LE MATEMATICHE
Vol. LXIX (2014) – Fasc. I, pp. 149–158
doi: 10.4418/2014.69.1.10

UNIFICATION OF ALMOST STRONGLY μ_θ-CONTINUOUS FUNCTIONS

BISHWAMBHAR ROY

We introduce a new type of functions called almost strongly μ_θ-continuous functions which unifies some weak forms of almost strongly θ-continuous functions and investigate their properties.

1. Introduction

In topology weak forms of open sets play an important role in the generalization of different forms of continuity. Using different forms of open sets many authors have introduced and studied various types of continuity. In this paper a unified version of some types of continuity from a generalized topological space to a topological space has been introduced where a generalized topology was first introduced by A. Császár (see [5–9]).

We recall some notions defined in [5]. Let X be a non-empty set, $\exp X$ denotes the power set of X. We call a class $\mu \subsetneqq \exp X$ a generalized topology [5], (briefly, GT) if $\varnothing \in \mu$ and union of elements of μ belongs to μ. A set X, with a GT μ on it is said to be a generalized topological space (briefly, GTS) and is denoted by (X, μ).

Entrato in redazione: 26 febbraio 2013

AMS 2010 Subject Classification: 54C08, 54C10.
Keywords: $\mu(\theta)$-open set, $\mu(\theta)$-closed set, Weakly μ-compact space, Almost strong μ_θ-continuity.
The author acknowledges the financial support from UGC, New Delhi.

For a GTS (X,μ), the elements of μ are called μ-open sets and the complement of μ-open sets are called μ-closed sets. For $A \subseteqq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A, i.e., the smallest μ-closed set containing A; and by $i_\mu(A)$ the union of all μ-open sets contained in A, i.e., the largest μ-open set contained in A (see [5, 6]).

It is easy to observe that i_μ and c_μ are idempotent and monotonic, where $\gamma : \exp X \to \exp X$ is said to be idempotent iff $A \subseteqq B \subseteqq X$ implies $\gamma(\gamma(A)) = \gamma(A)$ and monotonic iff $\gamma(A) \subseteqq \gamma(B)$. It is also well known from [6, 7] that if μ is a GT on X and $A \subseteqq X$, $x \in X$, then $x \in c_\mu(A)$ iff $x \in M \in \mu \Rightarrow M \cap A \neq \varnothing$ and $c_\mu(X \setminus A) = X \setminus i_\mu(A)$.

The $\mu(\theta)$-closure [7] (resp. $\mu(\theta)$-interior [7]) of a subset A in a GTS (X,μ) is denoted by $c_{\mu(\theta)}(A)$ (resp. $i_{\mu(\theta)}(A)$) and is defined to be the set $\{x \in X : c_\mu(U) \cap A \neq \varnothing$ for each $U \in \mu(x)\}$ (resp. $\{x \in X :$ there exists $U \in \mu(x)$ such that $c_\mu(U) \subseteqq A\}$), where $\mu(x) = \{U \in \mu : x \in U\}$. It is well known from [13] that in a GTS (X,μ), $X \setminus c_{\mu(\theta)}(A) = i_{\mu(\theta)}(X \setminus A)$ and $X \setminus i_{\mu(\theta)}(A) = c_{\mu(\theta)}(X \setminus A)$. A set A is called $\mu(\theta)$-closed if $c_{\mu(\theta)}(A) = A$. The complement of a $\mu(\theta)$-closed set is called $\mu(\theta)$-open. For any μ-open set U in a GTS (X,μ), $c_\mu(U) = c_{\mu(\theta)}(U)$ (see [17]).

Let (X,τ) be a topological space. The δ-closure [21] of a subset A of a topological space (X,τ) is denoted by $cl_\delta(A)$ and is defined by $\{x \in X : A \cap U \neq \varnothing$ for all regular open sets U containing $x\}$, where a subset A is called regular open if $A = int(cl(A))$. A is called δ-closed if $cl_\delta(A) = A$. The complement of a δ-closed set is called δ-open. It is known from [21] that the family of all δ-open sets forms a topology on X which is smaller than the original topology. A subset A of X is called semi-open [11] (resp. preopen [12], β-open [1]) if $A \subseteqq cl(int(A))$ (resp. $A \subseteqq int(cl(A)), A \subseteqq cl(int(cl(A)))$). The complement of a semi-open (resp. preopen, β-open) set is called a semi-closed (resp. preclosed, β-closed) set. The semi-closure of a set A is denoted by $scl(A)$ and defined by the smallest semi-closed set containing A [4].

Hereafter, throughout the paper we shall use (X,μ) to mean a generalized topological space and (Y,σ) to be a topological space unless otherwise stated.

