\right)\left(\sum_{n=0}^{\infty} x^n(\ln c)^n\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty} y^j(\ln c)^j\frac{t^{2j}}{j!}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(\ln c)^{n-k}B_k^{(\alpha)}(x;a,b,c)\frac{t^n}{n!}\left(\sum_{j=0}^{\infty} y^j(\ln c)^j\frac{t^{2j}}{j!}\right) \tag{31}$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(\ln c)^{n-k+j}B_k^{(\alpha)}(x;a,b,c)\frac{t^{n+2j}}{n!j!}$$

Replacing n by $n-2j$, we have

$$\sum_{n=0}^{\infty} B_n^{(\alpha)}(x+1,y;a,b,c)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{[\frac{n}{2}]}\sum_{k=0}^{n-2j}\binom{n-2j}{k}(\ln c)^{n-k-j}B_k^{(\alpha)}(x;a,b,c)\frac{t^n}{(n-2j)!j!} \tag{32}$$

Combining (31) (conjectured reference) and (10) and equating their coefficients of t^n leads to formula (29). □

Theorem 3.15. *Let $a,b,c>0$ and $a\neq b$. Then for $x,y\in\mathbb{R}$ and $n\geq 0$:*

$${}_HB_n^{(\alpha)}(x,y;a,b,e) = \sum_{n=0}^{m}\binom{m}{n}B_{n-m}^{(\alpha-1)}(a,b)\,{}_HB_m^{(\alpha)}(x,y;a,b,e) \tag{33}$$

Proof. By the definition of generalized Hermite-Bernoulli polynomials,
we have (sono vere queste uguaglianze?)

$$\frac{t}{a^t-b^t}\left(\frac{t}{a^t-b^t}\right)^{\alpha} e^{xt+yt^2} = \frac{t}{a^t-b^t}\sum_{n=0}^{\infty}{}_HB_n^{(\alpha)}(x,y;a,b,e)\frac{t^n}{n!}$$

$$\left(\frac{t}{a^t-b^t}\right)^{\alpha} e^{xt+yt^2} = \frac{t}{a^t-b^t}\sum_{n=0}^{\infty}{}_HB_n^{(\alpha)}(x,y;a,b,e)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}B_n(a,b)\frac{t^n}{n!}\sum_{m=0}^{\infty}{}_HB_m^{(\alpha)}(x,y;a,b,e)\frac{t^m}{m!}$$

Now replacing n by $n-m$ and equating the coefficients of t^n leads to formula (33). □

Theorem 3.16. *For arbitrary real or complex parameter α, the following implicit summation formula involving generalized Bernoulli polynomials $B_n^{(\alpha)}(x,y;a,b,c)$ holds true:*

$$B_n^{(\alpha)}(x+1,y;a,b,c)-B_n^{(\alpha)}(x,y;a,b,c)=\sum_{k=0}^{n-1}\binom{n}{k}(\ln c)^{n-k}B_k^{(\alpha)}(x;a,b,c) \quad (34)$$

Proof. By the definition of generalized Bernoulli polynomials, we have

$$\sum_{n=0}^{\infty}B_n^{(\alpha)}(x+1,y;a,b,c)\frac{t^n}{n!}-\sum_{n=0}^{\infty}B_n^{(\alpha)}(x,y;a,b,c)\frac{t^n}{n!}$$

$$=\left(\frac{t}{a^t-b^t}\right)^{\alpha}e^{xt+yt^2}(c^t-1)$$

$$=\left(\sum_{k=0}^{\infty}B_k^{(\alpha)}(x,y;a,b,c)\frac{t^k}{k!}\right)\left(\sum_{n=0}^{\infty}(\ln c)^n\frac{t^n}{n!}\right)-\sum_{n=0}^{\infty}B_n^{(\alpha)}(x,y;a,b,c)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}(\ln c)^{n-k}t^{n-k}B_k^{(\alpha)}(x,y;a,b,c)\frac{t^k}{(k)!}-\sum_{n=0}^{\infty}B_n^{(\alpha)}(x,y;a,b,c)\frac{t^n}{n!}$$

Finally, equating the coefficients of the like powers of t^n, we get (34). $\square$

