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SOME PROPERTIES OF SKEW HURWITZ SERIES

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In this paper we show that, if R is a ring and σ an endomorphism of R, then the *skew Hurwitz series ring* $T = (HR, \sigma)$ is an n-clean ring if and only if R is an n-clean ring. Moreover, if R is an integral domain and a torsion-free $\mathbb{Z}$-module, then $T = (HR, \sigma)$ is a Prüfer domain if and only if R is a field. Also, we investigate when the ring $T = (HR, \sigma)$ is $g(x)$-clean, $(n, g(x))$-clean and a Neat ring.

1. Introduction

Throughout this paper R is an associative ring with identity 1, $U(R)$ its group of units, $Id(R)$ its set of idempotents and $C(R)$ its center and σ an endomorphism of the ring R.

In a series of papers ([15], [16], [17]) Keigher demonstrated that the ring HR of *Hurwitz series* over a commutative ring R with identity has many interesting applications in differential algebra.

Some properties which are shared between R and HR have been studied by Keigher [17], Zhongkui [24], Hassanein, et al in [12, 13], Benhissi [1, 2] and Ghanem [5].

The concept of Hurwitz series was extended by Hassanein in [11] to the ring of *skew Hurwitz series* as follows: the elements of $T = (HR, \sigma)$, the ring of *skew Hurwitz series*, are the ordinary functions $f\colon \mathbb{N} \to R$ with component wise

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addition and the following operation of multiplication: For each two functions $f,g \in T = (HR,\sigma)$,

$$(fg)(n) = \sum_{k=0}^{n} \binom{n}{k} f(k)\sigma^k(g(n-k)).$$

Define the mappings $h_n\colon \mathbb{N} \to R$ via $h_n(n-1) = 1$ and $h_n(m) = 0$ for each $m \neq n-1$ in $\mathbb{N}$. And $h_r^{'}\colon \mathbb{N} \to R$ via $h_r^{'}(0) = r$ and $h_r^{'}(n) = 0$ for each $0 \neq n$ in $\mathbb{N}$ and $r \in R$. It can be easily shown that $T = (HR,\sigma)$ is a ring with identity h_1, defined by $h_1\colon \mathbb{N} \to R$ via $h_1(0) = 1$ and $h_1(n) = 0$ for each $n \neq 0$ in $\mathbb{N}$ and $1 \in R$.

There is a ring homomorphism $\lambda_R : R \to T = (HR,\sigma)$ defined for any $r \in R$ by $\lambda_R(r) = h_r^{'}$. So, the ring R is canonically embedded as a subring of T via $r \in R \mapsto h_r^{'} \in T$. Note also that there is a ring homomorphism $\varepsilon_R : T = (HR,\sigma) \to R$ defined for any $f \in T = (HR,\sigma)$ by $\varepsilon_R(f) = f(0)$. Clearly, $\varepsilon_R \circ \lambda_R = \mathrm{id}_R$.

Let $\mathrm{supp}(f)$ denote the support of $f \in T = (HR,\sigma)$, i.e.,

$$\mathrm{supp}(f) = \{i \in \mathbb{N} \,|\, 0 \neq f(i) \in R\},$$

$\pi(f)$ denote the minimal element in $\mathrm{supp}(f)$. See [10] for more details.

Recently, Hassanein [10, 11, 14], Handam [9] and Yu-juan, et al [23] studied the transfer of some algebraic properties between R and $T = (HR,\sigma)$.

The motivation of this paper is to show that and extend the results in [5] to the ring $T = (HR,\sigma)$ of skew Hurwitz series over the ring R. Neat skew Hurwitz rings are also considered.

2. *n*-clean skew Hurwitz ring.

An element $r \in R$ is called *clean* if it can be expressed as a sum of an idempotent and a unit in R. This definition was introduced by Nicholson [19].

According to Xiao and Tong [21], an element x of a ring R is called n-clean, where n is a positive integer, if $x = e + u_1 + u_2 + \ldots + u_n$ where $e \in Id(R)$ and $u_i \in U(R)$; $i = 1,2,\ldots,n$. The ring R is called n-clean if every element of R is n-clean for some fixed positive integer n.