2. Almost strongly μ_θ-continuous functions

Definition 2.1. A function $f : (X,\mu) \to (Y,\sigma)$ is said to be almost strongly μ_θ-continuous (resp. strongly μ-θ-continuous [17]) at $x \in X$ if for each open set V of Y containing $f(x)$, there exists $U \in \mu(x)$ such that $f(c_\mu(U)) \subseteqq scl(V)$ (resp. $f(c_\mu(U)) \subseteq V$). If f is almost strongly μ_θ-continuous (resp. strongly μ-θ-continuous) at each point of X then f is called almost strongly μ_θ-continuous (resp. strongly μ-θ-continuous) on X.

It follows from Definition 2.1 that strong θ-μ-continuity $\Rightarrow$ almost strong μ_θ-continuity. The implication is not reversible as shown by the next example.

Example 2.2. Let $X = Y = \{a,b,c\}$, $\mu = \{\varnothing,\{a\},\{a,b\},\{b,c\},X\}$ and $\sigma = \{\varnothing,\{a\},\{c\},\{a,c\},\{b,c\},X\}$. Then μ is a GT on X and σ is a topology on X. It can be easily verified that the identity function $f : (X,\mu) \to (Y,\sigma)$ is not strongly θ-μ-continuous but almost strongly μ_θ-continuous.

Theorem 2.3. *For a function $f : (X,\mu) \to (Y,\sigma)$ the followings are equivalent:*
(i) f is almost strongly μ_θ-continuous;
(ii) $f^{-1}(V)$ is $\mu(\theta)$-open for each regular open set V of Y;
(iii) $f^{-1}(K)$ is $\mu(\theta)$-closed for each regular closed set K of Y;
(iv) For each $x \in X$ and each regular open set V of Y containing $f(x)$, there exists $U \in \mu(x)$ such that $f(c_\mu(U)) \subseteqq V$;
(v) $f^{-1}(V)$ is $\mu(\theta)$-open for each δ-open set V of Y;
(vi) $f^{-1}(F)$ is $\mu(\theta)$-closed for each δ-closed set F of Y;
(vii) $f(c_{\mu(\theta)}(A)) \subseteqq cl_\delta(f(A))$ for each subset A of X;
(viii) $c_{\mu(\theta)}(f^{-1}(B)) \subseteqq f^{-1}(cl_\delta(B))$ for each subset B of Y;
(ix) $f^{-1}(int_\delta(B)) \subseteq i_{\mu(\theta)}(f^{-1}(B))$ for each subset B of Y;
(x) $f^{-1}(V) \subseteqq i_{\mu(\theta)}(f^{-1}(scl(V)))$ for each open set V of Y.

Proof. (i) $\Rightarrow$ (ii) Let V be a regular open set in Y and $x \in f^{-1}(V)$. Then by (i) there exists a μ-open set U containing x such that $f(c_\mu(U)) \subseteqq scl(V) = V$ (as V is regular open). Thus $x \in U \subseteqq c_\mu(U) \subseteqq f^{-1}(V)$. Thus $x \in i_{\mu(\theta)}(f^{-1}(V))$ and hence $f^{-1}(V) \subseteqq i_{\mu(\theta)}(f^{-1}(V))$. Thus $f^{-1}(V) = i_{\mu(\theta)}(f^{-1}(V))$ showing $f^{-1}(V)$ to be $\mu(\theta)$-open.

(ii) $\Rightarrow$ (iii) Let K be a regular closed set of Y. Then by (ii), $X \setminus f^{-1}(K) = f^{-1}(Y \setminus K)$ is $\mu(\theta)$-open showing $f^{-1}(K)$ to be $\mu(\theta)$-closed.

(iii) $\Rightarrow$ (iv) Let $x \in X$ and V be a regular open set in Y with $f(x) \in V$. Then $X \setminus f^{-1}(V) = f^{-1}(Y \setminus V) = c_{\mu(\theta)}(f^{-1}(Y \setminus V))$ (by (iii)) $= X \setminus i_{\mu(\theta)}(f^{-1}(V))$. Thus $f^{-1}(V) = i_{\mu(\theta)}(f^{-1}(V))$. So there exists a μ-open set U containing x such that $c_\mu(U) \subseteqq f^{-1}(V)$. Hence $f(c_\mu(U)) \subseteqq V$.