Remark 3.17. The generalization of some of the classical Bernoulli polynomials have ramifications of interest. Perhaps the most important property of the Bernoulli polynomials is that

$$B_n(x+1)-B_n(x)=nx^{n-1}, \qquad n\geq 1$$

which happens to be a special case of (34). Some more examples which are special case of (34) are

$$\sum_{m=0}^{n-1}\binom{n}{m}B_{n-m}(x)=nx^n, \qquad n\geq 1$$

Remark 3.18. To obtain the generalization of the result [2, p. 96] involving Bernoulli numbers B_n

$$-t=\sum_{n=0}^{\infty}[1-(-1)^n]B_n\frac{t^n}{n!}$$

and

$$B_n(x+1)=(-1)^nB_n(x)$$

we obtain the following theorem.

Theorem 3.19. *For arbitrary real or complex parameter α, the following implicit summation formula involving generalized Bernoulli polynomials $B_n^{(\alpha)}(x,y;a,b,c)$ holds true:*

$$\sum_{m=0}^{n}\binom{n}{m}(\ln ab)^m(\alpha)^m B_{n-m}^{(\alpha)}(-x,y;a,b,c)=(-1)^n B_n^{(\alpha)}(x,y;a,b,c) \tag{35}$$

$$B_n^{(\alpha)}(x,y;a,b,c)=(-1)^n B_n^{(\alpha)}(\alpha-x,y;a,b,c) \tag{36}$$

Proof. We replace t by $-t$ in (10) and then subtract the result from (10) itself finding

$$c^{yt^2}\left[\left(\frac{t}{a^t-b^t}\right)^{\alpha}(c^{xt}-(ab)^{\alpha t}c^{-xt})\right]=\sum_{n=0}^{\infty}[1-(-1)^n]B_n^{(\alpha)}(x,y;a,b,c)\frac{t^n}{n!} \tag{37}$$

which is equivalent to

$$\begin{aligned}&\sum_{n=0}^{\infty}B_n^{(\alpha)}(x,y;a,b,c)\frac{t^n}{n!}\\&-\left(\sum_{m=0}^{\infty}(\alpha)^m(\ln ab)^m\frac{t^m}{m!}\right)\sum_{n=0}^{\infty}B_n^{(\alpha)}(-x,y;a,b,c)\frac{t^n}{n!}\sum_{n=0}^{\infty}B_n^{(\alpha)}(x,y;a,b,c)\frac{t^n}{n!}\\&-\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}(\alpha)^m(\ln ab)^m\right)B_{n-m}^{(\alpha)}(-x,y;a,b,c)\frac{t^n}{(n-m)!}\frac{t^m}{m!}\\&=\sum_{n=0}^{\infty}[1-(-1)^n]B_n^{(\alpha)}(x,y;a,b,c)\frac{t^n}{n!}\end{aligned}$$

and thus by equating coefficients of like powers of t^n, we get (35). In order to get (36), we write (37) in the form

$$\begin{aligned}&c^{yt^2}\left[\left(\frac{t}{a^t-b^t}\right)^{\alpha}c^{xt}-\left(\frac{t}{(\frac{c}{b})^t-(\frac{c}{a})^t}\right)^{\alpha t}(c^{(\alpha-x)t})\right]\\&\qquad=\sum_{n=0}^{\infty}[1-(-1)^n]B_n^{(\alpha)}(x,y;a,b,c)\frac{t^n}{n!}\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\sum_{n=0}^{\infty}B_n^{(\alpha)}(x,y;a,b,c)\frac{t^n}{n!}-\sum_{n=0}^{\infty}B_n^{(\alpha)}(\alpha-x,y;\frac{c}{b},\frac{c}{a},c)\frac{t^n}{n!}\\&\qquad=\sum_{n=0}^{\infty}[1-(-1)^n]B_n^{(\alpha)}(x,y;a,b,c)\frac{t^n}{n!}\end{aligned} \tag{38}$$

Now comparing the coefficients of t^n in (38) and (36).

□

Remark 3.20. The formula

$$\sum_{m=0}^{n}\binom{n}{m}(\alpha)^{m}{}_{H}B_{n-m}^{(\alpha)}(-x,y)=(-1)^{n}{}_{H}B_{n-m}^{(\alpha)}(x,y)$$

follows if in (35) we set $a=c=e$ and $b=1$.