We need the following construction. Let R be a ring and let ${}_RV_R$ be an R-bimodule. Then the ideal extension $I(R;V)$ of R by V is defined to be the additive abelian group $I(R;V) = R \oplus V$ with multiplication given as follows: for all $v,w \in V$ and $r,s \in R$, we get,

$$(r,v)(s,w) = (rs, rw + vs + vw).$$

Note that if S is a ring and $S = R \oplus A$, where R is a subring of S and A is a two sided ideal of S, then $S \cong I(R;A)$.

Proposition 2.1. *Let R be a ring and σ an endomorphism of the ring R, then:*
1) $A = \{f \in T \,|\, f(0) = 0\}$ is a two sided ideal of T.
2) For each two sided σ-ideal I of R we have $H_I = \left\{h_r^{'} \in T \,|\, r \in I\right\}$ is a two sided ideal in T and

$$(HR,\sigma)\big/ (H_I + A) \cong (H(R/I),\sigma).$$

In particular, if I is a maximal σ-ideal of R, then $H_I + A$ is a maximal σ-ideal of T.

Proof. The proof of (1) is clear and that of (2) follows from Proposition 3.2 in [10]. □

Proposition 2.2 ([11]). *Let R be a ring and $\sigma \in \mathrm{End}(R)$. Then $T = (HR,\sigma) \cong I(R;A)$, where $A = \{f \in T \,|\, f(0) = 0\}$ is a two sided ideal of T.*

In the following Theorem shows us how the n-clean property shared between R and $T = (HR,\sigma)$.

Theorem 2.3. *Let R be a ring and $\sigma \in \mathrm{End}(R)$. Then $T = (HR,\sigma)$ is an n-clean ring if and only if R is an n-clean ring.*

Proof. Since $\langle h_2\rangle = Th_2 = \{fh_2 \,|\, f \in T\}$ is an ideal of T and clearly $(fh_2)(0) = 0$, by Proposition 2.2, we have $T \cong I(R;\langle h_2\rangle)$. Since $R \cong T\big/\langle h_2\rangle$, by Proposition 2.4 in [21], we conclude that if $T = (HR,\sigma)$ is an n-clean ring, then its homomorphic image R is.

Conversely, suppose that R is an n-clean ring and $f \in T$, hence $f(0) \in R$, therefore we can write

$$f(0) = e + u_1 + u_2 + ... + u_n,$$

where $e \in Id(R)$ and $u_i \in U(R)$; $i = 1,2,...,n$. Then

$$f = h_e^{'} + g + h_{u_2}^{'} + ... + h_{u_n}^{'}$$

where $g \in T$ defined by

$$g(0) = u_1 \text{ and } g(n) = f(n) \text{ for each } n \geq 1.$$

Since $g(0) = u_1$ is a unit in R, then, by Proposition 2.2 in [10], g is a unit in T. Also, we can easily check that $h_{u_2}^{'},...,h_{u_n}^{'} \in U(T)$; $i = 2,...,n$ and $h_e^{'} \in Id(T)$. Thus, we conclude that $T = (HR,\sigma)$ is an n-clean ring. □

Taking $\sigma = \mathrm{id}_R$, the identity automorphism on R, we get the next result

Corollary 2.4. *Let R be a ring, then the ring of Hurwitz series HR is an n-clean ring if and only if R is an n-clean ring.*

The previous corollary generalizes the following result due to Ghanem [5].

Theorem 2.5. *Suppose R is a commutative ring and n is a positive integer. Then HR is an n-clean ring if and only if R is an n-clean ring.*

3. $g_H(x)$-clean skew Hurwitz ring.

