LE MATEMATICHE
Vol. LXI (2006) - Fasc. I, pp. 157-162

ISOMORPHISM OF MODULAR GROUP ALGEBRAS OF FINITE TORSION-FREE RANK COPRODUCTS OF p-MIXED COUNTABLE ABELIAN GROUPS

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Suppose G is with finite torsion-free rank a coproduct of p-mixed countable abelian groups and F is a field with characteristic p such that the group algebras FG and FH are F-isomorphic for another group H. Then G and H are isomorphic.

I. Introduction and Principle Known Facts.

Traditionally, assume that FG is the group algebra of an abelian group G, written via multiplicative record as is the custom when regarding group rings, over a field F of prime characteristic; for instance p.

As a point of departure, we shall give a brief chronological citation of the best known results concerning the bounded direct products (named *coproducts* as well) of countable abelian p-groups and their non-trivial generalizations in the p-mixed situation.

It is only within the last few years that research on commutative

Entrato in redazione il 17 gennaio 2006.
2000 Mathematics Subject Classification: 20C07, 16S34, 16U60, 20K21.
Keywords and phrases: group algebras, finite torsion-free rank, p-mixed groups, coproducts of countable groups, isomorphisms.

modular group algebras of p-mixed groups has begun to be a major part of group rings theory. Notice that these group algebras are a natural extension of the group algebras of p-groups. It is for this reason, I think, that the direction of modular group algebras of p-mixed groups has always occupied a central place in the group algebras theory. There have ever been a variety of ideas and open questions available, as well as more high points, the most recent and important problem being the following:

Isomorphism Question: Given that G is a p-mixed abelian group and F is a field of characteristic p. If $FG \cong FH$ are F-isomorphic group algebras for some other group H, then does it follow that $G \cong H$?

There have been considerable periods in which there has been a very little activity; however the theory of commutative semi-simple group algebras has a rather more sporadic history than that of the previous one - the difficulty has been a paucity of usable methods and approachable problems.

Nevertheless, concentrating on the (purely) modular case, we are seeing a great deal of activity on two fronts: the results on p-mixed groups of torsion-free rank one with maximal torsion subgroup which belongs to a concrete group class and the results on p-mixed groups that are members of quite large classes of groups. In this succession, we shortly list in the sequel the more remarkable achievements.

In 1997, Hill and Ullery [9] established that any p-mixed group of torsion-free rank 1 with totally projective component of length $< \omega_1 + \omega_0$ can be determined by its group algebra of prime characteristic p, thus extending a result of Ullery [14] when the component is countable. We enlarged in ([1], Theorem 14) this assertion to lengths $< \omega_1^2$. In ([7], Main Theorem) we also have discussed the result of Ullery by dropping off the limitation on the torsion-free rank but by an additional requiring of conditions on the p-component which is equivalent to its divisibility and on the coefficient field.

In the other direction mentioned, in ([4], p. 50, Isomorphism Theorem) and ([5], p. 914, Theorem 1) we argued that the p-mixed Warfield group can be recovered by the group algebra of characteristic p, thus improving a theorem of May ([12], p. 404, Theorem 3) for the p-local variant. Recently, we refine in ([6], Theorem 11) the main statements from ([4], [5]) to a subclass of the class of so-called W-groups that subclass properly contains the class of all Warfield groups and quite general sorts of p-torsion groups. We indicate that the results in [4] and [5] are very strong and generalize the aforementioned one from [1] to arbitrary length of the totally projective p-component since every Warfield group include those rank one groups whose torsion subgroups are coproducts of totally projective groups (see, for more details, [15, pp. 10-11]).

It is the purpose of the present paper to combine the two preceding directions by examining some specific sorts of p-mixed groups of finite torsion-free rank which is, possibly, more than one.

II. Group Technicalities.

In this auxiliary section, we quote certain useful uniqueness theorems, of mixed abelian groups with finite torsion-free rank, for further applications. We emphasize that because there exist many examples of a p-mixed countable group of finite torsion-free rank strictly greater than 1 which is not a Warfield group, the attainments presented below are independent and are not deducible from the results already discussed above.

The next isomorphism criterion, which properly due to Warfield [15], is essential.

Theorem (Warfield, 1976). *If G and H are countable groups of finite torsion-free rank with the same Ulm-Kaplansky invariants such that there are torsion groups T and S with the property that $G \times T \cong H \times S$, then $G \cong H$.*

Now we have at our disposal all the machinery needed to prove the following common strengthening to the p-mixed case of the alluded to above Warfield's assertion.

