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RELATION BETWEEN DUAL *S*-ALGEBRAS AND *BE*-ALGEBRAS

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In this paper, we investigate the relationship between dual (weak) subtraction algebras, Heyting algebras and *BE*-algebras. In fact, the purpose of this paper is to show that *BE*-algebra is a generalization of Heyting algebras and dual (weak) subtraction algebras. Also, we show that a bounded commutative self distributive *BE*-algebra is equivalent to the Heyting algebra.

1. Introduction

H. S. Kim and Y. H. Kim introduced the notion of a *BE*-algebra as a generalization of a dual *BCK*-algebra (2007). A. Rezaei et al. got some results on *BE*-algebras and introduced the notion of commutative ideals in *BE*-algebras and proved several characterizations of such ideals [6, 7]. A. Walendziak investigated the relationship between *BE*-algebras, implication algebras, and *J*-algebras [9]. Moreover, he defined commutative *BE*-algebras and stated that these algebras are equivalent to the commutative dual *BCK*-algebras. Also, in [8], we proved that every Hilbert algebra is a self distributive *BE*-algebra and commutative self distributive *BE*-algebra is a Hilbert algebra and we showed

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that can not remove the conditions of a commutativity and a self distributivity. A. Borumand Saeid [1], proved that CI-algebras are equivalent to dual Q-algebras (2012). Heyting algebras generalize the well-known idea of Boolean algebras and most simply defined as a certain type of lattice. Recently, algebras including Heyting algebras, have played an important role and have its comprehensive applications in many aspects including genetic code of biology and dynamical systems.

Now, our aim is to investigate the relationship between BE-algebras and Heyting algebras. In this paper, we show that a Heyting algebra is equivalent to the bounded commutative $self$–distributive BE–algebra. Furthermore, we show that every dual S-algebra is a commutative BE-algebras but the converse may be not true.

2. Preliminaries

Definition 2.1 ([5])**.** By a CI-algebra we shall mean an algebra $(X;*,1)$ of type $(2,0)$ satisfying the following axioms:

$(CI1)$ $x*x=1$,

$(CI2)$ $1*x=x$,

$(CI3)$ $x*(y*z)=y*(x*z)$, for all $x,y,z\in X$.

We introduce a relation "$\leq$" on X by $x\leq y$ if and only if $x*y=1$. A CI–algebra X is said to be a BE-algebra if (BE) $x*1=1$, for all $x\in X$. A BE-algebra X is said to be $self$-distributive if $x*(y*z)=(x*y)*(x*z)$, for all $x,y,z\in X$ ([3]). We say that X is commutative if $(x*y)*y=(y*x)*x$, for all $x,y\in X$.

We note that "$\leq$" is reflexive by $(CI1)$. If X is a self distributive BE-algebra, then relation "$\leq$" is a transitive order set on X. Because if $x\leq y$ and $y\leq z$, then

$$x*z=1*(x*z)=(x*y)*(x*z)=x*(y*z)=x*1=1$$

and so $x\leq z$. If X is commutative, $x\leq y$ and $y\leq x$, then

$$x=1*x=(y*x)*x=(x*y)*y=1*y=y.$$

Hence "$\leq$" is antisymmetric. Therefore, if X is a commutative self distributive BE-algebra, then "$\leq$" is a partially ordered set on X. X is called bounded if there exists the smallest element 0 of X (i.e., $0*x=1$, for all $x\in X$).

Given a bounded BE-algebra X with 0 as the smallest element, we denote $x*0$ by Nx, then N can be regarded as a unary operation on X. If $NNx=x$, then x is called an involution of X. A bounded BE-algebra X is called involutory if any element of X is involution ([2]).

Proposition 2.2 ([2]). *Let X be a bounded BE-algebra with the smallest element 0. Then the following hold:*

(i) $N0 = 1$ *and* $N1 = 0$,

(ii) $x \leq NNx$,

(iii) $x * Ny = y * Nx$, *for all* $x, y \in X$.

(iv) *if X is commutative, then X is involutory.*

Theorem 2.3 ([2]). *Let X be an involutory self distributive BE-algebra. Then the following are equivalent:*

(i) $(X;\leq)$ *is an upper semi-lattice,*

(ii) $(X;\leq)$ *is a lower semi-lattice,*

(iii) $(X;\leq)$ *is a lattice. Moreover, if $(X;\leq)$ is a lattice, then the following identities hold:*

$$x \wedge y = N(Nx \vee Ny) \text{ and } x \vee y = N(Nx \wedge Ny).$$

Definition 2.4 ([10]). A bounded lattice $(L;\leq)$ is said to be a Heyting algebra if for any $a, b \in L$, there is an element $a \to b \in L$ satisfying

$$c \wedge a \leq b \;\; if \text{ and only } if \;\; c \leq a \to b,$$

for some $c \in L$.

