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RELATIVE APPROXIMATE CONTROLLABILITY OF FRACTIONAL STOCHASTIC DELAY EVOLUTION EQUATIONS WITH NONLOCAL CONDITIONS

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In this paper, we study the relative approximate controllability of nonlinear fractional stochastic evolution equations with time delays and nonlocal conditions, in Hilbert space, via new fixed point analysis approach. An example is provided to show the application of our result.

1. Introduction

The notion of controllability has played a central role throughout the history of modern control theory. Conceived by Kalman, controllability study was started systematically at the beginning of the sixties. Since then various researches have been carried out extensively in the context of finite-dimensional linear systems, nonlinear systems and infinite-dimensional systems using different kinds of approaches (e.g., [3, 4, 18, 26]).

Fractional dynamical equations have recently proved to be valuable tools in the modeling of anomalous relaxation and diffusion processes. The fact that fractional derivatives introduce a convolution integral with a power-law memory kernel makes the fractional differential equations an important one to describe

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memory effects in complex systems [10]. The increasing interest of fractional equations is motivated by their applications in various fields of science such as physics, fluid mechanics, viscoelasticity, heat conduction in materials with memory, chemistry and engineering [5, 6, 9].

Stochastic differential equations (SDEs) are used to model diverse phenomena such as fluctuating stock prices or physical systems subject to thermal fluctuations. In the literature, there are different definitions of controllability for SDEs, both for linear and nonlinear dynamical systems [4].

For linear systems, results were obtained about three types of stochastic controllability: approximate, complete, and S-controllability in Banach spaces and Hilbert spaces, respectively, in [15, 16]. With the help of backward SDEs and dual technique, Goreac [8] characterized the approximate controllability of linear SDEs. Sirbu and Tessitore [27] were concerned with the exact null controllability of infinite dimensional linear SDEs in Hilbert space. In particular, Klamka [12] generalized the results in [13] from the deterministic case to the stochastic one, and investigated the controllability of linear SDEs with delay in control.

In the setting of nonlinear SDEs, Arapostathis et al. [1] obtained sufficient conditions that guarantee weak and strong controllability. Assuming the corresponding linear SDEs are controllable, Mahmudov and Zorlu [14] studied the controllability of nonlinear SDEs. Later, Mahmudov [17] gave a characterization of weaker concept-approximate controllability for nonlinear SDEs. And recently, results in [11] were generalized by Balachandran et al. [3] about controllability on nonlinear SDEs with distributed delays in control. Complete controllability property of a nonlinear stochastic control system with jumps in a separable Hilbert space has been investigated in [23].

In the theory of dynamical systems with delays in control, it is necessary to distinguish between two fundamental concepts of controllability, namely relative controllability (relative approximate controllability) and controllability (approximate controllability), see [3, 11, 12, 25] for more details. Controllability problems for fractional deterministic and stochastic dynamical systems have drawn considerable attention recently [22, 28]. Sakthivel and Ren [19] studied the approximate controllability for a class of nonlinear fractional differential equations with state-dependent delays. Balachandran et al. [2] investigated the global relative controllability of fractional dynamical systems with multiple delays in control for finite dimensional spaces. Sakthivel et al. [20] discussed the approximate controllability for a class of dynamic control systems described by nonlinear fractional stochastic differential equations in Hilbert spaces. The approximate controllability of neutral stochastic fractional integro-differential equation with infinite delay in a Hilbert space has been studied in [21]. However, to the best of our knowledge, there are no relevant reports on the relative controllability of fractional stochastic delay dynamical systems. Inspired by the above mentioned works [2, 20, 21, 25], in this paper we are interested in studying the relative approximate controllability of nonlinear fractional stochastic evolution equations with time delays and nonlocal conditions in Hilbert space, via the fixed point theorem of Schaefer. The paper is organized as follows. Some preliminary facts are recalled in Section 2. Section 3 is devoted to sufficient condition on the relative approximate controllability of nonlinear fractional stochastic differential equations in Hilbert spaces. In section 4, examples is discussed to illustrate the effectiveness of our results.

2. Preliminaries

In this section, we provide definitions, lemmas and notations necessary to establish our main results. Throughout this paper, we use the following notations. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space equipped with a normal filtration $\mathcal{F}_t, t\in J=[0,T]$ satisfying the usual conditions (i.e., right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). We consider three real separable spaces X, E and U, and Q-Wiener process on $(\Omega,\mathcal{F}_T,\mathbb{P})$ with the linear bounded covariance operator Q such that $trQ<\infty$. We assume that there exists a complete orthonormal system $\{e_n\}_{n\geq 1}$ on E, a bounded sequence of non-negative real numbers $\{\lambda_n\}$ such that $Qe_n=\lambda_n e_n$, $n=1,2,\ldots$ and a sequence $\{\beta_n\}_{n\geq 1}$ of independent Brownian motions such that

$$\langle w(t),e\rangle=\sum_{n=1}^{\infty}\sqrt{\lambda_n}\langle e_n,e\rangle\beta_n(t),\quad e\in E, t\in[0,T],$$

and $\mathcal{F}_t=\mathcal{F}_t^w$, where $\mathcal{F}_t^w$ is the sigma algebra generated by $\{w(s):0\leq s\leq t\}$. Let $L_2^0=L_2(Q^{1/2}E;X)$ be the Banach space of all $\mathcal{F}_T$-measurable square integrable random variables with values in the Hilbert space X. Let $\mathbb{E}(.)$ denotes the expectation with respect to the measure $\mathbb{P}$.

Let $\mathcal{C}([0,T];L^2(\mathcal{F},X))$ be the Banach space of continuous maps from $[0,T]$ into $L^2(\mathcal{F},X)$ satisfying $\sup_{t\in J}\mathbb{E}\|x(t)\|^2<\infty$. Let $H_2([0,T];X)$ is a closed subspace of $\mathcal{C}([0,T];L^2(\mathcal{F},X))$ consisting of measurable and $\mathcal{F}_t$-adapted X-valued process $x\in\mathcal{C}([0,T];L^2(\mathcal{F},X))$ endowed with the norm

$$\|x\|_{H_2}=(\sup_{t\in J}\mathbb{E}\|x(t)\|_X^2)^{1/2}.$$

The purpose of this paper is to investigate the relative (approximate) controllability for a class of nonlinear fractional stochastic differential equation of the form

$$
\begin{aligned}
{}^cD_t^q x(t) &= Ax(t) + Bu(t) + f(t, x(t), x(\sigma(t))) \\
&\quad + g(t, x(t), x(\sigma(t)))\frac{dw(t)}{dt}, \quad t \in (0, T], \; x(0) + h(x) = x_0, \quad (1)
\end{aligned}
$$

where $0 < q < 1$; ${}^cD_t^q$ denotes the Caputo fractional derivative operator of order q; $x(.)$ takes its values in the Hilbert space X; A is the infinitesimal generator of a compact semigroup of uniformly bounded linear operators $\{S(t), t \geq 0\}$; the control function $u(.)$ is given in $L^2_{\mathcal{F}}([0, b], U)$ of admissible control functions, U is a Hilbert space. B is a bounded linear operator from U into X; $f : J \times X \times X \to X$ and $g : J \times X \times X \to L_2^0$ are appropriate functions; x_0 is $\mathcal{F}_0$ measurable X-valued random variables independent of w; $\sigma : J \to J$ is continuous function such that $\sigma(t) \leq t$, $\forall t \in J$ and $h : \mathcal{C}(J, X) \to X$ is a given function.

