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NILRADICALS OF SKEW HURWITZ SERIES RINGS

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For a ring endomorphism α of a ring R, Krempa called α a rigid endomorphism if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is called rigid if there exists a rigid endomorphism of R. In this paper, we extend the α-rigid property of a ring R to the upper nilradical $Nil^*(R)$ of R. For an endomorphism α and the upper nilradical $Nil^*(R)$ of a ring R, we introduce the condition $(*)$: $Nil^*(R)$ is an α-ideal of R and $a\alpha(a) \in Nil^*(R)$ implies $a \in Nil^*(R)$ for $a \in R$. We study characterizations of a ring R with an endomorphism α satisfying the condition $(*)$, and we investigate their related properties. The connections between the upper nilradical of R and the upper nilradical of the skew Hurwitz series ring (HR, α) of R are also investigated.

1. Introduction

Rings of formal power series have been of interest and have had important applications in many areas, one of which has been differential algebra. In an earlier paper by Keigher [8], a variant of the ring of formal power series was considered, and some of its properties, especially its categorical properties, were studied. In the papers [9], [10], Keigher demonstrated that the ring of Hurwitz series has many interesting applications in differential algebra. While there are

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many studies of these rings over a commutative ring, very little is known about them over a noncommutative ring. In the present paper we study Hurwitz series over a noncommutative ring with identity, examine its structure and properties.

Throughout this paper R denotes an associative ring with identity. We use $Nil_*(R)$, $Nil^*(R)$ and $Nil(R)$ to represent the lower nilradical (i.e., the prime radical), the upper nilradical (i.e., sum of all nil ideals) and the set of all nilpotent elements of R, respectively.
A ring R is called 2-*primal* [2] if the ring's prime radical coincides with the set of nilpotent elements of the ring.

Every reduced ring (i.e., $Nil(R) = 0$) is obviously a 2-primal ring. Observe that R is a 2-primal ring if and only if $Nil_*(R) = Nil^*(R) = Nil(R)$ if and only if $Nil_*(R)$ is a completely semiprime ideal (i.e., $a^2 \in Nil_*(R)$ implies $a \in Nil_*(R)$ for $a \in R$) of R. Also, $Nil^*(R) = Nil(R)$ if and only if $Nil^*(R)$ is completely semiprime.

Hence the class of rings which satisfy $Nil^*(R) = Nil(R)$ properly contains the class of 2-primal rings; while there exists a ring R with $Nil^*(R) = Nil(R)$ which is not 2-primal [[3], Example 3.3]. We refer to [[3], [5] and [6]] for more details on 2-primal rings.

For a ring endomorphism α of a ring R, Krempa [11] called α a *rigid* endomorphism if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. We called R an α-*rigid* ring [6] if the endomorphism α of R is rigid. Note that any rigid endomorphism is a monomorphism, and α-rigid rings are reduced rings. But there exists an endomorphism of a reduced ring which is not rigid [[6],Example 9]. However, if α is an inner automorphism (i.e., there exists an invertible element $u \in R$ such that $\alpha(r) = u^{-1}ru$ for any $r \in R$) of a reduced ring R, then α is rigid.

The ring $T = (H(R), \alpha)$, or simply by $T = (HR, \alpha)$, of skew Hurwitz series over a ring R is defined as follows: the elements of $T = (HR, \alpha)$ are functions $f : \mathbb{N} \to R$, where $\mathbb{N}$ is the set of all natural numbers. Let $supp(f)$ denote the support of $f \in T$, i.e. $supp(f) = \{i \in \mathbb{N} : 0 \neq f(i) \in R\}$, $\Pi(f)$ denote the minimal element in $supp(f)$ and $\Delta(f)$ denote the maximal element in $supp(f)$ if it exists, the operation of addition in T is componentwise and the operation of multiplication for each $f, g \in T$ is defined by:

$$(fg)(n) = \sum_{k=0}^{n} \binom{n}{k} f(k)\alpha^k(g(n-k)) \text{ for all } n \in \mathbb{N}.$$

where $\binom{n}{k}$ is the binomial coefficient.

Define the mappings $h_n : \mathbb{N} \to R$, $n \geq 1$ via $h_n(n-1) = 1$ and $h_n(m) = 0$ for each $(n-1) \neq m \in \mathbb{N}$ and $h'_r : \mathbb{N} \to R$ via $h'_r(0) = r$ and $h'_r(n) = 0$ for each $0 \neq n \in \mathbb{N}$ and $r \in R$. It can be easily shown that $T = (HR, \alpha)$ is a ring with identity h_1.

