NEW GENERALIZATIONS OF SOME INEQUALITIES
FOR k-SPECIAL AND q,k-SPECIAL FUNCTIONS

SABRINA TAF - BOCHRA NEFZI - LATIFA RIAHI

In this work, we establish some new inequalities involving some k-special and q,k-special functions, by using the technique of A. McD. Mercer [11].

1. Introduction

Let \( L \) be a positive linear functional defined on a subspace \( C^*(I) \subset C(I) \), where \( I \) is the interval \((0,a)\) with \( a > 0 \) or \((0, +\infty)\).

Let \( f \) and \( g \) be two functions continuous on \( I \) which are strictly increasing and strictly positive on \( I \).

In [11], A. Mcd. Mercer, posed the following result:

Supposing that \( f, g \in C^*(I) \) such that \( f(x) \to 0, g(x) \to 0 \) as \( x \to 0^+ \) and \( \frac{f}{g} \) is strictly increasing, we define the function \( \phi \) by

\[
\phi = g \frac{L(f)}{L(g)},
\]

and let \( F \) be a function defined on the ranges of \( f \) and \( g \) such that the compositions \( F(f) \) and \( F(g) \) each belong to \( C^*(I) \).

a) If \( F \) is convex then

\[
L[F(f)] \geq L[F(\phi)]. \quad (1)
\]
b) If $F$ is concave then

$$L[F(f)] \leq L[F(\phi)].$$  \hspace{1cm} (2)

The objective of this paper is to use the technique of A. McD. Mercer [11] to develop some new inequalities for $k$-Gamma, $k$-Beta and $k$-Zeta functions then $q,k$-Gamma and $q,k$-Beta functions which are a generalization of some inequalities studied in [11, 13].

Note that for $\alpha \in \mathbb{R}$, the function

$$F(t) = t^\alpha$$

is convex if $\alpha < 0$ or $\alpha > 1$ and concave if $0 < \alpha < 1$.

Then, for $f$ and $g$ satisfying the conditions (1) and (2), we have:

$L(f^{\alpha}) > L(\phi^{\alpha})$ if $\alpha < 0$ or $\alpha > 1$ and $L(f^{\alpha}) < L(\phi^{\alpha})$ if $0 < \alpha < 1$.

Substituting for $\phi$ this reads:

$$\frac{[L(g)]^\alpha}{L(g^{\alpha})} > \text{resp.} < \frac{[L(f)]^\alpha}{L(f^{\alpha})},$$

if $\alpha < 0$ or $\alpha > 1$ (resp. $0 < \alpha < 1$). In particular, if we take $f(x) = x^\beta$ and $g(x) = x^\delta$ with $\beta > \delta > 0$, we obtain the following useful inequality:

$$\frac{[L(x^\delta)]^\alpha}{L(x^{\alpha \delta})} \geq \frac{[L(x^\beta)]^\alpha}{L(x^{\alpha \beta})},$$  \hspace{1cm} (3)

where $\geq$ correspond to the case ($\alpha < 0$ or $\alpha > 1$) and ($0 < \alpha < 1$) respectively.

In recent years, many authors have studied Gamma and Beta functions. For more information see [1], [2], [3], [5], [6], [10], [13].

2. Basic Results

Throughout this paper, we will fix $q \in (0,1)$. For the convenience of the reader, we provide in this section a summary of the mathematical notions and definitions used in this paper (see [7], [9], [12]). We write for $a \in \mathbb{C}$,

$$[a]_q = \frac{1 - q^a}{1 - q}.$$

The $q$-Jackson integrals from 0 to $a$ and from 0 to $\infty$ are defined by (see [8])

$$\int_0^a f(x)d_qx = (1 - q)a \sum_{n=0}^{\infty} f(aq^n)q^n,$$
\[\int_0^\infty f(x) d_q x = (1 - q) \sum_{n = -\infty}^\infty f(q^n) q^n,\]

provided the sums converge absolutely.