Camilo and Simon in [3] introduced the $g(x)$-clean ring for a polynomial $g(x) \in C(R)[x]$. A ring R is said to be $g(x)$-clean if every element of R is a sum of a unit and a root of the polynomial $g(x)$. Nicholson and Zhou in [20] showed that $\mathrm{End}({}_RM)$ is a $g(x)$-clean where ${}_RM$ is a semisimple left R-module and $g(x) \in (x-a)(x-b)C(R)[x]$ where $a, b \in C(R)$ and $b, b-a \in U(R)$. Fan and Yang [4] investigated $g(x)$-clean rings and obtained several important results. Clearly, any clean ring is n-clean and $g(x)$-clean. The following example shows us that the converse need not be true:

Example 3.1 (Example 3.1, [22])**.** Let G be a cyclic group of order 3, then the group ring $\mathbb{Z}_{(7)}G$ is not clean, while Theorem 2.3, in [21], illustrates that $\mathbb{Z}_{(7)}G$ is a 2-clean ring. Hence, n-clean ring need not be clean.

Next, we give a characterization of $g_H(x)$-clean of skew Hurwitz series rings.

Theorem 3.2. *Let R be a ring, $\sigma \in \mathrm{End}(R)$ and $g(x) = a_0 + a_1x + \ldots + a_mx^m \in C(R)[x]$. Then the ring R is $g(x)$-clean if and only if $T = (HR, \sigma)$ is $g_H(x)$-clean, where*

$$g_H(x) = h'_{a_0} + h'_{a_1}x + \ldots + h'_{a_m}x^m \in C(T)[x].$$

Proof. Suppose R is a $g(x)$-clean ring and $f \in T$. Hence $f(0) = u + s$ where $u \in U(R)$ and $g(s) = 0$. Therefore, $f = v + h'_s$ where $v \in T$ defined by $v(0) = u$ and $v(n) = f(n)$ for each $n \geq 1$. Since $v(0) = u$ is a unit of R, then v is a unit of T, by Proposition 2.2, in [10]. Clearly, h'_s is a root of the polynomial $g_H(x) \in C(T)[x]$. Therefore, T is a $g_H(x)$-clean ring.

Conversely, suppose T is a $g_H(x)$-clean ring and $r \in R$, then $\lambda_R(r) \in T$. Hence $\lambda_R(r) = f + q$ where $f \in U(T)$ and $g_H(q) = 0$. Therefore $\varepsilon_R(f) \in U(R)$, by Proposition 2.2, in [10], and $g(\varepsilon_R(q)) = 0$. Moreover, $r = \varepsilon_R(f) + \varepsilon_R(q)$. So, R is a $g(x)$-clean ring. □

Taking $\sigma = \mathrm{id}_R$, the identity automorphism on R, we get the next result

Corollary 3.3. *Let R be a ring and $g(x) \in C(R)[x]$. The ring of Hurwitz series HR is a $g_H(x)$-clean ring if and only if R is a $g(x)$-clean ring.*

The previous corollary generalizes the following result due to Ghanem [5].

Theorem 3.4. *Suppose R is a commutative ring and $g(x) \in C(R)[x]$. The ring of Hurwitz series HR is a $g_H(x)$-clean ring if and only if R is a $g(x)$-clean ring.*

4. $(n, g_H(x))$-clean skew Hurwitz ring.

In [8], Handam extended the definition of $g(x)$-clean ring to obtain a larger class of rings, call it $(n, g(x))$-clean. A ring R is said to be $(n, g(x))$-clean if every element of R can be written as a sum of a root of the polynomial $g(x)$ and n-units. The following two examples are due to Handam in [8]:

Example 4.1. Let R be the ring of all 3×3 upper triangular matrices over $\mathbb{Z}_2$. Since

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are units in R and

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}^2 + \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}^3 = 0.$$

Hence, $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is a $\left(2, x^2 + x^3\right)$-clean element.

Clearly, clean rings are $\left(1, x^2 - x\right)$-clean rings, n-clean rings are $\left(n, x^2 - x\right)$-clean rings and $g(x)$-clean rings are $(1, g(x))$-clean rings. Thus, the classes of n-clean and $g(x)$-clean rings are proper subclasses of $(n, g(x))$-clean rings.