For a ring R, the subring R' of $T = (HR, \alpha)$ defined by $\{h'_r : r \in R\}$, we have $R \cong R'$. For any subset A of R define $A' = \{h'_r : r \in A\}$. If A is an ideal of R, then A' is an ideal of R'.

The ring (hR, α) of skew Hurwitz polynomials over a ring R is a subring of (HR, α) that consists elements of the form $f \in (HR, \alpha)$ that $\Delta(f) < \infty$.

In this paper, we investigate the relationship between the upper nilradical $Nil^*(R)$ of a ring R and the upper nilradical $Nil^*(HR, \alpha)$ of the skew Hurwitz series ring (HR, α) of R.
We shall always assume that α is an endomorphism of a given ring and it is a nonzero and non-identity endomorphism, unless especially noted.

2. α-completely semiprime ideals

In this section, we introduce α-completely semiprime ideals of a ring R, and then we investigate their equivalent conditions and related properties.
A ring R is said to be *prime* if $AB \neq 0$ for any nonzero ideals A, B of R. An ideal P of R is *prime* if R/P is a prime ring. R is said to be *strongly prime* if R is prime with no nonzero nil ideals. An ideal P of R is *strongly prime* if R/P is a strongly prime ring. An ideal P of a ring R is *minimal strongly prime* if P is minimal among strongly prime ideals of R.

We can show that there exists a minimal strongly prime ideal of a ring R using Zorn's lemma. Observe that for a ring R, $Nil^*(R) = \{a \in R : (a)$ is a nil ideal of $R\} = \bigcap\{P : P$ is a strongly prime ideal of $R\}$ [15].

An ideal P of a ring R is *completely prime* if $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$. Observe that every completely prime ideal of R is strongly prime and every strongly prime ideal is prime, but the converses do not hold, in general.

Recall that an ideal I of R is called an *α-ideal* if $\alpha(I) \subseteq I$, and I is called *α-invariant* if $\alpha^{-1}(I) = I$. Note that every α-invariant ideal is a α-ideal.

Let $mSpecs(R)$ be the set of all (minimal) strongly prime ideals of a ring R.

Proposition 2.1. *Let R be a ring.*

(1) If P is α-invariant for each $P \in mSpecs(R)$, then $Nil^*(R)$ is α-invariant;

(2) If P is an α-ideal for each $P \in mSpecs(R)$, then $Nil^*(R)$ is an α-ideal;

(3) If R satisfies $Nil^*(R) = Nil(R)$, then

(a) $Nil^*(R)$ is an α-ideal of R, and

(b) $Nil^*(R)$ is α-invariant, where α is a monomorphism.

Proof. (1) Let $a \in \alpha^{-1}(Nil^*(R))$. Then $\alpha(a) \in Nil^*(R) \subset P$ and so $a \in \alpha^{-1}(P)$ $= P$ for all $P \in mSpecs(R)$. Thus $a \in Nil^*(R)$ and so $\alpha^{-1}(Nil^*(R)) \subset Nil^*(R)$. Now, if $a \in Nil^*(R)$, then $a \in P = \alpha^{-1}(P)$ and so $\alpha(a) \in P$ for all $P \in mSpecs(R)$. Thus $\alpha(a) \in Nil^*(R)$ and so $a \in \alpha^{-1}(Nil^*(R))$. Therefore

$Nil^*(R) \subset \alpha^{-1}(Nil^*(R))$ and so $Nil^*(R)$ is α-invariant.

(2) Let $a \in Nil^*(R)$. Then $a \in P$ for all $P \in mSpecs(R)$. Thus $\alpha(a) \in \alpha(P) \subset P$. Therefore $\alpha(a) \in Nil^*(R)$ and so $Nil^*(R)$ is an α-ideal.

(3) Recall that a ring R satisfies $Nil^*(R) = Nil(R)$ if and only if $Nil^*(R)$ is a completely semiprime ideal of R.

(a) Let $a \in Nil^*(R)$. Then $a^n = 0$ for some positive integer n. Thus $(\alpha(a))^n = \alpha(a^n) = \alpha(0) = 0$, and so $\alpha(a) \in Nil(R) = Nil^*(R)$. Therefore $Nil^*(R)$ is an α-ideal.