Theorem 2.5 ([10]). *Let $(L;\leq)$ be a bounded lattice. Then L is a Heyting algebra if and only if there is a map $\_ \to \_ : L \times L \to L \quad ((a,b) \to (a \to b))$ satisfying the following:*

(i) $a \to a = 1$,

(ii) $a \wedge (a \to b) = a \wedge b$,

(iii) $b \wedge (a \to b) = b$,

(iv) $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$, *for all* $a, b, c \in L$.

Proposition 2.6 ([10]). *Let $(L;\leq)$ be a Heyting algebra. Then*

(i) $a \leq b$ *if and only if* $a \to b = 1$,

(ii) $b \leq a \to b$,

(iii) $b \le c$ *implies* $a \to b \le a \to c$ *and* $c \to a \le b \to a$,

(iv) $a \to (b \to c) = (a \wedge b) \to c$,

(v) $a \wedge (b \to c) = a \wedge ((a \wedge b) \to (a \wedge c))$,

(vi) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$,

(vii) $a \to (b \to c) = b \to (a \to c)$, *for all* $a, b, c \in L$.

Definition 2.7 ([11]). Let X be a set with a binary operation "$-$". Then $(X; -)$ is called a subtraction algebra (shortly, S-algebra) if it satisfies the following conditions:

$(S1)$ $x - (y - x) = x$,

$(S2)$ $x - (x - y) = y - (y - x)$,

$(S3)$ $(x - y) - z = (x - z) - y$, for all $x, y, z \in X$.

For any $x \in X$ we define the element 0 of X by $x - x = 0$. Then 0 does not depend on x. K. J. Lee et al. introduced to the notion of weak subtraction algebras ([4]).

Definition 2.8 ([4]). An algebra $(X; -, 0)$ of type $(2, 0)$ is called a weak subtraction algebra (shortly, WS-algebra) if it satisfies the following conditions:

$(WS1)$ $(x - y) - z = (x - z) - y$,

$(WS2)$ $(x - y) - z = (x - z) - (y - z)$,

$(WS3)$ $x - 0 = x$,

$(WS4)$ $x - x = 0$, for all $x, y, z \in X$.

Note that every S-algebra is a WS–algebra. There exists WS-algebras that are not S-algebras.

Example 2.9 ([4]). Let $X := \{0, a, b, c\}$ be a set with the following table.

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	0
c	c	c	0	0

Then $(X; -, 0)$ is a WS-algebra but it is not a S-algebra, since

$$b = b - (b - c) \neq c - (c - b) = c.$$

3. Heyting algebras and *BE*-algebras

We denote a *BE*–algebra X by $(X;*,1)$ as an algebra of type $(2,0)$ and a Heyting algebra L by $(L;\leq,\wedge,\rightarrow,1,0)$ as an algebra of type $(2,2,2,0,0)$.

Theorem 3.1. *If* $(X;\leq,\wedge,\rightarrow,0,1)$ *is a Heyting algebra, then* $(X;\rightarrow,1)$ *is a BE-algebra.*

The following example shows that the converse of Theorem 3.1 may not hold, generally.

Example 3.2. Let $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$ and $*$ be the binary operation of $\mathbb{N}_0$ defined by:

$$x*y = \begin{cases} 0 & \text{if } y \leq x \\ y-x & \text{if } x < y \end{cases}$$

Then $(\mathbb{N}_0;*,0)$ is a commutative *BE*-algebra but it is not a Heyting algebra, because it is not bounded.

Theorem 3.3. *Let* $(X;*,0,1)$ *be a bounded commutative self distributive BE-algebra. Then* $(X;*,\vee,\wedge,0,1)$ *is a Heyting algebra.*

Proof. It is sufficient to show that for any $x,y\in X$, $z\wedge x\leq y$ if and only if $z\leq x*y$ for some $z\in X$.
($\Rightarrow$). Let $z\wedge x\leq y$ for some $z\in X$. By using Proposition 2.2(iii), (iv) and Theorem 2.3, we have $Ny\leq N(z\wedge x) = Nx\vee Nz = (Nz*Nx)*Nx$. Hence $Nz*Nx\leq Ny*Nx$. On the other hand since $z\leq x*z$, we have $z\leq x*z = Nz*Nx\leq Ny*Nx = x*y$.
($\Leftarrow$). Let $z\leq x*y$ for some $z\in X$. Then $z*(x*y) = 1$ and so

$$\begin{aligned}(z\wedge x)*y = N(Nz\vee Nx)*y &= Ny*(Nz\vee Nx)\\ &= Ny*((Nz*Nx)*Nx)\\ &= (Nz*Nx)*(Ny*Nx)\\ &= (x*z)*(x*y)\\ &= x*(z*y) = z*(x*y) = 1.\end{aligned}$$

Therefore, $z\wedge x\leq y$. □

Example 3.4. Let $X := \{0,a,b,1\}$. Define a binary operation "$\rightarrow$" on X as the following table:

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

If we consider an ordering "$\leq$" as a and b are incomparable, $0 \leq a, b \leq 1$, then $(X; \leq, \wedge, \rightarrow, 1, 0)$ is a Heyting algebra and $(X; \rightarrow, 1)$ is a commutative BE-algebra.