Let us recall the following known definitions. For more details see [9]

Definition 2.1. Riemann-Liouville derivative of order β with lower limit zero for a function $f : [0, \infty) \to \mathbb{R}$ can be written as

$$
{}^LD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0, n-1 < \alpha < n. \quad (2)
$$

Definition 2.2. The Caputo derivative of order α for a function $f : [0, \infty) \to \mathbb{R}$ can be written as

$$
{}^cD^\alpha f(t) = {}^LD^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^k(0) \right), \quad t > 0, n-1 < \alpha < n. \quad (3)
$$

If $f(t) \in C^n[0, \infty)$, then

$$
{}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^n(s) ds = I^{n-\alpha} f^n(s), t > 0, n-1 < \alpha < n.
$$

Definition 2.3. The fractional integral of order β with the lower limit 0 for a function f is defined as

$$
I^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(s)}{(t-s)^{1-\beta}} ds, \quad t > 0, \beta > 0
$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the gamma function.

The following results will be used throughout this paper.

Lemma 2.4 ([20]). *Let $g : [0,T] \times \Omega \to L_2^0$ be a strongly measurable mapping such that $\int_0^T \mathbb{E}\|g(t)\|_{L_2^0}^p dt < \infty$. Then*

$$\mathbb{E}\left\|\int_0^t g(s)dw(s)\right\|^p \le L_g \int_0^t \mathbb{E}\|g(s)\|_{L_2^0}^p ds$$

for all $0 \le t \le T$ and $p \ge 2$, where L_g is the constant involving p and T.

Now, we present the mild solution of the problem (1).

Definition 2.5 ([7]). A stochastic process $x \in H_2([0,b],X)$ is a mild solution of (1) if for each $u \in L^2_{\mathcal{F}}([0,b],U)$, it satisfies the following integral equation,

$$\begin{aligned} x(t) = \psi(t)(x_0 - h(x)) &+ \int_0^t (t-s)^{q-1}\varphi(t-s)[Bu(s) + f(s,x(s),x(\sigma(s)))]ds \\ &+ \int_0^t (t-s)^{q-1}\varphi(t-s)g(s,x(s),x(\sigma(s))dw(s), \end{aligned} \tag{4}$$

where $\psi(t) = \int_0^\infty \xi_q(\theta)S(t^q\theta)d\theta$; $\varphi(t) = q\int_0^\infty \theta\xi_q(\theta)S(t^q\theta)d\theta$; $S(t)$ is a C_0-semigroup generated by a linear operator A on X; ξ_q is a probability density function defined on $(0,\infty)$, that is $\xi_q(\theta) \ge 0$, $\theta \in (0,\infty)$ and $\int_0^\infty \xi_q(\theta)d\theta = 1$.

For each $0 \le t < T$, the operator $\alpha(\alpha I + \phi_0^T)^{-1} \to 0$ in the strong operator topology as $\alpha \to 0^+$, where $\phi_0^T = \int_0^T (T-s)^{2(q-1)}\varphi(T-s)BB^\star\varphi^\star(T-s)ds$ is the controllability Gramian, here B^* denotes the adjoint of B and $\varphi^\star(t)$ is the adjoint of $\varphi(t)$. Observe that linear fractional control system

$${}^cD_t^q x(t) = Ax(t) + (Bu)(t) + g(t)\frac{dw(t)}{dt}, \quad t \in [0,T], \tag{5}$$

corresponding to (1) is relatively approximately controllable on $[0,T]$ iff the operator $\alpha(\alpha I + \phi_0^b)^{-1} \to 0$ strongly as $\alpha \to 0^+$.

Definition 2.6 ([25]). The controlled system (1) is said to be relatively controllable at T if for every initial function $x_0 \in L^2(\Omega,\mathcal{F}_0,X)$, there is some control $u \in \mathcal{U}_{ad} = L^2_{\mathcal{F}}([0,T],U)$ such that $\mathcal{R}(T;x_0,u) = L_2(\Omega,\mathcal{F}_T,X)$, where the reachable set $\mathcal{R}(T;x_0,u)$ is defined as

$$\mathcal{R}(T;x_0,u) = \{x(T) = x(T;x_0,u) : u \in \mathcal{U}_{ad}, x_0 \in L^2(\Omega,\mathcal{F}_0,X)\}.$$

Definition 2.7 ([25]). The control system (1) is said to be relatively approximately controllable if for every initial function $x_0 \in L^2(\Omega,\mathcal{F}_0,X)$ there is some control $u \in \mathcal{U}_{ad}$ such that $\overline{\mathcal{R}(T;x_0,u)} = L_2(\Omega,\mathcal{F}_T,X)$.

Here $\mathcal{U}_{ad} = L^2_{\mathcal{F}}([0,T],U)$, is the closed subspace of $L^2_{\mathcal{F}}([0,T]\times\Omega,U)$, consisting of all $\mathcal{F}_t$ adapted, U-valued stochastic processes.

The following lemma is required to define the control function. The reader can refer to [20] for the proof.

Lemma 2.8. *For any $\tilde{x}_T \in L^2(\mathcal{F}_T,X)$, there exists $\tilde{g} \in L^2_{\mathcal{F}}(\Omega;L^2(0,T;L_2^0))$ such that $\tilde{x}_T = \mathbb{E}\tilde{x}_T + \int_0^T \tilde{g}(s)dw(s)$.*

Now for any $\alpha > 0$ and $\tilde{x}_T \in L^2(\mathcal{F}_T,X)$, we define the control function in the following form

$$\begin{aligned}
&u^\alpha(t,x) = B^\star(T-t)^{q-1}\varphi^\star(T-t)\\
&\times\left[(\alpha I+\phi_0^T)^{-1}[\mathbb{E}\tilde{x}_T - \psi(T)(x_0-h(x))] + \int_0^t (\alpha I+\phi_0^T)^{-1}\tilde{g}(s)dw(s)\right]\\
&-B^\star(T-t)^{q-1}\varphi^\star(T-t)\\
&\times\int_0^t (\alpha I+\phi_0^T)^{-1}(T-s)^{q-1}\varphi(T-s)f(s,x(s),x(\sigma(s)))ds\\
&-B^\star(T-t)^{q-1}\varphi^\star(T-t)\\
&\times\int_0^t (\alpha I+\phi_0^T)^{-1}(T-s)^{q-1}\varphi(T-s)g(s,x(s),x(\sigma(s)))dw(s).
\end{aligned}$$