(b) By (a), we have $Nil^*(R) \subset \alpha^{-1}(Nil^*(R))$. Thus it suffices to show $\alpha^{-1}(Nil^*(R)) \subset Nil^*(R)$. Let $a \in \alpha^{-1}(Nil^*(R))$. Then $\alpha(a) \in Nil^*(R)$ and so $(\alpha(a))^n = 0$ for some positive integer n. Thus $\alpha(a^n) = 0 = \alpha(0)$ and so $a^n = 0$, because α is a monomorphism. Hence $a \in Nil(R) = Nil^*(R)$. Thus $\alpha^{-1}(Nil^*(R)) \subset Nil^*(R)$. Consequently, $Nil^*(R)$ is α-invariant. $\square$

Corollary 2.2. *If R is a 2-primal ring, then $Nil_*(R)$ is an α-ideal of R and $Nil_*(R)$ is α-invariant when α is a monomorphism.*

In the next example, parts (1) and (2) show that the converses of Proposition 2.1 (1) and (3) (a) do not hold, respectively; while part (3) illustrates that the converse of Proposition 2.1 (2) does not hold, and that the condition "α is a monomorphism" in Proposition 2.1 (3) (b) is not superfluous.

Example 2.3. (1) Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then R is a commutative reduced ring. Define $\alpha : R \to R$ by $\alpha(a,b) = (b,a)$. Then $Nil^*(R) = \{(0,0)\}$ is α-invariant since α is an automorphism. However, $P = \{0\} \oplus \mathbb{Z}_2 \in mSpecs(R)$ is not α-invariant: For $(0,1) \in P$, $\alpha(0,1) \notin P$. Hence P is not an α-ideal and so it is not α-invariant.

(2) Let $R = Mat_2(F)$ be the 2×2 full matrix ring over a field F. Let $\alpha : R \to R$ be an endomorphism defined by $\alpha\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$. Then $Nil^*(R) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a strongly prime ideal of R and $Nil^*(R)$ is an α-ideal. But $Nil^*(R) \neq Nil(R)$.

(3) Let $R = F[x]$ be the polynomial ring over a field F. Then R is a commutative domain, and so it satisfies $Nil^*(R) = Nil(R) = \{0\}$. Let $\alpha : R \to R$ be an endomorphism defined by $\alpha(f(x)) = f(0)$. Clearly $Nil^*(R)$ is an α-ideal. However, $Nil^*(R)$ is not α-invariant. For $ax \in \alpha^{-1}(Nil^*(R))$ but $ax \notin Nil^*(R)$, where $ax \in R$ and $0 \neq a \in F$. Moreover, for $P = \langle x+1 \rangle = \{g(x)(x+1) : g(x) \in R\} \in mSpecs(R)$, we have $x+1 \in P$, but $\alpha(x+1) = 1 \notin P$. Thus P is not an α-ideal.

Now we extend the α-rigid property of a ring R to its upper nilradical $Nil^*(R)$ to study the connection of the upper nilradical $Nil^*(R)$ of a ring R and the upper nilradical $Nil^*(HR,\alpha)$ of the skew Hurwitz series ring (HR,α) of R as follows.

Let α be an endomorphism and $Nil^*(R)$ be an α-ideal of a ring R. $Nil^*(R)$ is called to be *α-completely semiprime* if $a\alpha(a) \in Nil^*(R)$ implies $a \in Nil^*(R)$ for $a \in R$.

Note that if R is an α-rigid ring, then $Nil^*(R)$ is an α-completely semiprime ideal of R but the converse does not hold by the next example.

Example 2.4. Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. Then $Nil^*(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} = Nil(R)$. Let $\alpha : R \to R$ be defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then obviously α is not a monomorphism. Thus R is not α-rigid. Now we show that $Nil^*(R)$ is an α-completely semiprime ideal of R:

Clearly $Nil^*(R)$ is an α-ideal of R. If $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) \in Nil^*(R)$, then $\begin{pmatrix} a^2 & bc \\ 0 & c^2 \end{pmatrix} \in Nil^*(R)$. Since $Nil^*(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$, we obtain $a^2 = 0$ and $c^2 = 0$. Thus $a = 0 = c$ and so $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in Nil^*(R)$.