For \( k > 0 \), the \( \Gamma_k \) function is defined by (see [4], [10])

\[\Gamma_k(x) = \lim_{n \to \infty} \frac{n! k^n (nk)^{x-1}}{(x)_{n,k}}, x \in \mathbb{C} \setminus k\mathbb{Z}^- ,\]

where \( (x)_{n,k} = x(x+k)(x+2k)\ldots(x+(n-1)k) \).

The above definition is a generalization of the definition of \( \Gamma(x) \) function.

For \( x \in \mathbb{C} \) with \( \Re(x) > 0 \), the function \( \Gamma_k(x) \) is given by the integral (see [4])

\[\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t}{x}} dt ,\]

and satisfies the following properties: (see [5], [10])

1. \( \Gamma_k(x + k) = x \Gamma_k(x) \)

2. \( (x)_{n,k} = \frac{\Gamma_k(x + nk)}{\Gamma_k(x)} \)

3. \( \Gamma_k(k) = 1 \)

**Definition 2.1.** Let \( x, y, s, t \in \mathbb{R} \) and \( n \in \mathbb{N} \), we note by

1. \( (x+y)_{q,k}^n := \prod_{j=0}^{n-1} (x + q^j y) \)

2. \( (1 + x)_{q,k}^\infty := \prod_{j=0}^\infty (1 + q^j x) \)

3. \( (1 + x)^t_{q,k} := \frac{(1 + x)_{q,k}^\infty}{(1 + q^t x)_{q,k}^\infty} \).

We have \( (1 + x)^{s+t}_{q,k} = (1 + x)^s_{q,k} (1 + q^k x)^t_{q,k} \).

We recall the two \( q, k \)-analogues of the exponential functions (see [5])

\[E^x_{q,k} = \sum_{n=0}^{\infty} q^{\frac{k(n-1)}{2}} \frac{x^n}{[n]_{q^k}!} = (1 + (1 - q^k)x)^{\infty}_{q,k} \]

and

\[e^x_{q,k} = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{q^k}!} = \frac{1}{(1 - (1 - q^k)x)^{\infty}_{q,k}} .\]

These \( q, k \)-exponential functions satisfy the following relations:
\[ D_{q}e_{q,k}^{x} = e_{q,k}^{x}, \quad D_{q}E_{q,k}^{x} = E_{q,k}^{q^{x}} \quad \text{and} \quad E_{q,k}^{-x}e_{q,k}^{x} = e_{q,k}^{x}E_{q,k}^{-x} = 1. \]

The \( q,k \)-Gamma function is defined by [5]

\[
\Gamma_{q,k}(x) = \frac{(1 - q^{k})_{q,k}^{\infty}}{(1 - q^{x})_{q,k}^{\infty}(1 - q)^{\frac{x}{k}} - 1}, \quad x > 0.
\]

When \( k = 1 \) it reduces to the known \( q \)-Gamma function \( \Gamma_{q} \).

It satisfies the following functional equation:

\[
\Gamma_{q,k}(x + k) = [x]_{q} \Gamma_{q,k}(x), \quad \Gamma_{q,k}(k) = 1
\]

and it has the following integral representation (see[5])

\[
\Gamma_{q,k}(x) = \int_{0}^{\left[\frac{k}{q}\right]} \left(1 - q^{k}\right)^{\frac{x}{k}} t^{x-1}E_{q,k}^{-q^{k}x} dt, \quad x > 0.
\]

The \( k \)-Beta function is defined by (see [4])

\[
B_{k}(t,s) = \int_{0}^{\infty} x^{t-1}(1 + x^{k})^{-\frac{t+s}{k}} dx, \quad t,s > 0.
\]

By using the following change of variable \( x^{k} = u \) and \( u = \frac{y}{1 - y} \), we obtain

\[
B_{k}(t,s) = \frac{1}{k} \int_{0}^{1} u^{t-1}(1 - u)^{\frac{s}{k}} du, \quad s,t,k > 0.
\]

It is well-known that

\[
B_{k}(s,t) = \frac{\Gamma_{k}(s)\Gamma_{k}(t)}{\Gamma_{k}(s+t)}.
\]