3. Controllability results

Now, let us present the main result of this paper.
In this section, we formulate and prove conditions for relative approximate controllability of the fractional stochastic dynamical control system (1) using the fixed point theorem of Schaefer. In particular, we establish approximate controllability of nonlinear fractional stochastic control system (1) under the assumptions that the corresponding linear system is relatively approximately controllable.
We first impose the following conditions on data of the problem:
(i) For any fixed $t \geq 0$, $\psi(t)$ and $\varphi(t)$ are bounded linear operators, i.e., for any $x \in X$,

$$\|\psi(t)\| \leq M, \qquad \|\varphi(t)\| \leq \frac{Mq}{\Gamma(q+1)}.$$

(ii) For each $t \in J$ the functions $f(t,\cdot,\cdot): X\times X \to X$ and $g(t,\cdot,\cdot): X\times X \to L_2^0$ are continuous and for each $(x,y) \in X\times X$ the functions $f(\cdot,x,y): J \to X$ and $g(\cdot,x,y): J \to L_2^0$ are strongly $\mathcal{F}_t$-measurable.
(iii) For every positive integer k there exists $\beta_k \in L^2(J)$ such that for a.a. $t \in J$

$$\sup_{|x|^2,|y|^2\le k} \mathbb{E}\|f(t,x,y)\|^2 \le \beta_k,$$
$$\sup_{|x|^2,|y|^2\le k} \mathbb{E}\|g(t,x,y)\|^2_{L_2^0} \le \beta_k.$$

(iv) There exists a constant $\tilde{\beta}$ such that $\|h(x)\|^2 \le \tilde{\beta}$, for $x \in X$.
(v) There exist continuous functions $\lambda : J \to \mathbb{R}$ and $\tilde{\lambda} : J \to \mathbb{R}$ such that
$$\mathbb{E}\|f(t,x(t),x(\sigma(t)))\|^2 \le \lambda(t)\vartheta(\mathbb{E}\|x\|^2), \quad \forall t \in J, x \in X,$$
and
$$\mathbb{E}\|g(t,x(t),x(\sigma(t)))\|^2_{L_2^0} \le \tilde{\lambda}(t)\vartheta(\mathbb{E}\|x\|^2), \quad \forall t \in J, x \in X,$$
where $\vartheta : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function with

$$\int_0^T \hat{\lambda}(s)ds < \int_c^\infty \frac{ds}{s+\vartheta(s)}$$

where $c = 8M^2(\|x_0\|^2 + \tilde{\beta}) + \hat{\beta}$ and $\hat{\lambda}(t) = \max\{z, \hat{N}e^{zt}[\lambda(t) + \tilde{\lambda}(t)]\}$ for all $z \in \mathbb{R}, t \in J$ with $\hat{N}, \hat{\beta}$ are positive real constants.
(vi) $\varphi(t)$, $t > 0$ is compact.
(vii) The set $\{w_0 - h(w), w \in D_k\}$ where $D_k = \{w \in K : \|w\|^2 \le k\}$ is precompact in X, with K is a Banach space.
(viii) The linear fractional stochastic system (5) is relatively approximately controllable on $[0,T]$.

The consideration of this paper are based on the following fixed point result of Schaefer [24].

Lemma 3.1. *Let K be a Banach space and $\mathcal{P} : K \to K$ be completely continuous map. If the set*

$$I := \{x \in K : \varepsilon x = \mathcal{P}x \quad \textit{for som} \;\; \varepsilon > 1\}$$

is bounded, then $\mathcal{P}$ has a fixed point.

Lemma 3.2. *There exists a positive real constant $\hat{M}$ such that for all $x \in X$, we have*

$$\mathbb{E}\|u^\alpha(t,x)\|^2 \le \frac{\hat{M}}{\alpha^2}\left(1 + \int_0^t \beta_k(s)ds\right). \tag{6}$$

Proof. Let $x \in X$. Then

$$\mathbb{E}\|u^\alpha(t,x)\|^2$$
$$\le 3\mathbb{E}\Big\|B^\star(T-t)^{q-1}\varphi^\star(T-t)\Big[(\alpha I + \phi_0^T)^{-1}(\mathbb{E}\tilde{x}_T - \psi(T)(x_0 - h(x))) +$$

$$
\begin{aligned}
&+\int_0^t (\alpha I+\phi_0^T)^{-1}\tilde{g}dw(s)\Big]\Big\|^2 \\
&+3\mathbb{E}\left\|B^\star(T-t)^{q-1}\varphi^\star(T-t)\right. \\
&\times\left.\int_0^t (\alpha I+\phi_0^T)^{-1}(T-s)^{q-1}\varphi(T-s)f(s,x(s),x(\sigma(s)))ds\right\|^2 \\
&+3\mathbb{E}\left\|B^\star(T-t)^{q-1}\varphi^\star(T-t)\right. \\
&\times\left.\int_0^t (\alpha I+\phi_0^T)^{-1}(T-s)^{q-1}\varphi(T-s)g(s,x(s),x(\sigma(s)))dw(s)\right\|^2.
\end{aligned}
$$

From the Hölder inequality, Lemma 2.4 and the assumption on the data, we obtain

$$
\begin{aligned}
&\mathbb{E}\|u^\alpha(t,x)\|^2 \\
&\le \frac{3}{\alpha^2}\|B\|^2T^{2q-2}\left(\frac{Mq}{\Gamma(q+1)}\right)^2\left[\mathbb{E}\|\tilde{x}_T\|^2-M(\|x_0\|^2-\tilde{\beta})\right] \\
&+\frac{3}{\alpha^2}\|B\|^2T^{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^2L_gl \\
&+\frac{6}{\alpha^2}\|B\|^2T^{2q-2}\left(\frac{Mq}{\Gamma(q+1)}\right)^4\frac{T^{2q-1}}{2q-1}(1+L_g)\int_0^t\beta_k(s)ds \\
&=\frac{c_1}{\alpha^2}+\frac{c_2}{\alpha^2}\int_0^t\beta_k(s)ds \\
&\le\frac{\hat{M}}{\alpha^2}\left(1+\int_0^t\beta_k(s)ds\right),
\end{aligned}
$$

where $l=\max\{\|\tilde{g}(s)\|^2 : s\in[0,T]\}$, $\hat{M}=\max\{c_1,c_2\}$, c_1 and c_2 are two positive constants. □

Now, we state and prove the following theorem, which will be used in the proof of the main result.

Theorem 3.3. *Assume that the conditions* (**i**)-(**vii**) *are satisfied. Then the fractional stochastic system* (1) *has at least one mild solution on* $[0,T]$.