(iv) $\Rightarrow$ (v) Let V be a δ-open set in Y and $x \in f^{-1}(V)$. So there exists a regular open set G in Y such that $f(x) \in G \subseteqq V$. Thus by (iv), there exists a μ-open set U containing x such that $f(c_\mu(U)) \subseteqq G$. So $x \in U \subseteqq c_\mu(U) \subseteqq f^{-1}(V)$. Thus $x \in i_{\mu(\theta)}(f^{-1}(V))$. Hence $f^{-1}(V) \subseteqq i_{\mu(\theta)}(f^{-1}(V))$. Thus $f^{-1}(V)$ is $\mu(\theta)$-open.

(v) $\Rightarrow$ (vi) Let F be a δ-closed set of Y. Then by (v), $f^{-1}(F) = X \setminus f^{-1}(Y \setminus F) = X \setminus i_{\mu(\theta)}(f^{-1}(Y \setminus F)) = c_{\mu(\theta)}(f^{-1}(F))$. Thus $f^{-1}(F)$ is $\mu(\theta)$-closed.

(vi) $\Rightarrow$ (vii) Let A be a subset of X. Then $cl_\delta(f(A))$ is δ-closed in Y. Thus by (vi), $f^{-1}(cl_\delta(f(A))) = c_{\mu(\theta)}(f^{-1}(cl_\delta(f(A))))$. Let $x \notin f^{-1}(cl_\delta(f(A)))$. Thus there exists a μ-open set U containing x such that $c_\mu(U) \cap f^{-1}(cl_\delta(f(A))) = \varnothing$. Hence $c_\mu(U) \cap A = \varnothing$. Hence $x \notin c_{\mu(\theta)}(A)$. Thus $f(c_{\mu(\theta)}(A)) \subseteqq cl_\delta(f(A))$.

(vii) $\Rightarrow$ (viii) Let B be a subset of Y. Then by (vii), $f(c_{\mu(\theta)}(f^{-1}(B))) \subseteqq cl_\delta(B)$. Hence $c_{\mu(\theta)}(f^{-1}(B)) \subseteqq f^{-1}(cl_\delta(B))$.

(viii) $\Rightarrow$ (ix) Let B be a subset of Y and $x \in f^{-1}(int_\delta(B))$. Then $f(x) \in int_\delta(B)$. So $f(x) \notin cl_\delta(Y \setminus B)$. Hence $x \notin f^{-1}(cl_\delta(Y \setminus B))$. Thus by (viii), $x \notin c_{\mu(\theta)}(f^{-1}(Y \setminus B)) = X \setminus i_{\mu(\theta)}(f^{-1}(B))$. Hence $x \in i_{\mu(\theta)}(f^{-1}(B))$.
Thus $f^{-1}(int_\delta(B)) \subseteq i_{\mu(\theta)}(f^{-1}(B))$.

(ix) $\Rightarrow$ (x) Let V be an open set of Y. Then $V \subseteqq int(cl(V)) \subseteqq int_\delta(scl(V))$ and hence by (ix), $f^{-1}(V) \subseteqq f^{-1}(int_\delta(scl(V))) \subseteqq i_{\mu(\theta)}(f^{-1}(scl(V)))$.

(x) $\Rightarrow$ (i) Let V be an open set in Y containing $f(x)$. Then $x \in f^{-1}(V) \subseteqq i_{\mu(\theta)}(f^{-1}(scl(V)))$. Thus there exists a μ-open set U containing x such that $x \in U \subseteq c_\mu(U) \subseteqq f^{-1}(scl(V))$. This shows that $f(c_\mu(U)) \subseteqq scl(V)$, showing that f is almost strongly μ_θ-continuous. □

Theorem 2.4. *For a function $f : (X, \mu) \to (Y, \sigma)$ the followings are equivalent:*
(a) f is almost strongly μ_θ-continuous;
(b) $c_{\mu(\theta)}(f^{-1}(cl(int(F)))) \subseteqq f^{-1}(F)$ for each closed subset F of Y;
(c) $c_{\mu(\theta)}(f^{-1}(cl(int(cl(B))))) \subseteqq f^{-1}(cl(B))$ for each subset B of Y;
(d) $f^{-1}(int(B)) \subseteqq i_{\mu(\theta)}(f^{-1}(int(cl(int(B)))))$ for each B subset of Y.

Proof. (a) $\Rightarrow$ (b) Let F be a closed subset of Y. Then $X \setminus f^{-1}(F) = f^{-1}(Y \setminus F)$ $\subseteqq i_{\mu(\theta)}(f^{-1}(int(cl(Y \setminus F)))) = i_{\mu(\theta)}(X \setminus f^{-1}(cl(int(F)))) =$
$X \setminus c_{\mu(\theta)}(f^{-1}(cl(int(F))))$. Therefore, $c_{\mu(\theta)}(f^{-1}(cl(int(F)))) \subseteqq f^{-1}(F)$.