Example 4.2. Let G be a cyclic group of order 3, then the group ring $\mathbb{Z}_{(7)}G$ is not clean, by [7], while Theorem 2.3, in [21], illustrates that $\mathbb{Z}_{(7)}G$ is a 2-clean ring. Hence, n-clean ring need not be clean. So, $\mathbb{Z}_{(7)}G$ is a $(2, x^2 - x)$-clean ring which is not a $(1, x^2 - x)$-clean ring. Thus we obtain an example which is a $(2, x^2 - x)$-clean ring but not a $(x^2 - x)$-clean ring.

Propositions 2.9 and 2.10, in [8], tell us the following: if R is an $(n,g(x))$-clean ring, then the power series ring $R[[x]]$ is an $(n,g(x))$-clean ring but its subring $R[x]$ is not an $(n,g(x))$-clean ring.

In the following we give the necessary and sufficient condition for the skew Hurwitz series ring $T=(HR,\sigma)$ to be $(n,g_H(x))$-clean ring:

Theorem 4.3. *Let R be a ring,* $\sigma\in\mathrm{End}(R)$*, n a positive integer and* $g(x)$ *a fixed polynomial in* $C(R)[x]$*. Then* $T=(HR,\sigma)$ *is an* $(n,g_H(x))$*-clean ring if and only if R is an* $(n,g(x))$*-clean ring.*

Proof. Since $\langle h_2\rangle = Th_2=\{fh_2\,|\,f\in T\}$ is an ideal of T and, by Proposition 2.2, we have $T\cong I(R;\langle h_2\rangle)=R\oplus\langle h_2\rangle$. If $T=(HR,\sigma)$ is an $(n,g_H(x))$-clean ring, then $R\cong T\big/\langle h_2\rangle$ is an $(n,g(x))$-clean ring, by Proposition 2.8 in [8].

Conversely, suppose that R is an $(n,g(x))$-clean ring and $f\in T$, hence $f(0)\in R$. Write

$$f(0)=s+u_1+u_2+\ldots+u_n,$$

where $u_i\in U(R)$; $i=1,2,\ldots,n$ and $g(s)=0$.

Then

$$f=h'_s+v+h'_{u_2}+\ldots+h'_{u_n}$$

where $v\in T$ defined by

$$v(0)=u_1 \text{ and } v(n)=f(n) \text{ for each } n\geq 1.$$

Since $v(0)=u_1$ is a unit in R, then, by Proposition 2.2 in [10], v is a unit in T. Also, we can easily check that $h'_{u_2},\ldots,h'_{u_n}\in U(T)$; $i=2,\ldots,n$ and $g(h'_s)=0$. Thus, we conclude that $T=(HR,\sigma)$ is an $(n,g_H(x))$-clean ring. □

Taking $\sigma=\mathrm{id}_R$, the identity automorphism on R, we get the next result

Corollary 4.4. *Let R be a ring, n a positive integer and* $g(x)$ *be a fixed polynomial in* $C(R)[x]$*. Then the ring of Hurwitz series HR is an* $(n,g_H(x))$*-clean ring if and only if R is an* $(n,g(x))$*-clean ring.*

The previous corollary generalizes the following result due to Ghanem [5].

Theorem 4.5. *Suppose R is a commutative ring, n a positive integer and* $g(x)$ *be a fixed polynomial in* $C(R)[x]$*. Then HR is an* $(n,g_H(x))$*-clean ring if and only if R is an* $(n,g(x))$*-clean ring.*

5. Neat skew Hurwitz ring.

One of the fundamental properties of a clean ring is that every homomorphic image of a clean ring is clean. McGovern [18] defined a neat ring to be: the ring in which every proper homomorphic image is clean. Clearly, every clean ring is a neat ring but the converse need not be true, for example any nonlocal PID is a neat ring but is not clean.

In the following we give the necessary and sufficient condition for the skew Hurwitz series ring $T = (HR, \sigma)$ to be a neat ring:

Theorem 5.1. *Let R be a ring and $\sigma \in \mathrm{End}(R)$. Then:*
1) $T = (HR, \sigma)$ is a neat ring if and only if R is a clean ring.
2) $T = (HR, \sigma)$ is a neat ring if and only it is a clean ring.