Proof. We transform the problem (1) into a fixed point problem. Consider the map $\mathcal{P}_\alpha : Y := H_2([0,T];X)\to Y$ defined by

$$
\begin{aligned}
&(\mathcal{P}_\alpha x)(t)=\psi(t)(x_0-h(x)) \\
&+\int_0^t(t-s)^{q-1}\varphi(t-s)\Big[f(s,x(s),x(\sigma(s)))+Bu^\alpha(s,x)\Big]ds \\
&+\int_0^t(t-s)^{q-1}\varphi(t-s)g(s,x(s),x(\sigma(s)))dw(s), \quad t\in J.
\end{aligned}
$$

We shall prove that the operator $\mathcal{P}_\alpha$ is a completely continuous operator.

Let $Y_k = \{x \in Y : \|x\|^2 \le k\}$ for some $k \ge 1$. We first show that $\mathcal{P}_\alpha$ maps Y_k into an equicontinuous family. Let $x \in Y_k$ and $t_1, t_2 \in J$. Then if $0 < \varepsilon < t_1 < t_2 \le T$

$$
\begin{aligned}
&\mathbb{E}\|(\mathcal{P}_\alpha)(t_1) - (\mathcal{P}_\alpha)(t_2)\|^2 \le 8\|\psi(t_1) - \psi(t_2)\|^2(\mathbb{E}\|x_0\|^2 + \mathbb{E}\|h(x)\|^2) \\
&+ 12\mathbb{E}\left\|\int_0^{t_1-\varepsilon}(t_1-\tau)^{q-1}[\varphi(t_1-\tau) - \varphi(t_2-\tau)]f(\tau, x(\tau), x(\sigma(\tau)))d\tau\right\|^2 \\
&+ 12\mathbb{E}\left\|\int_{t_1-\varepsilon}^{t_1}(t_1-\tau)^{q-1}[\varphi(t_1-\tau) - \varphi(t_2-\tau)]f(\tau, x(\tau), x(\sigma(\tau)))d\tau\right\|^2 \\
&+ 12\mathbb{E}\left\|\int_{t_1}^{t_2}(t_2-\tau)^{q-1}\varphi(t_2-\tau)f(\tau, x(\tau), x(\sigma(\tau)))d\tau\right\|^2 \\
&+ 12\mathbb{E}\left\|\int_0^{t_1-\varepsilon}(t_1-\tau)^{q-1}[\varphi(t_1-\tau) - \varphi(t_2-\tau)]Bu^\alpha(\tau, x)d\tau\right\|^2 \\
&+ 12\mathbb{E}\left\|\int_{t_1-\varepsilon}^{t_1}(t_1-\tau)^{q-1}[\varphi(t_1-\tau) - \varphi(t_2-\tau)]Bu^\alpha(\tau, x)d\tau\right\|^2 \\
&+ 12\mathbb{E}\left\|\int_{t_1}^{t_2}(t_2-\tau)^{q-1}\varphi(t_2-\tau)Bu^\alpha(\tau, x)d\tau\right\|^2 \\
&+ 12\mathbb{E}\left\|\int_0^{t_1-\varepsilon}(t_1-\tau)^{q-1}[\varphi(t_1-\tau) - \varphi(t_2-\tau)]g(\tau, x(\tau), x(\sigma(\tau)))dw(\tau)\right\|^2 \\
&+ 12\mathbb{E}\left\|\int_{t_1-\varepsilon}^{t_1}(t_1-\tau)^{q-1}[\varphi(t_1-\tau) - \varphi(t_2-\tau)]g(\tau, x(\tau), x(\sigma(\tau)))dw(\tau)\right\|^2 \\
&+ 12\mathbb{E}\left\|\int_{t_1}^{t_2}(t_2-\tau)^{q-1}\varphi(t_2-\tau)g(\tau, x(\tau), x(\sigma(\tau)))dw(\tau)\right\|^2
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\|(\mathcal{P}_\alpha)(t_1) - (\mathcal{P}_\alpha)(t_2)\|^2_{Y_k} \le 8\|\psi(t_1) - \psi(t_2)\|^2(\|x_0\|^2 + \|h(x)\|^2) \\
&+ 12T\frac{t_1^{2q-1}}{2q-1}\int_0^{t_1-\varepsilon}\|\varphi(t_1-\tau) - \varphi(t_2-\tau)\|^2\beta_k(\tau)d\tau \\
&+ 12T\frac{t_1^{2q-1}}{2q-1}\int_{t_1-\varepsilon}^{t_1}\|\varphi(t_1-\tau) - \varphi(t_2-\tau)\|^2\beta_k(\tau)d\tau \\
&+ 12T\frac{(t_2-t_1)^{2q-1}}{1-2q}\int_{t_1}^{t_2}\|\varphi(t_2-\tau)\|^2\beta_k(\tau)d\tau \\
&\le 12\frac{t_1^{2q-1}}{2q-1}L_g\int_0^{t_1-\varepsilon}\|\varphi(t_1-\tau) - \varphi(t_2-\tau)\|^2\beta_k(\tau)d\tau +
\end{aligned}
$$

$$+12\frac{t_1^{2q-1}}{2q-1}L_g\int_{t_1-\varepsilon}^{t_1}\|\varphi(t_1-\tau)-\varphi(t_2-\tau)\|^2\beta_k(\tau)d\tau$$
$$+12\frac{(t_2-t_1)^{2q-1}}{1-2q}L_g\int_{t_1}^{t_2}\|\varphi(t_2-\tau)\|^2\beta_k(\tau)d\tau$$
$$+\frac{12}{\alpha^2}\tilde{M}\|B\|^2\frac{t_1^{2q-1}}{2q-1}\int_0^{t_1-\varepsilon}\|\varphi(t_1-\tau)-\varphi(t_2-\tau)\|^2(1+\tau\beta_k(\tau))d\tau$$
$$+\frac{12}{\alpha^2}\tilde{M}\|B\|^2\frac{t_1^{2q-1}}{2q-1}\int_{t_1-\varepsilon}^{t_1}\|\varphi(t_1-\tau)-\varphi(t_2-\tau)\|^2(1+\tau\beta_k(\tau))d\tau$$
$$+\frac{12}{\alpha^2}\tilde{M}\|B\|^2\frac{(t_2-t_1)^{2q-1}}{1-2q}\int_{t_1}^{t_2}\|\varphi(t_2-\tau)\|^2(1+\tau\beta_k(\tau))d\tau.$$

The right hand side is independent of $x\in Y_k$ and tend to zero as $t_2-t_1\to 0$ and ε sufficiently small, sine the compactness of $\varphi(t)$, $t>0$, implies the continuity in the uniform operator topology. Thus $\mathcal{P}_\alpha$ maps Y_k into an equicontinuous family of functions. It is easy to see that the family Y_k uniformly bounded.

Next, we show that $\overline{\mathcal{P}_\alpha Y_k}$ is compact. Since we have shown that $\mathcal{P}_\alpha Y_k$ is an equicontinuous collection, it suffices by Arzela-Ascoli theorem to show that $\mathcal{P}_\alpha$ maps Y_k into a precompact set in X.