(b) $\Rightarrow$ (c) Follows from (b), by taking $F = cl(B)$.

(c) $\Rightarrow$ (d) Let B be a subset of Y. Then $f^{-1}(int(B)) = X \setminus f^{-1}(cl(Y \setminus B)) \subseteqq X \setminus c_{\mu(\theta)}(f^{-1}(cl(int(cl(Y \setminus B))))) = i_{\mu(\theta)}(f^{-1}(int(cl(int(B)))))$.
Thus $f^{-1}(int(B)) \subseteqq i_{\mu(\theta)}(f^{-1}(int(cl(int(B)))))$.

(d) $\Rightarrow$ (a) Let V be any regular open set of Y. Then by (d), $f^{-1}(V) \subseteqq i_{\mu(\theta)}(f^{-1}(V))$ and hence $f^{-1}(V) = i_{\mu(\theta)}(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is $\mu(\theta)$-open. Hence by Theorem 2.3, f is almost strongly μ_θ-continuous. □

Theorem 2.5. *For a function $f : (X, \mu) \to (Y, \sigma)$ the followings are equivalent:*
(a) f is almost strongly μ_θ-continuous;
(b) $c_{\mu(\theta)}(f^{-1}(V)) \subseteqq f^{-1}(cl(V))$ for each β-open set V of Y;
(c) $c_{\mu(\theta)}(f^{-1}(V)) \subseteqq f^{-1}(cl(V))$ for each semi-open set V of Y;
(d) $f^{-1}(V) \subseteqq i_{\mu(\theta)}(f^{-1}(int(cl(V))))$ for each preopen set V of Y.

Proof. (a) $\Rightarrow$ (b) Let V be a β-open set of Y. Then by (a) and Theorem 2.4, $c_{\mu(\theta)}(f^{-1}(V)) \subseteqq c_{\mu(\theta)}(f^{-1}(cl(int(cl(V))))) \subseteqq f^{-1}(cl(V))$. Thus $c_{\mu(\theta)}(f^{-1}(V))$

$\subseteqq f^{-1}(cl(V))$.

(b) $\Rightarrow$ (c) This follows from the fact that every semi open set is β-open.

(c) $\Rightarrow$ (d) Let V be a pre-open set of Y. Then $Y \setminus V$ is preclosed in Y and hence $cl(int(Y \setminus V)) \subseteqq Y \setminus V$. Since $cl(int(Y \setminus V))$ is regular closed, it is semi-open in Y. Thus by (iii), $c_{\mu(\theta)}(f^{-1}(cl(int(Y \setminus V)))) \subseteqq f^{-1}(cl(int(Y \setminus V))) \subseteqq f^{-1}(Y \setminus V)$. Therefore, $f^{-1}(V) \subseteqq X \setminus c_{\mu(\theta)}(f^{-1}(cl(int(Y \setminus V)))) = X \setminus c_{\mu(\theta)}(X \setminus f^{-1}(int(cl(V)))) = i_{\mu(\theta)}(f^{-1}(int(cl(V))))$.

(d) $\Rightarrow$ (a) Let V be a regular open set of Y. Then V is pre-open and $f^{-1}(V) \subseteqq i_{\mu(\theta)}(f^{-1}(int(cl(V)))) = i_{\mu(\theta)}(f^{-1}(V))$. Thus $f^{-1}(V)$ is $\mu(\theta)$-open. Hence by Theorem 2.3, f is almost strongly μ_θ-continuous. □

Definition 2.6. A function $f : (X, \mu) \to (Y, \sigma)$ is said to be (μ, σ)-continuous at $x \in X$ if for each open set V of Y containing $f(x)$, there exists $U \in \mu$ containing x such that $f(U) \subseteqq V$.

It can be checked that a function $f : (X, \mu) \to (Y, \sigma)$ is (μ, σ)-continuous if and only if the primage of each open set in Y is μ-open in X.

Theorem 2.7. *Let (Y, σ) be a regular space. Then for a function $f : (X, \mu) \to (Y, \sigma)$ the followings are equivalent:*
(a) f is (μ, σ)-continuous;
(b) f is strongly μ-θ-continuous;
(c) f is almost strongly μ_θ-continuous.