Remark 3.21. For $a=c=e$ and $b=1$, (36) reduces to the following result involving generalized Hermite-Bernoulli polynomials

$${}_{H}B_{n}^{(\alpha)}(\alpha-x,y)=(-1)^{n}{}_{H}B_{n-m}^{(\alpha)}(x,y)$$

4. General Symmetry Identities

In this section, we give general symmetry identities for the generalized Bernoulli polynomials $B_n^{(\alpha)}(x,y;a,b,c)$ and $B_n^{(\alpha)}(a,b)$ by applying the generating function (6) and (8). The results extend some known identities of Zhang and Yang [20], Yang [19, Eqs. (9)] and Pathan [14]. Throughout this section α will be taken as an arbitrary real or complex parameter.

Theorem 4.1. *Let $a,b,c>0$ and $a\neq b$. For $x,y\in\mathbb{R}$ and $n\geq 0$. Then the following identity holds true:*

$$\sum_{k=0}^{n}\binom{n}{k}a^{k}b^{n-k}B_{n-k}^{(\alpha)}(bx,b^{2}y;a,b,c)B_{k}^{(\alpha)}(a,b)$$
$$=\sum_{k=0}^{n}\binom{n}{k}b^{k}a^{n-k}B_{n-k}^{(\alpha)}(ax,a^{2}y;b,a,c)B_{k}^{(\alpha)}(b,a) \quad (39)$$

Proof. Start with

$$g(t)=\left(\frac{abt^{2}}{(a^{at}-b^{at})(b^{bt}-b^{bt})}\right)^{\alpha}c^{abxt+a^{2}b^{2}yt^{2}}$$

Then the expression for $g(t)$ is symmetric in a and b and we can expand $g(t)$ into series in two ways to obtain

$$g(t)=\sum_{n=0}^{\infty}B_{n}^{(\alpha)}(bx,b^{2}y;a,b,c)\frac{(at)^{n}}{n!}\sum_{k=0}^{\infty}B_{k}^{(\alpha)}(a,b)\frac{(bt)^{k}}{k!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}B_{n-k}^{(\alpha)}(bx,b^{2}y;a,b,c)\frac{(a)^{n-k}}{(n-k)!}B_{k}^{(\alpha)}(a,b)\frac{(b)^{k}}{k!}\frac{(t)^{n}}{n!}$$

On the similar lines we can show that

$$g(t)=\sum_{n=0}^{\infty}B_n^{(\alpha)}(ax,b^2x;b,a,c)\frac{(bt)^n}{n!}\sum_{k=0}^{\infty}B_k^{(\alpha)}(b,a)\frac{(at)^k}{k!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}B_{n-k}^{(\alpha)}(ax,b^2y;b,a,c)\frac{(b)^{n-k}}{(n-k)!}B_k^{(\alpha)}(b,a)\frac{(b)^k}{k!}\frac{(t)^n}{n!}.$$

By comparing the coefficients of t^n on the right hand sides of the last two equations we arrive the desired result. □

Remark 4.2. For $\alpha=1$, the above result reduces to a known result of Pathan [14].

$$\sum_{k=0}^{n}B_{n-k}(bx,b^2y;a,b,c)B_k(a,b)\frac{a^kb^{n-k}}{(n-k)!k!}$$
$$=\sum_{k=0}^{n}B_{n-k}(ax,a^2y;b,a,c)B_k(b,a)\frac{b^ka^{n-k}}{(n-k)!k!}$$

Further by taking $c=e$ in Theorem 4.1, we immediately deduce the following result involving generalized Hermite-Bernoulli polynomials ${}_HB_n^{(\alpha)}(x,y;a,b,e)$ for nonnegative integer n

$$\sum_{k=0}^{n}{}_HB_{n-k}^{(\alpha)}(bx,b^2y;a,b,e)B_k^{(\alpha)}(a,b)\frac{a^kb^{n-k}}{(n-k)!k!}$$
$$=\sum_{k=0}^{n}{}_HB_{n-k}^{(\alpha)}(ax,a^2y;b,a,e)B_k^{(\alpha)}(b,a)\frac{b^ka^{n-k}}{(n-k)!k!}$$