4. Dual WS/S-algebras and BE-algebras

Definition 4.1. Let $(X; -, 0)$ be a WS/S-algebra and binary operation "$*$" on X is defined as follows:

$$x * y = y - x$$

Then $(X; *, 1)$ is called a dual WS/S-algebra, where $x * x = 1$.

In fact, the axioms of dual S-algebra are as follows:

$(DS1)$ $(x * y) * x = x$,

$(DS2)$ $(y * x) * x = (x * y) * y$,

$(DS3)$ $z * (y * x) = y * (z * x)$, for all $x, y, z \in X$.

And for dual WS-algebra are as follows:

$(DWS1)$ $z * (y * x) = y * (z * x)$,

$(DWS2)$ $z * (y * x) = (z * y) * (z * x)$,

$(DWS3)$ $1 * x = x$,

$(DWS4)$ $x * x = 1$, for all $x, y, z \in X$.

Proposition 4.2. *Let $(X; *, 1)$ be a dual S-algebra. Then*

(i) $1 * x = x$ *and* $x * 1 = 1$,

(ii) $x * (y * x) = 1$,

(iii) $(y * z) * ((z * x) * (y * z)) = 1$,

(iv) $y * ((y * x) * x) = 1$,

(v) $y * x = 1$ *and* $x * y = 1$ *imply* $x = y$,,

(vi) $z * (y * x) = (z * y) * (z * x)$, *fo all* $x, y, z \in X$.

Lemma 4.3. *Let $(X; *, 1)$ be a dual WS-algebra. Then the following conditions hold:*

(i) $x * 1 = 1$,

(ii) $x*(y*x) = 1,$

Theorem 4.4. *If* $(X;*,1)$ *is a dual S-algebra, then* $(X;*,1)$ *is a commutative self distributive BE–algebra.*

Proof. By using Proposition 4.2 and Lemma 4.3, the proof is obvious. □

Corollary 4.5. *If* $(X;*,1)$ *is a dual S-algebra, then* $(X;*,1)$ *is a commutative Hilbert algebra.*

The following example shows that the converse of Theorem 4.4 may not hold, generally.

Example 4.6. Let $\mathbb{N}$ be the set of all natural numbers and "$*$" be the binary operation on $\mathbb{N}$ defined by:

$$x*y = \begin{cases} y & \text{if } x = 1 \\ 1 & \text{if } x \neq 1 \end{cases}$$

Then $(\mathbb{N};*,1)$ is a *BE*-algebra but it is not a dual *S*–algebra, since

$$5*(4*5) = 5*1 = 1 \neq 5.$$

Example 4.7. Example 3.2 is not a dual WS-algebra, since

$$5*(7*9) = 0 \neq (5*7)*(5*9) = 2*4 = 2.$$

Theorem 4.8. *Every commutative self distributive* BE*-algebra is a dual* WS*-algebra.*

Corollary 4.9. (i) *Every Hilbert algebra is a dual* WS*-algebra.*

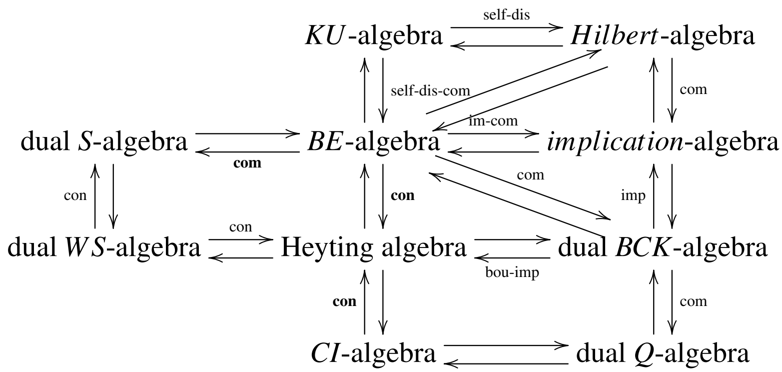
(ii) *Every Heyting algebra is a dual* WS*-algebra.*

Theorem 4.10. *Every commutative* BE*-algebra is a dual* S*-algebra.*

5. Conclusion

Now, in the following diagram we summarize the results of this paper and the past results in this field and we give the relations among BE-algebras, dual

$(WS/S/Q/BCK)$-algebras, Hilbert algebras, implication algebras, CI-algebras, KU-algebras and Heyting algebra. The mark $A \to B$ means that A implies B.



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