Observe that if I is an α-ideal, then $\overline{\alpha} : R/I \to R/I$ defined by $\overline{\alpha}(a+I) = \alpha(a)+I$ for $a \in R$ is also an endomorphism of R/I.

Proposition 2.5. *If $Nil^*(R)$ is an α-completely semiprime ideal of a ring R, then $Nil^*(R) = Nil(R)$, i.e., $Nil^*(R)$ is a completely semiprime ideal of R.*

Proof. Note that $a\alpha(a) \in Nil^*(R)$ if and only if $\overline{a}\overline{\alpha}(\overline{a}) = \overline{0}$ for $\overline{a} = a + Nil^*(R) \in R/Nil^*(R)$. Then $Nil^*(R)$ is an α-completely semiprime ideal of R if and only if the factor ring $R/Nil^*(R)$ is a $\overline{\alpha}$-rigid ring, where $\overline{\alpha} : R/Nil^*(R) \to R/Nil^*(R)$ defined by $\overline{\alpha}(a + Nil^*(R)) = \alpha(a) + Nil^*(R)$. Thus $R/Nil^*(R)$ is reduced, and so $Nil^*(R) = Nil(R)$. □

Example 2.3 (3) illustrates that the converse of Proposition 2.5 does not hold. Indeed, for the ring $R = F[x]$ which satisfies $Nil^*(R) = Nil(R) = \{0\}$ with α in Example 2.3 (3). But $Nil^*(R)$ is not α-completely semiprime:
For, if $0 \neq a \in F$ and $f(x) = ax \in R$, then $f(x)\alpha(f(x)) \in Nil^*(R)$, but $f(x) \notin Nil^*(R)$.

Under certain conditions, the converse of Proposition 2.5 can be done as the next result shows:

Lemma 2.6 ([5], Theorem 5, Theorem 8 and Corollary 13)**.** *Let R be a ring. Then the following are equivalent:*

(1) $Nil^*(R) = Nil(R)$;
(2) P *is a completely prime ideal of* R *for each* $P \in mSpecs(R)$;
(3) $P = \{a \in R : ab \in Nil^*(R)$ *for some* $b \in R \backslash P\}$ *for each* $P \in mSpecs(R)$.

Proposition 2.7. *Assume that for each* $P \in mSpecs(R)$, P *is an* α*-invariant ideal of a ring* R*. Then the following are equivalent:*
(1) $Nil^*(R) = Nil(R)$;
(2) $Nil^*(R)$ *is an* α*-completely semiprime ideal of* R.

Proof. It is enough to show that $(1) \Rightarrow (2)$. Suppose that $Nil^*(R) = Nil(R)$. Clearly, $Nil^*(R)$ is an α-ideal of R by Proposition 2.1 (3). Let $a\alpha(a) \in Nil^*(R)$ then $a\alpha(a) \in P$ for all $P \in mSpecs(R)$. Since P is completely prime and α-invariant by Lemma 2.6 and hypothesis, $a \in P$ for all $P \in mSpecs(R)$ and so $a \in Nil^*(R)$. Thus $Nil^*(R)$ is an α-completely semiprime ideal of R. $\square$

There exists a ring R with $Nil^*(R) \neq Nil(R)$, even though every strongly prime ideal of R is α-invariant (Example 2.3 (2)); while, the condition "P is an α-invariant ideal of a ring R for each $P \in mSpecs(R)$" cannot be replaced by the condition "$Nil^*(R)$ is an α-invariant ideal". In fact, for the α-invariant ideal $Nil^*(R) = \{0,0\} = Nil(R)$ in Example 2.3 (1), $(0,1)\alpha(0,1) = (0,0) \in Nil^*(R)$ but $(0,1) \notin Nil^*(R)$.