Remark 4.3. By setting $b=1$ in Theorem 4.1, we immediately get the following result

$$\sum_{k=0}^{n}B_{n-k}^{(\alpha)}(x,y;a,1,c)B_k^{(\alpha)}(a,1)\frac{a^k}{(n-k)!k!}$$
$$=\sum_{k=0}^{n}B_{n-k}^{(\alpha)}(ax,a^2y;1,a,c)B_k^{(\alpha)}(1,a)\frac{a^{n-k}}{(n-k)!k!}$$

Theorem 4.4. *Let $a,b,c>0$ and $a\neq b$. For $x,y\in\mathbb{R}$ and $n\geq 0$. Then the following identity holds true:*

$$\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}B_{n-k}^{(\alpha)}\left(bx+\frac{b}{a}i+j,b^2z;A,B,e\right)B_k^{\alpha}(ay;A,B,e)$$
$$=\sum_{k=0}^{n}\binom{n}{k}b^{n-k}a^k\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}B_{n-k}^{(\alpha)}\left(ax+\frac{a}{b}i+j,a^2z;A,B,e\right)B_k^{\alpha}(by;A,B,e)\quad(40)$$

Proof. Let

$$g(t) = \left(\frac{abt^2}{(A^{at}-B^{at})(A^{bt}-B^{bt})}\right)^{\alpha} \frac{(e^{abt}-1)^2 e^{ab(x+y)t+a^2b^2zt^2}}{(e^{at}-1)(e^{bt}-1)}$$

$$g(t) = \left(\frac{at}{(A^{at}-B^{at}}\right)^{\alpha} e^{abxt+a^2b^2zt^2}\left(\frac{e^{abt}-1}{e^{bt}-1}\right)\left(\frac{bt}{A^{bt}-B^{bt}}\right)^{\alpha} e^{abyt}\left(\frac{e^{abt}-1}{e^{at}-1}\right)$$

$$= \left(\frac{at}{(A^{at}-B^{at}}\right)^{\alpha} e^{abxt+a^2b^2zt^2}\sum_{i=0}^{a-1} e^{bti}\left(\frac{bt}{A^{bt}-B^{bt}}\right)^{\alpha} e^{abyt}\sum_{j=0}^{b-1} e^{atj} \tag{41}$$

$$= \left(\frac{at}{A^{at}-B^{at}}\right)^{\alpha} e^{abxt+a^2b^2zt^2}\sum_{i=0}^{a-1} e^{bti}\left(\frac{bt}{A^{bt}-B^{bt}}\right)^{\alpha} e^{abyt}\sum_{j=0}^{b-1} e^{atj}$$

$$= \left(\frac{at}{A^{at}-B^{at}}\right)^{\alpha} e^{a^2b^2zt^2}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1} e^{(bx+\frac{b}{a}i+j)at}\sum_{k=0}^{\infty} B_k^{\alpha}(ay;A,B,e)\frac{(bt)^k}{k!}$$

$$= \sum_{n=0}^{\infty}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1} B_n^{(\alpha)}\left(bx+\frac{b}{a}i+j, b^2z;A,B,e\right)\frac{(at)^n}{n!}\sum_{k=0}^{\infty} B_k^{\alpha}(ay;A,B,e)\frac{(bt)^k}{(k)!}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} a^{n-k}b^k\sum_{i=0}^{a-1}\sum_{j=0}^{b-1} B_{n-k}^{(\alpha)}\left(bx+\frac{b}{a}i+j, b^2z;A,B,e\right) B_k^{\alpha}(ay;A,B,e)t^n \tag{42}$$

On the other hand

$$g(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} b^{n-k}a^k\sum_{i=0}^{b-1}\sum_{j=0}^{a-1} B_{n-k}^{(\alpha)}\left(ax+\frac{a}{b}i+j, a^2z;A,B,e\right) B_k^{\alpha}(by;A,B,e)t^n \tag{43}$$

By comparing the coefficients of t^n on the right hand sides of the last two equations,we arrive at the desired result. □

Remark 4.5. For $\alpha = 1$, $A = e$ and $B = 1$ the above result reduces to a known result of Pathan [14, Eqs. (43)].

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