Let $0<t\le T$ be fixed and ε a real number satisfying $0<\varepsilon<t$. For $x\in Y_k$ we define

$$(\mathcal{P}_{\alpha,\varepsilon}x)(t)=\psi(t)(x_0-h(x))$$
$$+\int_0^{t-\varepsilon}(t-\tau)^{q-1}\varphi(t-\tau)\Big[f(\tau,x(\tau),x(\sigma(\tau)))+Bu^\alpha(\tau,x)\Big]d\tau+$$
$$+\int_0^{t-\varepsilon}(t-\tau)^{q-1}\varphi(t-\tau)g(\tau,x(\tau),x(\sigma(\tau)))dw(\tau)$$
$$=\psi(t)(x_0-h(x))$$
$$+\varphi(\varepsilon)\int_0^{t-\varepsilon}(t-\tau)^{q-1}\varphi(t-\tau-\varepsilon)\Big[f(\tau,x(\tau),x(\sigma(\tau)))+Bu^\alpha(\tau,x)\Big]d\tau$$
$$+\varphi(\varepsilon)\int_0^{t-\varepsilon}(t-\tau)^{q-1}\varphi(t-\tau-\varepsilon)g(\tau,x(\tau),x(\sigma(\tau)))dw(\tau).$$

Since $\varphi(t)$ is a compact operator, the set $Y_\varepsilon(t)=\{(\mathcal{P}_{\alpha,\varepsilon}x)(t):x\in Y_k\}$ is pre-compact in X, for every ε, $0<\varepsilon<t$. Moreover, for every $x\in Y_k$ we have

$$\mathbb{E}\|(\mathcal{P}_\alpha x)(t)-(\mathcal{P}_{\alpha,\varepsilon}x)(t)\|^2$$
$$\le 3\mathbb{E}\left\|\int_{t-\varepsilon}^{t}(t-\tau)^{q-1}\varphi(t-\tau)\Big[f(\tau,x(\tau),x(\sigma(\tau)))+Bu^\alpha(\tau,x)\Big]d\tau\right\|^2$$
$$+3\mathbb{E}\left\|\int_{t-\varepsilon}^{t}(t-\tau)^{q-1}\varphi(t-\tau)g(\tau,x(\tau),x(\sigma(\tau)))dw(\tau)\right\|^2.$$

Thus

$$\begin{aligned}
&\|(\mathcal{P}_\alpha x)(t)-(\mathcal{P}_{\alpha,\varepsilon}x)(t)\|^2_{Y_k}\\
&\le 3\frac{\varepsilon^{2q-1}}{1-2q}\left(\frac{Mq}{\Gamma(q+1)}\right)^2(T+L_g)\int_{t-\varepsilon}^{t}\beta_k(\tau)d\tau\\
&+\frac{3}{\alpha^2}\tilde{M}\|B\|^2\frac{\varepsilon^{2q-1}}{1-2q}\left(\frac{Mq}{\Gamma(q+1)}\right)^2\left(\varepsilon+\int_{t-\varepsilon}^{t}\beta_k(\tau)d\tau\right)
\end{aligned}$$

Therefore there are precompact sets arbitrary close to the set $\{(\mathcal{P}_\alpha x)(t) : x\in Y_k\}$. Hence the set $\{(\mathcal{P}_\alpha x)(t) : x\in Y_k\}$, according to assumption (**vii**) is precompact in X.

It remain to show that $\mathcal{P}_\alpha : Y\to Y$ is continuous. Let $\{v_n\}_0^\infty\subseteq Y$ with $v_n\to v$ in Y. Then there is an integer p such that $\|v_n(t)\|^2\le p$ for all n and $t\in J$, so $v_n\in Y_p$ and $v\in Y_p$. By (ii) we have for each $t\in J$

$$f(t,v_n(t),v_n(\sigma(t)))\to f(t,v(t),v(\sigma(t)))$$

and

$$g(t,v_n(t),v_n(\sigma(t)))\to g(t,v(t),v(\sigma(t))).$$

Since

$$\|f(t,v_n(t),v_n(\sigma(t)))-f(t,v(t),v(\sigma(t)))\|^2\le 2f_p(t)$$

and

$$\|g(t,v_n(t),v_n(\sigma(t)))-g(t,v(t),v(\sigma(t)))\|^2_{L_2^0}\le 2g_p(t),$$

we have by dominated convergence

$$\begin{aligned}
&\|(\mathcal{P}_\alpha v_n)(t)-(\mathcal{P}_\alpha v)(t)\|^2\le 3\frac{T^{2q}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^2\\
&\times\int_0^T\|f(\tau,v_n(\tau),v_n(\sigma(\tau)))-f(\tau,v(\tau),v(\sigma(\tau)))\|^2d\tau\\
&+3\frac{T^{2q-1}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^2L_g\\
&\times\int_0^T\|g(\tau,v_n(\tau),v_n(\sigma(\tau)))-g(\tau,v(\tau),v(\sigma(\tau)))\|^2d\tau\\
&+\frac{6}{\alpha^2}\|B\|^4T^{2q-2}\left(\frac{Mq}{\Gamma(q+1)}\right)^4\frac{T^{2q}}{2q-1}\times
\end{aligned}$$

$$\begin{aligned}
&\times \int_0^T \|f(\tau, v_n(\tau), v_n(\sigma(\tau))) - f(\tau, v(\tau), v(\sigma(\tau)))\|^2 d\tau \\
&+ \frac{6}{\alpha^2}\|B\|^4 T^{2q-2}\left(\frac{Mq}{\Gamma(q+1)}\right)^4 \frac{T^{2q}}{2q-1} L_g \\
&\times \int_0^T \|g(\tau, v_n(\tau), v_n(\sigma(\tau))) - g(\tau, v(\tau), v(\sigma(\tau)))\|^2 d\tau \to 0.
\end{aligned}$$

Thus $\mathcal{P}_\alpha$ is continuous. This completes the proof that $\mathcal{P}_\alpha$ is completely continuous.

Now, we shall prove that the set

$$I := \{x \in Y : \theta x = \mathcal{P}_\alpha x \quad \text{for some} \;\; \theta > 1\}$$

is bounded.