The \( q,k \)-Beta function is defined by (see [5])

\[
B_{q,k}(t,s) = [k]_{q}^{-\frac{t}{k}} \int_{0}^{[k]_{q}^{\frac{1}{k}}} x^{t-1}(1 - q^{k}x^{k})_{q,k}^{-\frac{s}{k}} dx, \quad s > 0, t > 0.
\]

By using the following change of variable \( u = \frac{x}{[k]_{q}^{\frac{1}{k}}} \), the last equation becomes

\[
B_{q,k}(t,s) = \int_{0}^{1} u^{t-1}(1 - q^{k}u^{k})_{q,k}^{-\frac{s}{k}} du, \quad s > 0, t > 0.
\]

It verifies the relation

\[
B_{q,k}(t,s) = \frac{\Gamma_{q,k}(t)\Gamma_{q,k}(s)}{\Gamma_{q,k}(t+s)}, \quad s,t > 0.
\]
The function $B_{q,k}$ satisfies the following formulas for $s, t > 0$

1. $B_{q,k}(t, \infty) = (1 - q)^{\frac{1}{\Gamma} - k} \Gamma_{q,k}(t)$
2. $B_{q,k}(t + k, s) = \left[ \frac{[s]}{[s + k]} \right] B_{q,k}(t, s + k)$
3. $B_{q,k}(t, s + k) = B_{q,k}(t, s) - q^s B_{q,k}(t + k, s)$
4. $B_{q,k}(t, s + k) = \left[ \frac{s}{s + [s]} \right] B_{q,k}(t, s)$
5. $B_{q,k}(t, k) = \frac{1}{[t]}$.

Definition 2.2. We define the function $\zeta_k$ as (see [10])

$$\zeta_k(s) = \frac{1}{\Gamma_k(s)} \int_0^\infty \frac{t^{s-k}}{e^t - 1} dt, \quad s > k.$$  

Note that when $k = 1$ we obtain the known Riemann Zeta function $\zeta(s)$.

3. Main results
We start with the following theorem:

Theorem 3.1. Let $f$ be the function defined by

$$f(x) = \frac{\Gamma_k^{(2n)}(k+x)^\alpha}{\Gamma_k^{(2n)}(k+\alpha x)}$$

then for all $\alpha > 1$ (resp. $0 < \alpha < 1$) $f$ is decreasing (resp. increasing) on $(0, \infty)$.

Proof. The $k$-Gamma function is infinitely differentiable on $(0, \infty)$ and we have

$$\Gamma_k^{(n)}(x) = \int_0^\infty t^{x-1} \log(t)^n e^{-\frac{t^k}{\Gamma}} dt, \quad n \in \mathbb{N}.$$ 

We consider the subspace $C^*(I)$ obtained from $C(I)$ by requiring its members to satisfy:

(i) $\omega(x) = O(x^\theta)$ (for any $\theta > -k$) as $x \to 0$,

(ii) $\omega(x) = O(x^\varphi)$ (for any finite $\varphi$) as $x \to +\infty$.

Then, for $\omega \in C^*(I)$ we define

$$L(\omega) = \int_0^\infty \omega(t)t^{k-1}(\log(t))^{2n} e^{-\frac{t^k}{\Gamma}} dt.$$
The operator $L$ is well-defined on $C^*(I)$ and it is a positive linear functional on $C^*(I)$.

Using the inequality (3), we obtain for $\beta > \delta > 0$

$$\frac{[\Gamma_k^{(2n)}(k+\delta)]^\alpha}{\Gamma_k^{(2n)}(k+\alpha\delta)} \geq \frac{[\Gamma_k^{(2n)}(k+\beta)]^\alpha}{\Gamma_k^{(2n)}(k+\alpha\beta)}.$$ 

Theorem 3.1 is thus proved.