Group Theorem (ISOMORPHISM). *Suppose that G and H are of finite torsion-free rank coproducts of countable p-mixed reduced abelian groups. If G and H possess equal Ulm-Kaplansky invariants and if there exist torsion groups T and S such that $G \times T \cong H \times S$, then $G \cong H$.*

Proof. Since G and H have finite torsion-free rank and since they are of the above form presented, we may write $G = \coprod_{i\in I} G_i \times A$ and $H = \coprod_{i\in I} H_i \times B$, where, for each index $i \in I$, the direct components G_i and H_i are countable p-groups and A as well as B are countable p-mixed groups with finite torsion-free rank. According to [13] (see, for example, volume II of [8, p. 85, Lemma 78.3] too), one can decompose G and H in such an appropriate way that $G = \coprod_{j\in J} G_j \times C$ and $H = \coprod_{j\in J} H_j \times D$ for $J \subseteq I$, where C and D are countable p-mixed groups of finite torsion-free rank, and such that the pairs $\coprod_{j\in J} G_j$ and $\coprod_{j\in J} H_j$ plus C and D have equal Ulm-Kaplansky invariants, respectively. Consequently, applying a result due to Kolettis ([10], p. 122, Theorem 3) (e.g. [8], volume II, p. 118, Theorem 83.3), we derive that $\coprod_{j\in J} G_j \cong \coprod_{j\in J} H_j$. Besides, under our assumption, there are torsion subgroups $T' = \coprod_{j\in J} G_j \times T$ and $S' = \coprod_{j\in J} H_j \times S$ so that $T' \times C \cong S' \times D$. We therefore utilize the foregoing statement due to Warfield to obtaining that $C \cong D$. We finally deduce that G is isomorphic to H, in fact. □

III. The Main Result and its Proof.

We have now accumulated much of the information necessary to proceed by proving the following.

Theorem (ISOMORPHISM). *Let G be a p-mixed abelian group of finite torsion-free rank and F a field of $char(F) = p \neq 0$. If G is a coproduct of countable groups, then $FH \cong FG$ as F-algebras for some group H implies $H \cong G$.*

Proof. With no harm of generality, we can presume that G and H are both reduced (cf. [3]). Besides, with the aid of [11] we know that H is also a p-mixed abelian group so that it has the same torsion-free rank and Ulm-Kaplansky p-invariants as G. On the other hand, referring to ([2], p. 257, Theorem (Isomorphism)), we infer that H is a coproduct of countable groups as well and, moreover, there exists a p-torsion group T with the property that $G \times T \cong H \times T$. We thus observe that the Group Theorem from the preliminary section works to conclude that G is isomorphic to H, indeed. □

In conclusion, we consider the following parallel to the construction from the Isomorphism Theorem of a p-mixed abelian group and its retrieving by the corresponding group algebra.

As usual, G_t denotes the maximal torsion subgroup (= the torsion part) of G.

Proposition (ISOMORPHISM). *Let G be a coproduct of countable abelian p-groups and a torsion-free group, and let F be a field of*

$char(F) = p > 0$. Then the existence of an F-isomorphism between FH and FG for any group H yields a group isomorphism between H and G.

Proof. Write down $G = \coprod_{i\in I} G_i \times E$, where, for each index $i \in I$, the members G_i of the restricted direct product are countable p-primary groups, while E is a torsion-free group. It is obvious that $G_t = \coprod_{i\in I} G_i$, whence G is a splitting p-mixed group with totally projective torsion-part. Employing now ([1], Theorem 8), we are done. □

Another approach to attack the last affirmation may be like this. First of all, it must be checked that H belongs to the same group sort as is G; note that this easily follows from the technique described in [1] and thereby we can write $H = \coprod_{j\in J} H_j \times L$, where L is torsion-free. After this, appealing to [11], we obtain that G and H have equal Ulm-Kaplansky p-invariants, hence, in virtue of ([8], volume I, p. 185, Exercise 8), the same holds true for $\coprod_{i\in I} G_i$ and $\coprod_{j\in J} H_j$. Without loss of generality, we may assume that both G and H are reduced groups ([3]), hence these two coproducts are reduced as well. Consequently, we wish apply ([10], p. 122, Theorem 3) to get that $\coprod_{i\in I} G_i \cong \coprod_{j\in J} H_j$. Henceforth, $FG \cong (F(\coprod_{i\in I} G_i))E$ and $FH \cong (F(\coprod_{i\in I} G_i))L$ assure that $(F(\coprod_{i\in I} G_i))E \cong (F(\coprod_{i\in I} G_i))L$, not necessarily isomorphic as $F(\coprod_{i\in I} G_i)$-algebras, which algebras are of necessity with characteristic p. Furthermore, as is well-known (e.g., [11], p. 142, Lemma 2), $E \cong L$. Finally, $G \cong H$, as wanted.

Acknowledgment.

The author would like to thank the competent referee for the helpful suggestions made.

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