Let $x \in I$. Then $\theta x = \mathcal{P}_\alpha x$ for some $\theta > 1$. Then

$$\begin{aligned}
x(t) &= \theta^{-1}\psi(t)(x_0 - h(x)) \\
&\quad + \theta^{-1}\int_0^t (t-s)^{q-1}\varphi(t-s)\Big[f(s, x(s), x(\sigma(s))) + Bu^\alpha(s,x)\Big] ds \\
&\quad + \theta^{-1}\int_0^t (t-s)^{q-1}\varphi(t-s) g(s, x(s), x(\sigma(s))) dw(s), \quad t \in J.
\end{aligned} \tag{7}$$

We have

$$\begin{aligned}
\mathbb{E}\|x(t)\|^2 &\le 8\|\psi(t)\|^2[\mathbb{E}\|x_0\|^2 + \mathbb{E}\|h(x)\|^2] \\
&\quad + 4\mathbb{E}\left\|\int_0^t (t-\tau)^{q-1}\varphi(t-\tau) f(\tau, x(\tau), x(\sigma(\tau))) d\tau\right\|^2 \\
&\quad + 4\mathbb{E}\left\|\int_0^t (t-\tau)^{q-1}\varphi(t-\tau) Bu^\alpha(\tau, x) d\tau\right\|^2 \\
&\quad + 4\mathbb{E}\left\|\int_0^t (t-\tau)^{q-1}\varphi(t-\tau) g(\tau, x(\tau), x(\sigma(\tau))) dw(\tau)\right\|^2 \\
&\le 8M^2[\mathbb{E}\|x_0\|^2 + \mathbb{E}\|h(x)\|^2] \\
&\quad + 4\frac{T^{2q}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^2 \int_0^t \mathbb{E}\|f(\tau, x(\tau), x(\sigma(\tau)))\|^2 d\tau \\
&\quad + 4\frac{T^{2q}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^2 L_g \int_0^t \mathbb{E}\|g(\tau, x(\tau), x(\sigma(\tau)))\|_{L_2^0}^2 d\tau \\
&\quad + 4\|B\|^2\frac{T^{2q}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^2 \int_0^t \mathbb{E}\|u^\alpha(\tau, x)\|^2 d\tau.
\end{aligned}$$

Using the same procedure as the proof of Lemma 3.2, we get

$$\begin{aligned}
\mathbb{E}\|x(t)\|^2 &\le 8M^2[\mathbb{E}\|x_0\|^2+\mathbb{E}\|h(x)\|^2]\\
&+4\frac{T^{2q}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^2\int_0^t \mathbb{E}\|f(\tau,x(\tau),x(\sigma(\tau)))\|^2 d\tau\\
&+4\frac{T^{2q}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^2 L_g\int_0^t \mathbb{E}\|g(\tau,x(\tau),x(\sigma(\tau)))\|^2_{L_2^0} d\tau\\
&+4\frac{c_1}{\alpha^2}\|B\|^2\frac{T^{2q}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^2\\
&+\frac{12}{\alpha^2}T^{2q-1}\|B\|^4\frac{T^{2q-1}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^8\int_0^t \mathbb{E}\|f(\tau,x(\tau),x(\sigma(\tau)))\|^2 d\tau\\
&+\frac{12}{\alpha^2}T^{2q-1}L_g\|B\|^4\frac{T^{2q-1}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^8\int_0^t \mathbb{E}\|g(\tau,x(\tau),x(\sigma(\tau)))\|^2_{L_2^0} d\tau
\end{aligned}$$

or

$$\begin{aligned}
\mathbb{E}\|x(t)\|^2 &\le 8M^2[\mathbb{E}\|x_0\|^2+\mathbb{E}\|h(x)\|^2]\\
&+4\frac{T^{2q}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^2\int_0^t \lambda(\tau)\vartheta(\mathbb{E}\|x(\tau)\|^2)d\tau\\
&+4L_g\frac{T^{2q}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^2\int_0^t \tilde{\lambda}(\tau)\vartheta(\mathbb{E}\|x(\tau)\|^2)d\tau\\
&+4\frac{c_1}{\alpha^2}\|B\|^2\frac{T^{2q}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^2\\
&+\frac{12}{\alpha^2}T^{2q-1}\|B\|^4\frac{T^{2q-1}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^8\int_0^t \lambda(\tau)\vartheta(\mathbb{E}\|x(\tau)\|^2)d\tau\\
&+\frac{12}{\alpha^2}T^{2q-1}L_g\|B\|^4\frac{T^{2q-1}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^8\int_0^t \tilde{\lambda}(\tau)\vartheta(\mathbb{E}\|x(\tau)\|^2)d\tau.
\end{aligned}$$

That is

$$\begin{aligned}
\|x(t)\|^2_{Y_k} &\le 8M^2\left[\|x_0\|^2+\tilde{\beta}\right]\\
&+4\frac{T^{2q}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^2\int_0^t \lambda(\tau)\vartheta(\|x(\tau)\|^2)d\tau\\
&+4L_g\frac{T^{2q}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^2\int_0^t \tilde{\lambda}(\tau)\vartheta(\|x(\tau)\|^2)d\tau
\end{aligned}$$

$$+4\frac{c_1}{\alpha^2}\|B\|^2\frac{T^{2q}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^2$$
$$+\frac{12}{\alpha^2}T^{2q-1}\|B\|^4\frac{T^{2q-1}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^8\int_0^t\lambda(\tau)\vartheta(\mathbb{E}\|x(\tau)\|^2)d\tau$$
$$+\frac{12}{\alpha^2}T^{2q-1}L_g\|B\|^4\frac{T^{2q-1}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^8\int_0^t\tilde{\lambda}(\tau)\vartheta(\|x(\tau)\|^2)d\tau.$$

Denoting by $\nu(t)$ the right-hand side of the above inequality we have $\nu(0)=8M^2(\|x_0\|^2+\tilde{\beta})+\hat{\beta}=c$, and $\|x(t)\|^2\leq\nu(t)$, $t\in J$ with $\hat{\beta}=4\frac{c_1}{\alpha^2}\|B\|^2\frac{T^{2q}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^2$, and

$$\begin{aligned}
\nu'(t)&=4\frac{T^{2q}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^2\lambda(t)\vartheta(\|x(t)\|^2)\\
&+4L_g\frac{T^{2q}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^2\tilde{\lambda}(t)\vartheta(\|x(t)\|^2)\\
&+\frac{12}{\alpha^2}T^{2q-1}\|B\|^4\frac{T^{2q-1}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^8\lambda(t)\vartheta(\|x(t)\|^2)\\
&+\frac{12}{\alpha^2}T^{2q-1}L_g\|B\|^4\frac{T^{2q-1}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^8\tilde{\lambda}(t)\vartheta(\|x(t)\|^2)\\
&\leq 4\frac{T^{2q}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^2\lambda(t)\vartheta(\nu(t))\\
&+4L_g\frac{T^{2q}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^2\tilde{\lambda}(t)\vartheta(\nu(t))\\
&+\frac{12}{\alpha^2}T^{2q-1}\|B\|^4\frac{T^{2q-1}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^8\lambda(t)\vartheta(\nu(t))\\
&+\frac{12}{\alpha^2}T^{2q-1}L_g\|B\|^4\frac{T^{2q-1}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^8\tilde{\lambda}(t)\vartheta(\nu(t)).
\end{aligned}$$