**Corollary 3.2.** For all $x \in [0,k]$, we have:

$$\frac{[\Gamma_k^{(2n)}(2k)]^\alpha}{\Gamma_k^{(2n)}(k+\alpha k)} \leq \frac{[\Gamma_k^{(2n)}(k+x)]^\alpha}{\Gamma_k^{(2n)}(k+\alpha x)} \leq [\Gamma_k^{(2n)}(k)]^{\alpha-1}, \quad \alpha \geq 1,$$

and

$$[\Gamma_k^{(2n)}(k)]^{\alpha-1} \leq \frac{[\Gamma_k^{(2n)}(k+x)]^\alpha}{\Gamma_k^{(2n)}(k+\alpha x)} \leq \frac{[\Gamma_k^{(2n)}(2k)]^\alpha}{\Gamma_k^{(2n)}(k+\alpha k)}, \quad 0 < \alpha \leq 1.$$

**Corollary 3.3.** For all $x \in [0,k]$, we have

$$\frac{k^\alpha}{\Gamma_k(\alpha k + k)} \leq \frac{[\Gamma_k(2k)]^\alpha}{\Gamma_k(k + \alpha k)} \leq \frac{[\Gamma_k^{(2n)}(k+x)]^\alpha}{\Gamma_k^{(2n)}(k+\alpha x)}, \quad \alpha \geq 1,$$

and

$$1 \leq \frac{[\Gamma_k(k+x)]^\alpha}{\Gamma_k(k+\alpha x)} \leq \frac{k^\alpha}{\Gamma_k(\alpha k + k)}, \quad 0 < \alpha \leq 1.$$

**Theorem 3.4.** Let $f$ be the function defined by

$$f(x) = \frac{[\Gamma_{q,k}^{(2n)}(k+x)]^\alpha}{\Gamma_{q,k}^{(2n)}(k+\alpha k)}$$

then for all $\alpha > 1$ (resp. $0 < \alpha < 1$) $f$ is decreasing (resp. increasing) on $(0, \infty)$.

**Proof.** Since that $\Gamma_{q,k}$ is an infinitely differentiable function on $(0, +\infty)$, we have

$$\Gamma_{q,k}^{(n)}(x) = \int_0^{[\frac{x}{(1-q^k)}]} t^{x-1}(\log(t))^n E_{q,k}^{\frac{k^\alpha}{\alpha q}} d_q t, \quad x > 0, \ n \in \mathbb{N}.$$ 

We consider $I = (0, \frac{[k]_q}{1-q^k})$ and the subspace $C^*(I)$ obtained from $C(I)$ by requiring its members to satisfy:

(i) $\omega(x) = O(x^\theta)$ (for any $\theta > -k$) as $x \to 0$,
(ii) $\omega(x) = O(1)$ as $x \to \left(\frac{|k|}{1-q^k}\right)^{\frac{1}{k}}$.

Then we have,

$$L(\omega) = \int_{0}^{\left(\frac{|k|}{1-q^k}\right)^{\frac{1}{k}}} \omega(t)t^{k-1}(\log(t))^{2n}E_{q,k}^{-\frac{q^k}{q}} \, dt.$$ 

The operator $L$ is a positive linear functional on $C^*(I)$.

By applying the inequality (3), we obtain for $\beta > \delta > 0$

$$\frac{[\Gamma_{q,k}^{(2n)}(k+\delta)]^\alpha}{\Gamma_{q,k}^{(2n)}(k+\alpha\delta)} \leq \frac{[\Gamma_{q,k}^{(2n)}(k+\beta)]^\alpha}{\Gamma_{q,k}^{(2n)}(k+\alpha\beta)}.$$ 

Theorem 3.4 is thus proved. \qed

In particular, we have the following results

**Corollary 3.5.** For all $x \in [0,k]$, we have:

$$\frac{[\Gamma_{q,k}^{(2n)}(2k)]^\alpha}{\Gamma_{q,k}^{(2n)}(k+\alpha k)} \leq \frac{[\Gamma_{q,k}^{(2n)}(k+x)]^\alpha}{\Gamma_{q,k}^{(2n)}(k+\alpha x)} \leq [\Gamma_{q,k}^{(2n)}(k)]^{\alpha-1}, \quad \alpha \geq 1,$$

and

$$[\Gamma_{q,k}^{(2n)}(k)]^{\alpha-1} \leq \frac{[\Gamma_{q,k}^{(2n)}(k+x)]^\alpha}{\Gamma_{q,k}^{(2n)}(k+\alpha x)} \leq \frac{[\Gamma_{q,k}^{(2n)}(2k)]^\alpha}{\Gamma_{q,k}^{(2n)}(k+\alpha k)}, \quad 0 < \alpha \leq 1.$$ 