However, we have the following:

Theorem 2.8. *For a ring* R*, the following are equivalent:*
(1) $Nil^*(R)$ *is an* α*-completely semiprime ideal of* R;
(2) $Nil^*(R) = Nil(R)$ *and* P *is* α*-invariant for each* $P \in mSpecs(R)$;
(3) P *is an* α*-ideal such that* $a\alpha(a) \in P$ *implies* $a \in P$ *for each* $P \in mSpecs(R)$

Proof. $(1) \Rightarrow (2)$ Observe that $Nil^*(R) = Nil(R)$ by Proposition 2.5. Let $P \in mSpecs(R)$ and $a \in \alpha^{-1}(P)$. Then $\alpha(a) \in P$. By Lemma 2.6, there exists $b \in R \backslash P$ such that $\alpha(a)b \in Nil^*(R)$ since $Nil^*(R) = Nil(R)$. Thus $b\alpha(a) \in Nil^*(R)$ and so $ab\alpha(ab) = ab\alpha(a)\alpha(b) \in Nil^*(R)$. Since $Nil^*(R)$ is an α-completely semiprime ideal of R, $ab \in Nil^*(R)$ and so $ab \in P$. Thus $a \in P$ by Lemma 2.6 and therefore $\alpha^{-1}(P) \subseteq P$. Now, we show that $P \subseteq \alpha^{-1}(P)$. Let $a \in P$, then there exists $b \in R \backslash P$ such that $ab \in Nil^*(R)$ by Lemma 2.6 and $Nil^*(R)$ is an α-ideal by Proposition 2.1 (3). Thus $\alpha(a)\alpha(b) = \alpha(ab) \in Nil^*(R)$. Then $\alpha(a)\alpha(b) \in P$ and hence $\alpha(a) \in P$, by Lemma 2.6. Therefore $a \in \alpha^{-1}(P)$ and so P is α-invariant.
$(2) \Rightarrow (3)$ Follows from Lemma 2.6.
$(3) \Rightarrow (1)$ Clearly $Nil^*(R)$ is an α-ideal of R by Proposition 2.1(2). Let $a\alpha(a) \in Nil^*(R)$ then $a\alpha(a) \in P$ for all $P \in mSpecs(R)$. By hypothesis, $a \in P$ for all $P \in mSpecs(R)$ and so $a \in Nil^*(R)$. Thus $Nil^*(R)$ is an α-completely semiprime ideal of R. $\square$

From Theorem 2.8, observe that if R is an α-rigid ring, then P is α-invariant for each $P \in mSpecs(R)$, equivalently,
P is an α-ideal of R for each $P \in mSpecs(R)$ and α is a monomorphism.

Hence, we have the following.

Corollary 2.9. *The following are equivalent:*

(1) R is an α-rigid ring;

(2) R is a reduced ring, α is a monomorphism and P is an α-ideal for each $P \in mSpecs(R)$.

Proof. It is enough to show (2) $\Rightarrow$ (1). Let $P \in mSpecs(R)$ and $a\alpha(a) = 0$ for $a \in R$. Then $a\alpha(a) \in P$. Since R is reduced, P is a completely prime ideal of R. Thus $a \in P$ or $\alpha(a) \in P$. If $a \in P$, then $\alpha(a) \in P$ because P is an α-ideal. Thus $\alpha(a) \in Nil^*(R) = \{0\}$ and so $a = 0$ because α is a monomorphism. Therefore R is an α-rigid ring. $\square$

3. The upper nilradical of the skew Hurwitz series ring

In this section, we characterize the upper nilradical $Nil^*(HR,\alpha)$ of the skew Hurwitz series ring (HR,α) of a ring R using the upper nilradical $Nil^*(R)$ of R.

Proposition 3.1. *Let $Nil^*(R)$ be an α-completely semiprime ideal of a ring R. For $a,b \in R$ we have the following.*

(1) If $ab \in Nil^*(R)$, then $a\alpha^n(b), \alpha^n(a)b \in Nil^*(R)$ for any positive integer n;

(2) If $a\alpha^k(b)$ or $\alpha^k(a)b \in Nil^*(R)$ for some positive integer k, then $ab \in Nil^*(R)$.