Proof. (a) $\Rightarrow$ (b) Let $x \in X$ and V be an open set of Y containing $f(x)$. Since Y is regular, there exists an open set G such that $f(x) \in G \subseteqq cl(G) \subseteqq V$. Since f is (μ, σ)-continuous, there exists $U \in \mu$ containing x such that $f(U) \subseteqq G$. We shall show that $f(c_\mu(U)) \subseteqq cl(G)$. Let $y \notin cl(G)$. Then there exists an open set H containing y such that $G \cap H = \varnothing$. Since f is (μ, σ)-continuous, $f^{-1}(H) = i_\mu(f^{-1}(H))$ and $f^{-1}(H) \cap U = \varnothing$. We shall now show that $f^{-1}(H) \cap c_\mu(U) = \varnothing$. If $f^{-1}(H) \cap c_\mu(U) \neq \varnothing$ then $i_\mu(f^{-1}(H)) \cap c_\mu(U) \neq \varnothing$. Let $z \in i_\mu(f^{-1}(H)) \cap c_\mu(U) \neq \varnothing$. Then $z \in i_\mu(f^{-1}(H))$ and $z \in c_\mu(U)$. Then there exists $V \in \mu$ containing z such that $V \subseteqq f^{-1}(H)$. Also $V \cap U \neq \varnothing$ (as $z \in c_\mu(U)$) implies that $f^{-1}(H) \cap U \neq \varnothing$ - a contradiction. Thus $H \cap f(c_\mu(U)) = \varnothing$. Hence $y \notin f(c_\mu(U))$ so that $f(c_\mu(U)) \subseteqq cl(G) \subseteqq V$. Thus f is strongly μ-θ-continuous.

(b) $\Rightarrow$ (c) This is trivial.

(c) $\Rightarrow$ (a) Let $x \in X$ and V be an open set of Y containing $f(x)$. Since (Y, σ) is regular, there exists an open set G in Y such that $x \in G \subseteqq cl(G) \subseteqq V$. Since f is almost strongly μ_θ-continuous, there exists $U \in \mu$ containing x such that $f(c_\mu(U)) \subseteqq cl(G) \subseteqq V$; hence $f(U) \subseteqq V$. Thus f is (μ, σ)-continuous. □

Definition 2.8. A topological space (X,τ) is said to be almost regular [20] if for any regular closed set F and any point $x \in X \setminus F$, there exist disjoint open sets U and V such that $x \in U$ and $F \subseteqq V$.

Definition 2.9. A function $f : (X,\mu) \to (Y,\sigma)$ is called μ-θ-continuous at $x \in X$ if for each open set V of Y containing $f(x)$, there exists $U \in \mu$ containing x such that $f(c_\mu(U)) \subseteqq cl(V)$.

Theorem 2.10. *If a function $f : (X,\mu) \to (Y,\sigma)$ is μ-θ-continuous and (Y,σ) is almost regular, then f is almost strongly μ_θ-continuous.*

Proof. Let $x \in X$ and V be an open set of Y containing $f(x)$. Since (Y,σ) is almost regular, there exists a regular open set G of Y, such that $f(x) \in G \subseteqq cl(G) \subseteqq int(cl(V))$ (see [20]). Since f is μ-θ-continuous, there exists $U \in \mu$ containing x such that $f(c_\mu(U)) \subseteqq cl(G) \subseteqq int(cl(V)) = scl(V)$. Thus f is almost strongly μ_θ-continuous. □

Definition 2.11. Let (X,μ) be a GTS. A subset A of X said to be weakly μ-compact relative to X [18] if every cover of A by μ-open subsets of X has a finite subfamily, the union of the μ-closures of whose members covers A. If $A = X$, then we say X is weakly μ-compact.

A topological space (X,τ) is called quasi H-closed [2] if every open cover of X has a finite subfamily whose closures cover X.

Theorem 2.12. *If a function $f : (X,\mu) \to (Y,\sigma)$ is μ-θ-continuous from a weakly μ-compact space (X,μ) to a Uryshon space (Y,σ), then f is almost strongly μ_θ-continuous.*

Proof. We shall first show that (Y,σ) is quasi H-closed. Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of X. Then for each $x \in X$ there exists $\alpha(x) \in \Lambda$ such that $f(x) \in V_{\alpha(x)}$. Since f is μ-θ-continuous, there exists a μ-open set U_x containing x such that $f(c_\mu(U_x)) \subseteqq cl(V_{\alpha(x)})$. Then the family $\{U_x : x \in X\}$ is a cover of X by μ-open sets of X. Since (X,μ) is weakly μ-compact, there exist finite number of points, say x_1, x_2, ..., x_n in X such that $X = \cup\{c_\mu(U_{x_i}) : i = 1,2,...,n\}$. Thus we have $Y = f(X) = \cup\{f(c_\mu(U_{x_i})) : i = 1,2,...,n\} \subseteqq \cup\{cl(V_{x_i}) : i = 1,2,...,n\}$. Since every quasi H-closed Urysohn space is almost regular [16], by Theorem 2.10, f is almost strongly μ_θ-continuous. □

3. Properties of almost strong μ_θ-continuity

Definition 3.1. A GTS (X,μ) is said to be μ-Urysohn if for any two distinct points x and y in X, there exist U and V in μ containing x and y respectively such that $c_\mu(U) \cap c_\mu(V) = \varnothing$.