**Corollary 3.6.** For all $x \in [0,k]$, we have

$$\frac{k^\alpha}{\Gamma_{q,k}(\alpha k+k)} \leq \frac{[\Gamma_{q,k}(k+x)]^\alpha}{\Gamma_{q,k}(k+\alpha x)} \leq 1, \quad \alpha \geq 1,$$

and

$$1 \leq \frac{[\Gamma_{q,k}(k+x)]^\alpha}{\Gamma_{q,k}(k+\alpha x)} \leq \frac{k^\alpha}{\Gamma_{q,k}(\alpha k+k)}, \quad 0 < \alpha \leq 1.$$ 

**Remark 3.7.** Applying Theorem 3.1 and Theorem 3.4 for $k = 1$, we obtain the Theorem 2.1 and Theorem 3.2 in [13].

**Theorem 3.8.** For $s > 0$, let $f$ be the function defined by

$$f(x) = \frac{[B_k(k(x+1),s)]^\alpha}{B_k(k(\alpha x+1),s)}$$

then for all $\alpha > 1$ (resp. $0 < \alpha < 1$) $f$ is decreasing (resp. increasing) on $(0,\infty)$. 

Proof. The \( k \)-Beta function is defined by
\[
B_{k}(t, s) = \frac{1}{k} \int_{0}^{1} x^{\frac{k}{t} - 1} (1 - x)^{\frac{k}{s} - 1} dx.
\]
We consider the interval \( I = (0, 1) \) and the subspace \( C^*(I) \) obtained from \( C(I) \) by requiring its members to satisfy:
(i) \( \omega(x) = O(x^\theta) \) (for \( \theta > -1 \)) as \( x \to 0 \)
(ii) \( \omega(x) = O(1) \) as \( x \to 1 \).

For \( \omega \in C^*(I) \), we define
\[
L(\omega) = \int_{0}^{1} \omega(x)(1 - x)^{\frac{k}{s} - 1} dx.
\]

\( L \) is a positive linear functional on \( C^*(I) \).

Applying the inequality (3), we obtain for \( \beta > \delta > 0 \)
\[
\left[ B_{k}(k(\delta + 1), s) \right]^\alpha \leq \frac{B_{k}(k(\beta + 1), s)}{B_{k}(k(\alpha \delta + 1), s)} \leq \left[ B_{k}(k(s) \right]^\alpha_1 \quad \alpha \geq 1.
\]

Corollary 3.9. For \( x \in [0, k] \) and \( s > 0 \), we have
\[
\frac{k^2(\alpha - 1)(\alpha k^2 + s)}{\alpha(\alpha k^2 + s)^\alpha} \left[ B_{k}(k^2, s) \right]^\alpha \leq \frac{B_{k}(k(x + 1), s)}{B_{k}(k(\alpha x + 1), s)} \leq \left[ B_{k}(k, s) \right]^\alpha_1 \quad \alpha \geq 1.
\]

Theorem 3.10. For \( s > 0 \), let \( f \) be the function defined by
\[
f(x) = \frac{B_{q,k}(k + x, s)}{B_{q,k}(k + qx, s)}
\]
then for all \( \alpha > 1 \) (resp. \( 0 < \alpha < 1 \)) \( f \) is decreasing (resp. increasing) on \((0, \infty)\).