We remark that for all $z\in\mathbb{R}$

$$\begin{aligned}
&\left[e^{zt}\nu(t)\right]'=e^{zt}\nu'(t)+ze^{zt}\nu(t)\\
&\leq e^{zt}\left[N\lambda(t)\vartheta(\nu(t))+\tilde{N}\tilde{\lambda}\vartheta(\nu(t))\right]+ze^{zt}\nu(t)\\
&\leq \hat{N}e^{zt}[\lambda(t)+\tilde{\lambda}(t)]\vartheta(\nu(t))+ze^{zt}\nu(t)\\
&\leq \hat{\lambda}(t)[\vartheta(\nu(t))+\nu(t)];\quad z\in\mathbb{R},t\in J,
\end{aligned}$$

where $\hat{N} = \max\{N, \tilde{N}\}, \hat{\lambda}(t) = \max\{ze^{zt}, \hat{N}e^{zt}[\lambda(t) + \tilde{\lambda}(t)]\}$
and N, $\tilde{N}$ are given by

$$N = \left[4\frac{T^{2q}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^2 + \frac{12}{\alpha^2}T^{2q-1}\|B\|^4\frac{T^{2q-1}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^8\right]$$

$$\tilde{N} = \left[4L_g\frac{T^{2q}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^2 + \frac{12}{\alpha^2}T^{2q-1}L_g\|B\|^4\frac{T^{2q-1}}{2q-1}\left(\frac{Mq}{\Gamma(q+1)}\right)^8\right].$$

This implies

$$\int_{\nu(0)}^{\nu(t)} \frac{ds}{s+\vartheta(s)} \le \int_0^T \hat{\lambda}(s)ds < \int_c^\infty \frac{ds}{s+\vartheta(s)}, \quad t \in J.$$

This inequality implies that there is a constant ρ such that $\nu(t) \le \rho$, $t \in J$ and hence $\|x\|^2 \le \rho$ where ρ depends only on T and the functions $\hat{\lambda}, \vartheta$.

As a consequence of Lemma 3.1, we deduce that $\mathcal{P}_\alpha$ has a fixed point, which is a mild solution of (1). $\square$

Theorem 3.4. *Assume that the assumptions (i)-(viii) hold. Then the system* (1) *is relatively approximately controllable on* $[0,T]$.

Proof. Let x_α be a fixed point of $\mathcal{P}_\alpha$. By using the stochastic Fubini theorem, it can be easily seen that

$$\begin{aligned}
x_\alpha(T) &= \tilde{x}_T - \alpha(\alpha I + \phi_0^T)^{-1}[\mathbb{E}\tilde{x}_T - \psi(T)(x_0 - h(x))] \\
&+ \alpha\int_0^T (\alpha I + \phi_s^T)^{-1}(T-s)^{q-1}\varphi(T-s)f(s, x_\alpha(s), x_\alpha(\sigma(s)))ds \\
&+ \alpha\int_0^T (\alpha I + \phi_s^T)^{-1}[(T-s)^{q-1}\varphi(T-s)g(s, x_\alpha(s), x_\alpha(\sigma(s))) - \tilde{g}(s)]dw(s).
\end{aligned}$$

By the assumption (ii) one can see that for each $t \in J$

$$f(t, x_\alpha(t), x_\alpha(\sigma(t))) \to f(t, x(t), x(\sigma(t))),$$

and

$$g(t, x_\alpha(t), x_\alpha(\sigma(t))) \to g(t, x(t), x(\sigma(t))).$$

From the above equation, we have

$$\begin{aligned}
\mathbb{E}\|x_\alpha(T) - \tilde{x}_T\|^2 &\le 6\|\alpha(\alpha I + \phi_0^T)^{-1}[\mathbb{E}\tilde{x}_T - \psi(T)(x_0 - h(x))]\|^2 \\
&+ 6\mathbb{E}\left(\int_0^T (T-s)^{q-1}\|\alpha(\alpha I + \phi_s^T)^{-1}\|\right. \\
&\times \|\varphi(T-s)(f(s, x_\alpha(s), x_\alpha(\sigma(s))) - f(t, x(t), x(\sigma(t))))\|ds\Big)^2
\end{aligned}$$

$$+6\mathbb{E}\left(\int_0^T (T-s)^{q-1}\|\alpha(\alpha I+\phi_s^T)^{-1}\varphi(T-s)f(t,x(t),x(\sigma(t))))\|ds\right)^2$$
$$+6\mathbb{E}\left(\int_0^T (T-s)^{q-1}\|\alpha(\alpha I+\phi_s^T)^{-1}\|\right.$$
$$\left.\times\|\varphi(T-s)(g(s,x_\alpha(s),x_\alpha(\sigma(s)))-g(t,x(t),x(\sigma(t))))\|_{L_2^0}^2 ds\right)$$
$$+6\mathbb{E}\left(\int_0^T (T-s)^{q-1}\|\alpha(\alpha I+\phi_s^T)^{-1}\varphi(T-s)g(t,x(t),x(\sigma(t))))\|_{L_2^0}^2 ds\right).$$

On the other hand, by assumption (viii), for all $0\le s<T$ the operator $\alpha(\alpha I+\phi_s^T)^{-1}\to 0$ strongly as $\alpha\to 0^+$ and moreover $\|\alpha(\alpha I+\phi_s^T)^{-1}\|\le 1$. by the Lebesgue dominated convergence theorem and asuumption (**vi**) implies that $\mathbb{E}\|x_\alpha(T)-\tilde{x}_T\|^2\to 0$ as $\alpha 0^+$. This gives the relative approximate controllability of (1). □

4. Example

Consider the following fractional stochastic control system of the form

$$\begin{aligned}
{}^cD_t^q x(t,z) &= \frac{\partial^2}{\partial z^2}x(t,z)+\mu(t,z)+\tilde{\nu}x(t-\tau,z)+\hat{\nu}x(t-\tau,z)\frac{d\hat{w}(t)}{dt},\\
x(t,0) &= x(t,\pi)=0, \quad t>0 \\
x(0,z) &= x_0-h(z), \quad z\in[0,\pi],
\end{aligned}\tag{8}$$

where $0<q<1$; $\hat{w}(t)$ is a two sided and standard one dimensional Brownian motion defined on the filtered probability space $(\Omega,\mathcal{F},\mathbb{P})$. To write the above system into the abstract form of (1), let $X=E=U=L^2[0,\pi]$. Define the operator $A:L^2[0,\pi]\to L^2[0,\pi]$ by $A\omega=\omega''$ with domain

$$\begin{aligned}
D(A)=\{\omega\in X;\omega,\omega' &\text{ are absolutely continuous, } \ \omega''\in X\\
&\text{and } \ \omega(0)=\omega(\pi)=0\}.\\
A\omega=\sum_{n=1}^\infty n^2(\omega,\omega_n)\omega_n,& \quad \omega\in D(A),
\end{aligned}$$

where $\omega_n(s)=\sqrt{2}\sin(ns)$, $n=1,2,\ldots$ is the orthogonal set of eigenvectors in A. It is well known that A generates a compact, analytic semigroup $\{S(t),t\ge 0\}$ in X and

$$S(t)\omega=\sum_{n=1}^\infty e^{-n^2(t)}(\omega,\omega_n)\omega_n, \qquad \|S(t)\|\le e^{-t} \quad \text{for all } t\ge 0.$$

