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UNIFIED REPRESENTATION OF A CERTAIN CLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY DZIOK-SRIVASTAVA OPERATOR

M. K. AOUF - R. M. EL-ASHWAH - F. M. ABDULKAREM

In this paper, we investigate several properties of the harmonic class $\overline{N}_H([\alpha_1],n,\gamma)$, defined by the modified Dziok-Sirvastava operator, obtain distortion theorem, extreme points, convolution condition, convex combinations and integral operator for this class. Some of our results generalize previously known results.

1. Introduction

A continuous complex valued functions $f = u + iv$ which is define in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write

$$f(z) = h(z) + \overline{g(z)}, \tag{1}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h^{'}(z)| > |g^{'}(z)|$ in D (see [5]).

Denote by S_H, the class of functions f of the form (2) that are harmonic univalent and sense preserving in the unit disc $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. For $f = h + \overline{g} \in S_H$, we may express

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$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k+\sum_{k=1}^{\infty}\overline{b_kz^k},\ |b_1|<1, \tag{2}$$

where the analytic functions h and g are of the form

$$h(z)=z+\sum_{k=2}^{\infty}|a_k|z^k,\ g(z)=\sum_{k=1}^{\infty}|b_k|z^k,\ |b_1|<1. \tag{3}$$

In 1984 Clunie and Sheil-Small [5] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds.
For positive real values of $\alpha_1,\dots,\alpha_q$ and $\beta_1,\dots,\beta_s$ ($\beta_j\notin\mathbb{Z}_0^-=\{0,-1,-2,\dots\}$; $j=1,2,\dots,s$), we now define the generalized hypergeometric function ${}_qF_s(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z)$ by (see, for example, [10, p.19])

$${}_qF_s(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\dots(\alpha_q)_k}{(\beta_1)_k\dots(\beta_s)_k}.\frac{z^k}{k!} \tag{4}$$

$$(q\le s+1;q,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\mathbb{N}=\{1,2,\dots\};z\in U),$$

where $(a)_m$ is the Pochhammer symbol defined by

$$(a)_m=\frac{\Gamma(a+m)}{\Gamma(a)}=\begin{cases}1 & (m=0),\\ a(a+1)\dots(a+m-1) & (m\in\mathbb{N}).\end{cases} \tag{5}$$

Corresponding to the function $h(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z)$ defined by

$$h(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z)=z\,{}_qF_s(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z), \tag{6}$$

which is defined by following Hadamard product (or convolution) for $\varphi(z)$ in the form:

$$\varphi(z)=z+\sum_{k=2}^{\infty}\phi_kz^k,$$

$$H(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s)\varphi(z)=h(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z)*\varphi(z). \tag{7}$$

or,

$$H(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s)\varphi(z)=z+\sum_{k=2}^{\infty}\Gamma_k(\alpha_1)\phi_kz^k, \tag{8}$$

where

$$\Gamma_k(\alpha_1)=\frac{(\alpha_1)_{k-1}\dots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\dots.(\beta_s)_{k-1}}\cdot\frac{1}{(1)_{k-1}}\,(k\ge 2). \tag{9}$$

If, for convenience, we write

$$H_{q,s}[\alpha_1]=H(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s). \tag{10}$$

The linear operator $H_{q,s}[\alpha_1]$ was introduced and studied by Dziok and Srivastava [6].
Al-Kharsani and Al-Khal [2] defined the modified Dziok-Srivastava operator of the harmonic function $f = h + \overline{g}$, where h and g given by (3) as follows:

$$H_{q,s}[\alpha_1]f(z) = H_{q,s}[\alpha_1]h(z) + \overline{H_{q,s}[\alpha_1]g(z)}. \tag{11}$$

Also let $S_{\overline{H}}$ denote the subclass of S_H consisting of functions $f = h + \overline{g}$ such that the functions h and g are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k,\ g(z) = \sum_{k=1}^{\infty} |b_k| z^k,\ |b_1| < 1. \tag{12}$$