Proof. Note that $Nil^*(R)$ is completely semiprime, since R satisfies $Nil^*(R) = Nil(R)$ by Theorem 2.8.
(1) It is enough to show that $a\alpha(b) \in Nil^*(R)$ for $ab \in Nil^*(R)$. If $ab \in Nil^*(R)$, then $b\alpha(a)\alpha(b\alpha(a)) = b\alpha(ab)\alpha^2(a) \in Nil^*(R)$ by hypothesis. Since $Nil^*(R)$ is an α-completely semiprime ideal of R, we have $b\alpha(a) \in Nil^*(R)$. Then $\alpha(a)b \in Nil^*(R)$, because $Nil^*(R)$ is completely semiprime. Similarly, using $ba \in Nil^*(R)$, we obtain $a\alpha(b) \in Nil^*(R)$.
(2) Suppose that $a\alpha^k(b) \in Nil^*(R)$ for some positive integer k. Then, by the previous part, we obtain $\alpha^k(ab) = \alpha^k(a)\alpha^k(b) \in Nil^*(R)$. Since $Nil^*(R)$ is α-invariant by Theorem 2.8 and Proposition 2.1(2), $\alpha^{k-1}(ab) \in \alpha^{-1}(Nil^*(R)) = Nil^*(R)$ and so $\alpha^{k-2}(ab) \in \alpha^{-1}(Nil^*(R)) = Nil^*(R)$. Continuing this process, we have $ab \in Nil^*(R)$ by induction. Similarly, $\alpha^k(a)b \in Nil^*(R)$ for some positive integer k implies $ab \in Nil^*(R)$. $\square$

Note that if $Nil^*(R)$ is an α-completely semiprime ideal of a ring R, then $(H(Nil^*(R)),\alpha)$ is an ideal of the skew Hurwitz series ring (HR,α) of R by Proposition 3.1.

Theorem 3.2. *Let $Nil^*(R)$ be an α-completely semiprime ideal of a ring R and $char(R)=0$. Assume that $f,g \in (HR,\alpha)$. Then the following are equivalent:*

(1) $fg \in (H(Nil^*(R)),\alpha)$;

(2) $f(i)g(j) \in Nil^*(R)$ for each $i \geq 0$ and $j \geq 0$.

Proof. (1) $\Rightarrow$ (2) Assume that $fg \in (H(Nil^*(R)),\alpha)$, then

$$(fg)(n) = \sum_{k=0}^{n} \binom{n}{k} f(k)\alpha^k(g(n-k)) \in Nil^*(R) \text{ for all } n \in \mathbb{N}.$$

We claim that $f(i)g(j) \in Nil^*(R)$ for all i,j. We proceed by induction on $i+j$. Then we obtain $(fg)(0) = f(0)g(0) \in Nil^*(R)$ and so this proves for $i+j=0$. Now suppose that our claim is true for $i+j \leq n-1$. We have

$$(fg)(n) = \sum_{k=0}^{n} \binom{n}{k} f(k)\alpha^k(g(n-k)) \in Nil^*(R).$$

Multiplying $f(0)$ from the right hand-side, we obtain $f(0)g(n)f(0) \in Nil^*(R)$ by Proposition 3.1. Since $Nil^*(R) = Nil(R)$ by Theorem 2.8, $f(0)g(n) \in Nil^*(R)$. Now we have

$$\sum_{k=1}^{n} \binom{n}{k} f(k)\alpha^k(g(n-k)) \in Nil^*(R).$$

Multiplying $\alpha(f(1))$ from the right hand-side, we obtain $\binom{n}{1} f(1)\alpha(g(n-1)f(1)) \in Nil^*(R)$ and since $char(R)=0$, $f(1)\alpha(g(n-1)f(1)) \in Nil^*(R)$. Hence $f(1)g(n-1) \in Nil^*(R)$ by Proposition 3.1. Continuing this process, we can prove $f(i)g(j) \in Nil^*(R)$ for all i,j with $i+j=n$. Therefore $f(i)g(j) \in Nil^*(R)$ for all i and j.

(2) $\Rightarrow$ (1) It follows directly from Proposition 3.1. □

Corollary 3.3. *If $Nil^*(R)$ is an α-completely semiprime ideal of a ring R and $char(R)=0$. Then $(H(Nil^*(R)),\alpha)$ is a completely semiprime ideal of $T = (HR,\alpha)$.*

Proof. Let $0 \neq f^2 \in (H(Nil^*(R)),\alpha)$ where $f \in T$. Then $(f(n))^2 \in Nil^*(R)$ for all n by Theorem 3.2. Since $Nil^*(R) = Nil(R)$ by Theorem 2.8, we have $f(n) \in Nil^*(R)$ for all n and so $f \in (H(Nil^*(R)),\alpha)$. Thus $(H(Nil^*(R)),\alpha)$ is a completely semiprime ideal of T. □

Example 2.3 (1) shows that the condition "$Nil^*(R)$ is an α-completely semiprime ideal of R" in Theorem 3.2 and Corollary 3.3 is not superfluous: indeed,

(1) $(1,0)\alpha(1,0) = (0,0) \in Nil^*(R)$, but $(1,0) \notin Nil^*(R)$. Thus $Nil^*(R)$ is not α-completely semiprime.