Theorem 3.2. *If $f:(X,\mu)\to(Y,\sigma)$ is an almost strongly μ_θ-continuous injection and (Y,σ) is Hausdorff, then (X,μ) is μ-Urysohn.*

Proof. Let x_1 and x_2 be any two distinct points of X. Then $f(x_1)\neq f(x_2)$. Thus there exist two open sets V_1 and V_2 containing $f(x_1)$ and $f(x_2)$ respectively such that $V_1\cap V_2=\varnothing$. Hence $scl(V_1)\cap scl(V_2)=\varnothing$. Since f is almost strongly μ_θ-continuous, there exist $U_i\in\mu$ containing x_i such that $f(c_\mu(U_i))\subseteqq scl(V_i)$ for $i=1,2$. Thus $c_\mu(U_1)\cap c_\mu(U_2)=\varnothing$. Hence (X,μ) is μ-Urysohn. □

Theorem 3.3. *Let (X,μ) be a GTS. If for any two distinct points there exists a function f from (X,μ) onto a Hausdorff space (Y,σ) such that (i) $f(x_1)\neq f(x_2)$ (ii) f is μ-θ-continuous at x_1 and (iii) f is almost strongly μ_θ-continuous at x_2 then (X,μ) is μ-Urysohn.*

Proof. Let x_1 and x_2 be any two distinct points of X. Then by hypothesis there exists a function $f:(X,\mu)\to(Y,\sigma)$ satisfying the given conditions, where (Y,σ) is Hausdorff. Let $y_i=f(x_i)$, for $i=1,2$. Then $y_1\neq y_2$. Thus there exist two disjoint open sets V_1 and V_2 containing y_1 and y_2 respectively such that $V_1\cap V_2=\varnothing$. Thus $cl(V_1)\cap scl(V_2)=\varnothing$. Since f is μ-θ-continuous at x_1, there exists $U_1\in\mu$ containing x_1 such that $f(c_\mu(U_1))\subseteqq cl(V_1)$. Since f is almost strongly μ_θ-continuous at x_2, there exists $U_2\in\mu$ containing x_2 such that $f(c_\mu(U_2))\subseteqq scl(V_2)$. Thus $c_\mu(U_1)\cap c_\mu(U_2)=\varnothing$ showing (X,μ) to be μ-Urysohn. □

Theorem 3.4. *Let μ and λ be two GT's on a non-empty set X such that $U\in\mu$ and $V\in\lambda$ implies that $U\cap V\in\mu$ and (Y,σ) be a Hausdorff space. If a function $f:(X,\mu)\to(Y,\sigma)$ is almost strongly μ_θ-continuous and a function $g:(X,\lambda)\to(Y,\sigma)$ is λ-θ-continuous, then $A=\{x\in X : f(x)=g(x)\}$ is $\mu(\theta)$-closed.*

Proof. Let $x\in X\setminus A$, then $f(x)\neq g(x)$. Since (Y,σ) is Hausdorff, there exist V_1 and V_2 containing $f(x)$ and $g(x)$ respectively such that $V_1\cap V_2=\varnothing$, hence $scl(V_1)\cap cl(V_2)=\varnothing$. Since f is almost strongly μ_θ-continuous at x, there exists $U_1\in\mu$ containing x such that $f(c_\mu(U_1))\subseteqq scl(V_1)$. Since g is λ-θ-continuous at x, there exists $U_2\in\mu$ containing x such that $g(c_\lambda(U_2))\subseteqq cl(V_2)$. Let $U=U_1\cap U_2$, then $x\in U\in\mu$ and $f(c_\mu(U_1))\cap g(c_\lambda(U_2))=\varnothing$. Therefore, $c_\mu(U)\cap A=\varnothing$. Thus $x\notin c_{\mu(\theta)}(A)$. Thus A is $\mu(\theta)$-closed. □

Definition 3.5. A function $f:(X,\mu)\to(Y,\sigma)$ is said to have an almost strongly μ_θ-closed graph if for each $(x,y)\in X\times Y\setminus G(f)$, there exist a μ-open set U in X containing x and an open set V in Y containing y such that $(c_\mu(U)\times scl(V))\cap G(f)=\varnothing$.