By using the modified Dziok-Srivastava operator $H_{q,s}[\alpha_1]$ defined by (11), Al-Khal [1] introduced and studied the class $\overline{S}_H(\alpha_1, \gamma)$, consisting of functions $f = h + \overline{g}$ such that h and g are given by (3) and f satisfies the condition

$$\Re\left\{(1+e^{i\alpha})\frac{z(H_{q,s}[\alpha_1]f(z))'}{z'H_{q,s}[\alpha_1]f(z)} - e^{i\alpha}\right\} \geq \gamma \quad (0 \leq \gamma < 1; > \alpha \in \mathbb{R}), \tag{13}$$

where $z' = \frac{\partial}{\partial\theta}\left(z = re^{i\theta}\right)$, $f'(z) = \frac{\partial}{\partial\theta}(f(z) = f(re^{i\theta}))$, $0 \leq r < 1$, and $0 \leq \theta < 2\pi$.
Also, Chandrashekar et al. [4] introduced and studied the class $T_{\mathcal{H}}([\alpha_1, \beta_1], \gamma)$, consisting of functions $f = h + \overline{g}$ such that h and g are given by (12) and f satisfies the condition

$$\Re\left\{1 + (1+e^{i\alpha})\frac{z(H_{q,s}[\alpha_1]h(z))'' + \overline{2z(H_{q,s}[\alpha]g(z))' + z^2(H_{q,s}[\alpha_1]g(z))''}}{z'\left(H_{q,s}[\alpha_1]f(z)\right)'}\right\} \geq \gamma, \tag{14}$$
$$(0 \leq \gamma < 1;\ \alpha \in \mathbb{R}),$$

where $H_{q,s}[\alpha_1](z)$ is the modified Dziok-Srivastava operator defined by (11).
To prove our results, we need the following lemmas.

Lemma 1.1 ([1]). *Let $f = h + \overline{g}$ such that h and g are given by (12). Then $f(z) \in \overline{S}_H(\alpha_1, \gamma)$ if and only if*

$$\sum_{k=1}^{\infty}\left(\frac{2k-1-\gamma}{1-\gamma}|a_k| + \frac{2k+1+\gamma}{1-\gamma}|b_k|\right)\Gamma_k(\alpha_1) \leq 2, \tag{15}$$

where $a_1 = 1,\ 0 \leq \gamma < 1$ and $\Gamma_k(\alpha_1)$ is given by (9).

Lemma 1.2 ([4]). *Let $f = h + \overline{g}$ such that h and g are given by (12). Then $f(z) \in T_{\mathcal{H}}([\alpha_1, \beta_1], \gamma)$ if and only if*

$$\sum_{k=1}^{\infty} k \left(\frac{2k-1-\gamma}{1-\gamma} |a_k| + \frac{2k+1+\gamma}{1-\gamma} |b_k| \right) \Gamma_k(\alpha_1) \leq 2, \tag{16}$$

where $a_1 = 1$, $0 \leq \gamma < 1$ and $\Gamma_k(\alpha_1)$ is given by (9).

In view of Lemma 1.1 and Lemma 1.2, we introduce and study an interesting unification of the classes $\overline{S}_H(\alpha_1, \gamma)$ and $T_{\mathcal{H}}([\alpha_1, \beta_1], \gamma)$.

Definition 1.3. Let the class $\overline{N}_H([\alpha_1], n, \gamma)$ ($n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $0 \leq \gamma < 1$) be the class of functions $f = h + \overline{g}$ such that h and g are given by (12), which satisfy the condition

$$\sum_{k=1}^{\infty} k^n \left(\frac{2k-1-\gamma}{1-\gamma} |a_k| + \frac{2k+1+\gamma}{1-\gamma} |b_k| \right) \Gamma_k(\alpha_1) \leq 2, \tag{17}$$

where $\Gamma_k(\alpha_1)$ is given by (9).
Specializing the parameters n, α_1 and γ, we obtain the following subclasses studied by various authors:
(i) $\overline{N}_H([1], 1, \gamma) = \mathcal{HLV}(k, \gamma)$ [with $k = 1$] (see Kim et al. [8]);
(ii) $\overline{N}_H([\alpha_1], 0, \gamma) = \overline{S}_H(\alpha_1, \gamma)$ (see Al-Khal [1]);
(iii) $\overline{N}_H([\alpha_1], 1, \gamma) = T_{\mathcal{H}}([\alpha_1, \beta_1], \gamma)$ (see Chandrashekar et al. [4]);
(iv) $\overline{N}_H([\alpha_1], 0, \gamma) = HGN(k, \alpha, \gamma)$ [with $k = 1, \gamma = 0$] (see El-Ashwah et al. [7]).

In this paper, we have obtained some properties for the class $\overline{N}_H([\alpha_1], n, \gamma)$. Further distortion theorems and extreme points are also obtained for functions in the class $\overline{N}_H([\alpha_1], n, \gamma)$.

2. Main Results

Unless otherwise mentioned, we assume in the reminder of this paper that $0 \leq \gamma < 1$, $q \leq s+1$, $q, s \in \mathbb{N}_0$, for positive real $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ ($\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}$; $j = 1, 2, \ldots, s$)and $n \in \mathbb{N}_0$.We begain with proving that the functions in the class $N_H([\alpha_1], n, \gamma)$ is sense-preserving, harmonic univalent.