(2) For $f = h'_{(1,0)} + h'_{(1,0)}h_1$ and $g = h'_{(0,1)} + h'_{(1,0)}h_1 \in (HR,\alpha)$, we have $fg = 0 \in (H(Nil^*(R)),\alpha)$ but $(1,0).(1,0) \notin Nil^*(R)$. Thus $(H(Nil^*(R)),\alpha)$ does not satisfy the conclusion of Theorem 3.2 (2).

(3) $(h'_{(1,0)}h_1)^2 = 0 \in (H(Nil^*(R)),\alpha)$ but $h'_{(1,0)}h_1 \notin (H(Nil^*(R)),\alpha)$ showing that the conclusions in Corollary 3.3 does not hold for $Nil^*(R)$.

(4) Moreover, $(h'_{(1,0)}h_1)^2 = 0 \in (H(P),\alpha)$ but $h'_{(1,0)}h_1 \notin (H(P),\alpha)$; this illustrates that not every completely semiprime ideal of a ring R can be lifted to a completely semiprime ideal of the skew Hurwitz series ring (HR,α) of R, in general. However, we have the following:

Lemma 3.4. *If $Nil^*(R)$ is an α-completely semiprime ideal of a ring R and $char(R) = 0$. Then $(H(P),\alpha)$ is a completely prime ideal of $T = (HR,\alpha)$ for each $P \in mSpecs(R)$.*

Proof. Note that P is an α-invariant ideal (as well as a completely prime ideal by Lemma 2.6) for each $P \in mSpecs(R)$ by Theorem 2.8. Let $h = fg \in (H(P),\alpha)$ with $g \notin (H(P),\alpha)$, where $f,g \in T$. Then

$$h(n) = (fg)(n) = \sum_{k=0}^{n} \binom{n}{k} f(k)\alpha^k(g(n-k)) \in P \text{ for all } n \in \mathbb{N}.$$

(1) If $g(0) \notin P$, then $f(0)g(0) \in P$ implies $f(0) \in P$ because P is completely prime. Thus $h(1) \in P$ implies $f(1) \in P$ because P is α-invariant. By the same method, $h(2) \in P$ implies $f(2) \in P$. Continuing this process, we have $f(n) \in P$ for each n and hence $f \in (H(P),\alpha)$.

(2) If $g(0), g(1), \cdots, g(n-1) \in P$ and $g(n) \notin P$, then we have

$$\sum_{k=0}^{n-1} fh'_{g(k)}h_{k+1} \in (H(P),\alpha).$$

Since $g(n) \notin P$, by the same method of the above (1), we have $f \in (H(P),\alpha)$.

(3) Continuing this process, $f \in (H(P),\alpha)$ and so $(H(P),\alpha)$ is a completely prime ideal of T. □

For the skew Hurwitz series ring $T = (HR,\alpha)$ of a ring R, if $Nil^*(T)$ is a completely semiprime ideal of T, i.e., $Nil^*(T) = Nil(T)$, then $Nil^*(R) = Nil(R)$:

For, if $a \in Nil(R)$, then $h'_a \in Nil(T) = Nil^*(T)$. Thus $\langle h'_a \rangle$ is a nil ideal of T generated by h'_a and so $\langle h'_a \rangle \bigcap R'$ is a nil ideal of R' and hence $\langle a \rangle \bigcap R$ is a nil ideal of R. Since $Nil^*(R)$ is the sum of all nil ideals of R, we have $a \in \langle a \rangle \bigcap R \subseteq Nil^*(R)$ and so $Nil(R) = Nil^*(R)$. Therefore R satisfies $Nil^*(R) = Nil(R)$. But the converse does not hold by the next example.