Proof. 1) Since $\langle h_2 \rangle = Th_2 = \{fh_2 \,|\, f \in T\}$ is a two-sided ideal of T, we have $T \cong I(R; \langle h_2 \rangle)$, by Proposition 2.2, if $T = (HR, \sigma)$ is a neat ring, then $R \cong T \big/ \langle h_2 \rangle$ is a clean ring. The converse direction is clear.

The conclusion (2) follows from (1) and Theorem 2.3. □

6. Prüfer domain of skew Hurwitz ring.

A commutative ring R is called Prüfer if every finitely generated ideal is invertible. An invertible ideal $A = \langle a_1, a_2, ..., a_m \rangle$ has the property that $A^n = \langle a_1^n, a_2^n, ..., a_m^n \rangle$ for each $n \in \mathbb{N}$. Thus it is clear that the Prüfer ring satisfies the following condition, if $a, b \in R$ and at least one of a and b is regular, then $ab \in \left\langle a^2, b^2 \right\rangle$. In [6], Gilmer called the ring satisfies the above condition a P-ring.

Throughout, unless otherwise stated, we assume that R is a commutative ring with identity 1 and D is an integral domain.

Proposition 6.1. *Suppose that R is a ring and $\sigma \in \mathrm{End}(R)$. If $T = (HR, \sigma)$ is a P-ring, then R is a von-Neumann regular ring.*

Proof. Assume T is a P-ring. Let $0 \neq r \in R$ be a regular element $1 \neq n \in \mathbb{N}$, whence h_n is a regular element of T and $h'_r h_n \in \left\langle h'_r, h_n \right\rangle^2 = \left\langle h'^2_r, h_n^2 \right\rangle = \left\langle h'_{r^2}, h_n^2 \right\rangle$. Hence $h'_r h_n = h'_{r^2} f + h_n^2 g$ for some $f, g \in T = (HR, \sigma)$. Since

$$\pi(h_n^2 g) = \pi(h_n^2) + \pi(g) = 2n - 2 + \pi(g)$$

and $\pi(h'_r h_n) = n - 1$, therefore, $r = (h'_r h_n)(n-1) = (h'_{r^2} f)(n-1) = r^2 f(n-1) \in r^2 R$. Since R is a commutative ring, then R is a von-Neumann regular ring. □

Proposition 6.2. *$T = (HR, \sigma)$ is an integral domain if and only if R is an integral domain and a torsion-free $\mathbb{Z}$-module.*

Proof. Let $T = (HR, \sigma)$ be an integral domain. Since R has a natural embedding in T, then clearly R is an integral domain. Now suppose that the ring R is a torsion-free $\mathbb{Z}$-module, then there is a positive integer m, such that $m1 = 0$. Now, we have

$$(h_2 h_m)(m) = \begin{pmatrix} 1+m-1 \\ 1 \end{pmatrix} h_2(1)\sigma(h_m(m-1)) = m1 = 0,$$

which implies that $h_2 h_{m-1} = 0$, a contradiction with the assumption that $T = (HR, \sigma)$ is an integral domain, so we conclude that R is a torsion-free $\mathbb{Z}$-module. The converse direction is clear. □

Theorem 6.3. *Let D be an integral domain and a torsion-free $\mathbb{Z}$-module. Then $T = (HR, \sigma)$ is a Prüfer domain if and only if D is a field.*

Proof. Using the same argument in the proof of Proposition 6.1, it can be easily shown that $d \in d^2 D$. Since D is an integral domain, then d is invertible and D must be a field.

Conversely, assume that D is a field, then, by Proposition 2.2, every element in the subset $J = \langle h_2 \rangle = Th_2 = \{fh_2 \,|\, f \in T\}$ satisfies $(fh_2)(0) = 0$, so J is a two sided ideal of T. We can easily check that J is the only non-zero maximal ideal of T and the other ideal are principal in the form $J_n = \langle h_n \rangle = Th_n$ for each $n \geq 3$. Hence T is a principal ideal domain, in particular, T is a Prüfer domain. □

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