Lemma 3.6. *A function* $f : (X,\mu) \to (Y,\sigma)$ *has an almost strongly* μ_θ*-closed graph if and only if for each* $(x,y) \in X \times Y \setminus G(f)$*, there exist a* μ*-open set* U *in* X *containing* x *and an open set* V *in* Y *containing* y *such that* $f(c_\mu(U)) \cap scl(V) = \varnothing$.

Theorem 3.7. *If a function* $f : (X,\mu) \to (Y,\sigma)$ *is almost* μ_θ*-continuous and* (Y,σ) *is Hausdorff, then* $G(f)$ *is almost strongly* μ_θ*-closed.*

Proof. Suppose that $(x,y) \in X \times Y \setminus G(f)$. Then $y \neq f(x)$. Since (Y,σ) is Hausdorff, there exist disjoint open sets V and H in Y containing y and $f(x)$ respectively, hence $scl(V) \cap scl(H) = \varnothing$. Since f is almost μ_θ-continuous at x there exists a μ-open set U containing x such that $f(c_\mu(U)) \subseteqq scl(H)$. So this implies that $f(c_\mu(U)) \cap scl(V) = \varnothing$. Thus by Lemma 3.6, $G(f)$ is almost strongly μ_θ-closed. □

Definition 3.8. A subset A of a topological space (Y,σ) is said to be N-closed relative to Y [3] if for any cover $\{V_\alpha : \alpha \in \Lambda\}$ of A by open sets of (Y,σ), there exists a finite subset Λ_0 of Λ such that $A \subseteqq \cup\{scl(V_\alpha) : \alpha \in \Lambda_0\}$. If Y is N-closed relative to (Y,σ), then (Y,σ) is said to be nearly compact [19].

Theorem 3.9. *If* $f : (X,\mu) \to (Y,\sigma)$ *is an almost strongly* μ_θ*-continuous function and* A *is weakly* μ*-compact relative to* (X,μ)*, then* $f(A)$ *is* N*-closed relative to* (Y,σ).

Proof. Let A be weakly μ-closed relative to (X,μ). Let $\{V_\alpha : \alpha \in \Lambda\}$ be any cover of $f(A)$ by open sets of (Y,σ). Then for each $x \in A$, there exists $\alpha(x) \in \Lambda$ such that $f(x) \in V_{\alpha(x)}$. Since f is an almost strongly μ_θ-continuous function, there exists a μ-open set U_x containing x such that $f(c_\mu(U_x)) \subseteqq scl(V_{\alpha(x)})$. Thus the family $\{U_x : x \in A\}$ is a cover of A by μ-open sets of X. Thus there exists a finite number of points, say, $x_1, x_2, ..., x_n$ in A such that $A \subseteqq \cup\{c_\mu(U_{x_i}) : x_i \in A, i = 1,2,...,n\}$. Therefore, we have $f(A) \subseteqq \cup\{f(c_\mu(U_{x_i})) : x_i \in A, i = 1,2,...,n\} \subseteqq \cup\{scl(V_{x_i}) : x_i \in A, i = 1,2,...,n\}$. Thus $f(A)$ is N-closed relative to (Y,σ). □

Corollary 3.10. *If* $f : (X,\mu) \to (Y,\sigma)$ *is an almost strongly* μ_θ*-continuous surjection and* X *is weakly* μ*-compact, then* Y *is nearly compact.*

Definition 3.11. A GTS (X,μ) is said to be μ-hyperconnected [10] if $c_\mu(U) = X$ for each μ-open set U of X. If (X,τ) be a topological space and $\mu = \tau$, then a μ-hyperconnected space reduces to a hyperconnected space i.e., a topological space (X,τ) is said to be hyperconnected [15] if every open set is dense in X.

Theorem 3.12. *If* $f : (X,\mu) \to (Y,\sigma)$ *is an almost strongly* μ_θ*-continuous surjection and* (X,μ) *is* μ*-hyperconnected, then* (Y,σ) *is hyperconnected.*

Proof. Let V be a non-empty open subset of Y. Since f is surjective, there exist $x \in f^{-1}(V)$ and $U \in \mu$ containing x such that $f(c_\mu(U)) \subseteq scl(V)$ i.e., $f(X) \subseteq scl(V)$ (as X is μ-hyperconnected). Thus by surjectiveness of f, $Y \subseteq scl(V) \subseteq cl(V)$. This shows that Y is hyperconnected. □

Definition 3.13. For any subset A of a GTS (X,μ), the $\mu_{(\theta)}$-frontier of A is denoted by $Fr_{\mu_{(\theta)}}(A)$ and defined by $Fr_{\mu_{(\theta)}}(A) = c_{\mu(\theta)}(A) \cap c_{\mu(\theta)}(X \setminus A)$.