Example 3.5. Let F be a field and let V be an infinite dimensional left vector space over F with $\{v_1, v_2, \cdots\}$ a basis. For the endomorphism ring $A = End_F(V)$, define

$$I = \{f \in A : rank(f) < \infty, f(v_i) \in \sum_{j<i} Fv_j\}.$$

Let R be the F-subalgebra of A generated by I and the identity 1_A of A. Note that $I = Nil(R) = Nil^*(R)$, and so $Nil^*(R)$ is a 1_R-completely semiprime ideal of R where 1_R is the identity endomorphism of R. Moreover $hI \subseteq HR$ since every element in I is strongly nilpotent in HR, where hI and HR denote the Hurwitz polynomial ring and the Hurwitz series ring over I and R, respectively. Let $f, g \in HR$ are defined as follows:

$$f(n) = e_{(2n+1)(2n+2)}, g(n) = e_{(2n+2)(2n+3)} \text{ for all } n \in \mathbb{N}.$$

where e_{ij} is the infinite matrix unit over F with (i, j)-entry 1 and 0 elsewhere. Then $f, g \in H(Nil^*(R))$ and $f^2 = 0 = g^2$. However, $(f+g)^k(n) = e_{i(k+i)}$, for each k, n and so it is not nilpotent. Hence $f \notin Nil^*(HR)$, or $g \notin Nil^*(HR)$. Therefore $H(Nil^*(R)) \nsubseteq Nil^*(HR)$.

We note that Example 3.5 also shows that

$$Nil^*(HR, \alpha) \neq (H(Nil^*(R)), \alpha).$$

even if $Nil^*(R)$ is an α-completely semiprime ideal of R, in general. However, we have the following:

Theorem 3.6. *Let $Nil^*(R)$ be an α-completely semiprime ideal of a ring R and $char(R) = 0$. Then the following are equivalent:*

(1) $Nil^*(HR, \alpha)$ is a completely semiprime ideal of $T = (HR, \alpha)$;

(2) $(H(Nil^*(R)), \alpha) = Nil^*(T)$.

Proof. $(1) \Rightarrow (2)$ Suppose that $Nil^*(T)$ is a completely semiprime ideal of $T = (HR, \alpha)$. It is enough to show that $(H(Nil^*(R)), \alpha) \subseteq Nil^*(T)$ by Lemma 3.4. Let $f \in (H(Nil^*(R)), \alpha)$. Since R satisfies $Nil(R) = Nil^*(R)$ as a subring, $h'_{f(n)}$ is a nilpotent element in T, and thus $h'_{f(n)} \in Nil(T) = Nil^*(T)$ for each n. Then $h'_{f(n)}h_{n+1} \in Nil^*(T)$ and hence $f \in Nil^*(T)$. Therefore $(H(Nil^*(R)), \alpha) \subseteq Nil^*(T)$.

$(2) \Rightarrow (1)$ It follows from Corollary 3.3. $\square$

Observe that if R is an α-rigid ring and $char(R) = 0$, then (HR, α) is a reduced ring by [[4], Proposition 2.8.] and thus $Nil^*(HR, \alpha) = \{0\}$. Moreover, we have the following consequence of Theorem 3.6.

Corollary 3.7. *If R is an α-rigid ring, then $Nil^*(HR, \alpha)$ is a completely semiprime ideal of (HR, α) if and only if $Nil^*(HR, \alpha) = \{0\}$.*

The following example shows that the condition “$Nil^*(R)$ is an α-completely semiprime ideal of R” in Theorem 3.6 is not superfluous.

Example 3.8. Consider the 2×2 full matrix ring $R = Mat_2(F)$ over a field F and the automorphism α of R is defined by $\alpha\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ in Example 2.3 (2). Clearly $Nil^*(R) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is not α-completely semiprime, and $(H(Nil^*(R)), \alpha) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. We claim that $Nil^*(HR, \alpha)$ is not a completely semiprime ideal of $T = (HR, \alpha)$, even though $(H(Nil^*(R)), \alpha) = Nil^*(T)$. First, we show that $Nil^*(T) = (H(Nil^*(R)), \alpha)$. Assume on the contrary that $Nil^*(T) \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Let $0 \neq f \in Nil^*(T)$, where $\prod(f) = s$. Note that $\alpha^s(R) = R$ and $R'h'_{f(s)}R' = R'$. Then $R'fR' = R'h'_{f(s)}R'h_{s+1} + \cdots$, and so there exists $g \in Nil^*(T)$ with $g(s) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. But g is not a nilpotent; which is a contradiction. Thus $Nil^*(T) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = (H(Nil^*(R)), \alpha)$. However, $Nil(T) \neq Nil^*(T)$ since $h'_A h_2 \in Nil(T)$, where $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Thus $Nil^*(T)$ is not a completely semiprime ideal of T.

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