Proof. The \( q,k \)-Beta function is defined by
\[
B_{q,k}(t, s) = \int_{0}^{1} x^{\frac{k}{t} - 1} (1 - q^k x^{k})^{\frac{k}{s} - 1} d_q x.
\]
We consider the interval \( I = (0, 1) \) and the subspace \( C^*(I) \) obtained from \( C(I) \) by requiring its members to satisfy:
(i) \( \omega(x) = O(x^\theta) \) (for any \( \theta > -k \)) as \( x \to 0 \)
(ii) \( \omega(x) = O(1) \) as \( x \to 1 \)

Then we put,
\[
L(\omega) = \int_{0}^{1} \omega(x)x^{\frac{1}{k} - 1} (1 - q^k x^{k})^{\frac{1}{s} - 1} d_q x.
\]
L is defined on $C^*(I)$ and it is a positive linear functional on $C^*(I)$.

Applying the inequality (3), we obtain for $\beta > \delta > 0$

$$\frac{[B_{q,k}(k + \delta, s)]^\alpha}{B_{q,k}(k + \alpha \delta, s)} \leq \frac{[B_{q,k}(k + \beta, s)]^\alpha}{B_{q,k}(k + \alpha \beta, s)}$$

Theorem 3.10 is thus proved. \(\square\)

**Corollary 3.11.** For $x \in [0,k]$ and $s > 0$, we have

$$\frac{[k]_q^{\alpha} [\alpha k + s]_q}{[k + s]_q^{\alpha} [\alpha k]_q} \frac{B_{q,k}(k, s)}{B_{q,k}(\alpha k, s)} \leq \frac{[B_{q,k}(k + x, s)]^\alpha}{B_{q,k}(k + \alpha x, s)} \leq \frac{[B_{q,k}(k, s)]^{\alpha - 1}}{[k]_q^{\alpha}} \quad \alpha \geq 1.$$

**Theorem 3.12.** Let $f$ be the function defined by

$$f(x) = \frac{[\zeta_k(x + k + 1) \Gamma_k(x + k + 1)]^\alpha}{\zeta_k(\alpha x + k + 1) \Gamma_k(\alpha x + k + 1)}$$

then for all $\alpha > 1$ (resp. $0 < \alpha < 1$) $f$ is decreasing (resp. increasing) on $(0,\infty)$.

**Proof.** From the definition of Zeta function, we can write

$$\zeta_k(s) \Gamma_k(s) = \int_0^\infty x^{s-k-1} \frac{1}{e^x - 1} dx, \quad s > k.$$  

We consider the subspace $C^*(I)$ obtained from $C(I)$ by requiring its members to satisfy:

(i) $\omega(x) = O(x^{\theta})$ (for any $\theta > -1$) as $x \to 0$

(ii) $\omega(x) = O(x^{\phi})$ (for any finite $\phi$) as $x \to +\infty$.

Then we have,

$$L(\omega) = \int_0^\infty \omega(x) \frac{x}{e^x - 1} dx.$$  

The linear functional $L$ is well-defined on $C^*(I)$ and it is positive.

Applying the inequality (3), we obtain $\beta > \delta > 0$

$$\frac{[\zeta_k(\delta + k + 1) \Gamma_k(\delta + k + 1)]^\alpha}{\zeta_k(\alpha \delta + k + 1) \Gamma_k(\alpha \delta + k + 1)} \leq \frac{[\zeta_k(\beta + k + 1) \Gamma_k(\beta + k + 1)]^\alpha}{\zeta_k(\alpha \beta + k + 1) \Gamma_k(\alpha \beta + k + 1)}$$

Theorem 3.12 is thus proved. \(\square\)

**Remark 3.13.** Applying Theorem 3.12 for $k = 1$, we obtain the inequality for $\zeta$ function proved in [11].
REFERENCES


SABRINA TAF
Department of Mathematics
Faculty SEI, UMAB University of Mostaganem, Algeria
e-mail: sabrina481@hotmail.fr

BOCHRA NEFZI
Department of Mathematics
Faculty of Science of Tunis, Tunisia
e-mail: asimellinbochra@gmail.com

LATIFA RIAHI
Department of Mathematics
Faculty of Science of Tunis, Tunisia
e-mail: riahilatifa2013@gmail.com