Theorem 3.14. *The set of all points $x \in X$ at which a function $f : (X,\mu) \to (Y,\sigma)$ is not almost strongly μ_θ-continuous is identical with the union of $\mu_{(\theta)}$-frontiers of the inverse images of regular open sets containing $f(x)$.*

Proof. Suppose that f is not almost strongly μ_θ-continuous at $x \in X$. Then there exists a regular open set V of Y containing $f(x)$ such that $f(c_\mu(U))$ is not contained in V for every μ-open set U containing x. Then $c_\mu(U) \cap X \setminus f^{-1}(V) \neq \varnothing$ for every μ-open set U containing x and hence $x \in c_{\mu(\theta)}(X \setminus f^{-1}(V))$. Also $x \in f^{-1}(V) \subseteq c_{\mu(\theta)}(f^{-1}(V))$ and hence $x \in Fr_{\mu_{(\theta)}}(f^{-1}(V))$.

Conversely, suppose that f is almost strongly μ_θ-continuous at $x \in X$ and let V be any regular open set of Y containing $f(x)$. Then by Theorem 2.3, $x \in f^{-1}(V) \subseteq i_{\mu(\theta)}(f^{-1}(scl(V))) = i_{\mu(\theta)}(f^{-1}(V))$. Therefore $x \notin Fr_{\mu_{(\theta)}}(f^{-1}(V))$ for every regular open set V containing $f(x)$. This completes the proof. □

Acknowledgements

The author is grateful to the referee for certain suggestions.

REFERENCES

[1] M.E. Abd El-Monsef - S.N. El-Deeb - R.A. Mahmoud, *β-open sets and β-continuous mappings*, Bull. Fac. Sci. Assiut Univ. 12 (1983), 77–90.

[2] D.E. Cameron, *Some maximal topologies which are Q. H. C.*, Proc. Amer. Math. Soc. 75 (1979), 149–156.

[3] D. Carnahan, *Locally-nearly compact spaces*, Boll. Un. Mat. Ital. 4 (6) (1972), 146–153.

[4] S.G. Crossley - S.K. Haldebrand, *Semi-closure*, Texas J. Sci. 22 (1971), 99–112.

[5] Á. Császár, *Generalized topology, generalized continuity*, Acta Math. Hungar. 96 (2002), 351–357.

[6] Á. Császár, *Generalized open sets in generalized topologies*, Acta Math. Hungar. 106 (2005), 53–66.

[7] Á. Császár, *δ- and θ-modifications of generalized topologies*, Acta Math. Hungar. 120 (3) (2008), 275–279.

[8] Á. Császár, *Separation axioms for generalized topologies*, Acta Math. Hungar. 104 (2004), 63–69.

[9] Á. Császár, *γ-connected sets*, Acta Math. Hungar. 101 (2003), 273–279.

[10] E. Ekici, *Generalized hyperconnectedness*, Acta Math. Hungar. 133 (2011), 140–147.

[11] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly 70 (1963), 36–41.

[12] A. S. Mashhour, M. E. Abd El-Monsef - S. N. El-Deeb, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt 53 (1982), 47–53.

[13] W. K. Min, *A note on $\theta(g, g')$-continuity in generalized topological spaces*, Acta Math. Hungar. 125 (2009), 387–393.

[14] O. Njaståd, *On some classes of nearly open sets*, Pacific Jour. Math. 15 (1965), 961–970.

[15] T. Noiri, *A note on hyperconnected sets*, Mat. Vesnik 3 (16) (31) (1979), 53–60.

[16] P. Papić, *Sur les espaces H fermés*, Glasnik Mat. Fiz. Astr., 14 (1959), 135–141.

[17] B. Roy - T. Noiri, *On unification of strongly θ-continuous functions*, (communicated).

[18] M. S. Sarsak, *Weakly μ-compact spaces*, Demonstratio Math., XLV (4) (2012), 929–938.

[19] A. R. Singal - S. P. Arya, *On nearly compact spaces*, Boll. Un. Mat. Ital., 4 (2) (1969), 702–710.

[20] A. R. Singal - S. P. Arya, *On almost regular spaces*, Glasnik Mat Ser. III 4 (24) (1969), 89–99.

[21] N. V. Veličko, *H-closed topological spaces*, Mat. Sb., 70 (1966), 98–112.

BISHWAMBHAR ROY
Department of Mathematics
Women's Christian College
6, Greek Church Row
Kolkata-700 026, INDIA
e-mail: bishwambhar_roy@yahoo.co.in