Especially, the operator $A^{1/2}$ is given by $A^{1/2}\omega = \sum_{n=1}^{\infty} n(\omega,\omega_n)\omega_n$, with domain

$$D(A^{1/2}) = \{\omega \in X : \sum_{n=1}^{\infty} n(\omega,\omega_n)\omega_n\}.$$

Define the bounded linear operator $B : U \to X$ by $Bu = \mu$, $0 \le z \le \pi$ and setting $f(x) = \nu x$, $g(x) = \tilde{\nu}x$ and $\sigma(t) = t - \tau$. Then, one can see that the problem (8) can be reformulated as follows

$$\begin{aligned} {}^{c}D_t^q x(t) &= Ax(t) + Bu(t) + f(x(\sigma(t))) + g(x(\sigma(t)))\frac{dw(t)}{dt}, \\ x(0) &= x_0 - h(x). \end{aligned}$$

On the other hand, it can be easily seen that the deterministic linear fractional control system corresponding to (8) is approximately controllable on $[0,\pi]$ (see, [16]). Therefore, with the above choices, the system (8) can be written to the abstract form (1) and all the conditions of Theorem 3.4 are satisfied. Thus by Theorem 3.4 fractional stochastic control system (8) is approximately controllable on $[0,\pi]$.

5. Conclusions

This paper has investigated the relative approximate controllability of nonlinear fractional stochastic evolution equations with time delays and nonlocal conditions in Hilbert space by using the assumption that the corresponding linear system is relatively approximately controllable. With the use of the fractional calculus and stochastic analysis technique, control function has been constructed. Moreover, the control function, together with operator semigroup, has helped us to obtain sufficient conditions for the relative approximate controllability of the control system via the fixed point theorem of Schaefer. An application is provided to illustrate the applicability of the new result. Our future work will try to make some the above results and study the relative approximate controllability for impulsive fractional neutral stochastic functional integro-differential inclusions with state-dependent delay.

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REFERENCES

[1] A. Arapostathis - R. K. George - M. K. Ghosh, *On the controllability of a class of nonlinear stochastic systems*, Systems & Control Lett. 44 (2001), 25–34.

[2] K. Balachandran - J. Kokila - J. J. Trujillo, *Relative controllability of fractional dynamical systems with multiple delays in control*, Computers & Mathematics with Applications 64 (10) (2012), 3037–3045.

[3] K. Balachandran - S. Karthikeyan - J. Y. Park, *Controllability of stochastic systems with distributed delays in control*, Internat. J. Control 82 (2009), 1288–1296.

[4] A. E. Bashirov - N. I. Mahmudov, *On concepts of controllability for deterministic and stochastic systems*, SIAM J. Control Optim. 37 (1999), 1808–1821.

[5] M. Belmekki - J. J. Nieto - R. Rodríguèz-López, *Existence of periodic solution for a nonlinear fractional differential equation*, Boudary Value Problems vol. 2009, Article ID 324561, 18 pages, 2009.

[6] M. Belmekki - K. Mekhelfi - S. K. Ntouyas, *Semilinear functional differential equations with fractional order and finite delay*, Malaya Journal of Mathematik 1 (1) (2012), 73–81.

[7] M. M. El-Borai, *Some probability densities and fundamental solutions of fractional evolution equations*, Chaos, Solitons and Fractals 14 (2002), 433–440.

[8] D. Goreac, *Approximate controllability for linear stochastic differential equations in infinite dimensions*, Appl. Math. Optim. 60 (2009), 105–132.

[9] A. A. Kilbas - H. M. Srivastava - J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[10] J. Klafter - S. C. Lim - R. Metzler, *Fractional Dynamics, Recent Advances*, World Scientific, Singapore, 2011.

[11] J. Klamka, *Relative controllability of nonlinear systems with distributed delay in control*, International Journal of Control 28 (1978), 307–312.

[12] J. Klamka, *Stochastic controllability of linear systems with delay in control*, Bulletin of the Polish Academy of Sciences: Technical Sciences 55 (2007), 23–29.

[13] J. Klamka, *On the controllability of linear systems with delays in control*, Internat. J. Control 25 (1977), 875–883.

[14] N. I. Mahmudov - S. Zorlu, *Controllability of non-linear stochastic systems*, Internat. J. Control 76 (2003), 95–104.

[15] N. I. Mahmudov, *Controllability of linear stochastic systems*, IEEE Trans. Autom. Contr. 46 (2001), 724–731.

[16] N. I. Mahmudov, *Controllability of linear stochastic systems in Hilbert spaces*, J. Math. Anal. Appl. 259 (2001), 64–82.

[17] N. I. Mahmudov, *Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces*, SIAM J. Control Optim. 42 (2003), 1604–1622.

[18] Y. Ren - H. Dai - R. Sakthivel, *Approximate controallability of stochastic differential system driven by a Levy process*, International Journal of Control 86 (2013),

1158–1164.

[19] R. Sakthivel - Y. Ren, *Approximate controllability of fractional differential equations with state-dependent delay*, Results. Math. 63 (1-2) (2013), 949–963.

[20] R. Sakthivel - S. Suganya - S. M. Anthoni, *Approximate controllability of fractional stochastic evolution equations*, Computers & Mathematics with Applications 63 (2012), 660–668.

[21] R. Sakthivel - R. Ganesh - S. Suganya, *Approximate controllability of fractional neutral stochastic system with infinite delay*, R. Math. Physics. 70 (2012), 291–311.

[22] R. Sakthivel - Y. Ren - N. I. Mahmudov, *On the approximate controllability of semilinear fractional differential systems*, Computers and Mathematics with Applications 62 (2011), 1451–1459.

[23] R. Sakthivel, *Complete controllability of stochastic evolution equations with jumps*, Reports on Mathematical Physics 68 (2011), 163–174.

[24] H. Schaefer, *Über die method der a priori Schranken*, Math. Ann. 129 (1955), 415–416.

[25] L. Shen - J. Sun, *Relative controllability of stochastic nonlinear systems with delay in control*, Nonlinear Analysis: RWA 13 (2012), 2880–2887.

[26] L. J. Shen - J. P. Shi - J. T. Sun, *Complete controllability of impulsive stochastic integro-differential systems*, Automatica 46 (2010), 1068–1073.

[27] M. Sirbu - G. Tessitore, *Null controllability of an infinite dimensional SDE with state- and control-dependent noise*, Systems & Control Lett. 44 (2001), 385–394.

[28] Y. Zang - J. Li, *Approximate controllability of fractional impulsive neutral stochastic differential equations with nonlocal conditions*, Boundary Val. Prob. (2013), 2013: 193.

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