Theorem 2.1. *Let $f = h + \overline{g}$, where h and g are given by (12)and $f(z) \in \overline{N}_H([\alpha_1], n, \gamma)$. Then $f(z)$ is sense-preserving, harmonic univalent in U.*

Proof. If $z_1 \neq z_2$,

$$\left|\frac{f(z_2)-f(z_1)}{h(z_2)-h(z_1)}\right| \geq 1-\left|\frac{g(z_2)-g(z_1)}{h(z_2)-h(z_1)}\right|$$
$$=1-\left|\frac{\sum\limits_{k=1}^{\infty} b_k\left(z_2^k-z_1^k\right)}{(z_2-z_1)-\sum\limits_{k=2}^{\infty} a_k\left(z_2^k-z_1^k\right)}\right| \geq 1-\frac{\sum\limits_{k=1}^{\infty} k\left|b_k\right|}{1-\sum\limits_{k=2}^{\infty} k\left|a_k\right|},$$

by using (17), we have

$$\left|\frac{f(z_2)-f(z_1)}{h(z_2)-h(z_1)}\right| \geq 1-\frac{\sum\limits_{k=1}^{\infty} k^n\left(\frac{2k+1+\gamma}{1-\gamma}\right)\Gamma_k(\alpha_1)\left|b_k\right|}{1-\sum\limits_{k=2}^{\infty} k^n\left(\frac{2k-1-\gamma}{1-\gamma}\right)\Gamma_k(\alpha_1)\left|a_k\right|}>0,$$

which proves the univalence. Also f is sense-preserving in U since

$$\begin{aligned}
|h'(z)| &\geq 1-\sum_{k=2}^{\infty} k\left|a_k\right||z|^{k-1}>1-\sum_{k=2}^{\infty} k\left|a_k\right| \\
&\geq 1-\sum_{k=1}^{\infty} k^n\left(\frac{2k-1-\gamma}{1-\gamma}\right)\left|a_k\right|\Gamma_k(\alpha_1) \\
&\geq \sum_{k=1}^{\infty} k^n\left(\frac{2k+1+\gamma}{1-\gamma}\right)\left|b_k\right|\Gamma_k(\alpha_1) \\
&\geq \sum_{k=1}^{\infty} k\left|b_k\right||z|^{k-1} \geq |g'(z)|.
\end{aligned}$$

The harmonic univalent functions

$$f(z)=z+\sum_{k=2}^{\infty} \frac{(1-\gamma)}{k^n(2k-1-\gamma)\Gamma_k(\alpha_1)} x_k z^k+\sum_{k=1}^{\infty} \frac{(1-\gamma)}{k^n(2k+1+\gamma)\Gamma_k(\alpha_1)} \overline{y_k z^k}, \quad (18)$$

where $\sum\limits_{k=2}^{\infty}|x_k|+\sum\limits_{k=1}^{\infty}|y_k|=1$, shows that the coefficient bound given by (17) is sharp. The functions of the form (18) are in the class $\overline{N}_H([\alpha_1], n, \gamma)$ because

$$\sum_{k=1}^{\infty} k^n\left[\frac{2k-1-\gamma}{1-\gamma}\left|a_k\right|+\frac{2k+1+\gamma}{1-\gamma}\left|b_k\right|\right] \Gamma_k(\alpha_1)=1+\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=2.$$

This completes the proof of Theorem 2.1. □

Next, we obtain the distortion theorems and extreme points of the closed convex hull for functions in the class $\overline{N}_H([\alpha_1], n, \gamma)$.

Theorem 2.2. *Let the function $f = h + \overline{g}$, where h and g are given by (12) be in the class $\overline{N}_H([\alpha_1], n, \gamma)$, then*

$$|f(z)| \geq (1 - |b_1|)r - \frac{1}{2^n \Gamma_2(\alpha_1)} \left[\frac{(1-\gamma)}{(3-\gamma)} - \frac{(3+\gamma)}{(3-\gamma)} |b_1|\right] r^2, \tag{19}$$

and

$$|f(z)| \leq (1 + |b_1|)r + \frac{1}{2^n \Gamma_2(\alpha_1)} \left[\frac{(1-\gamma)}{(3-\gamma)} - \frac{(3+\gamma)}{(3-\gamma)} |b_1|\right] r^2. \tag{20}$$

The equalities in (19) and (20) are attained for the functions f given by

$$f(z) = (1 - b_1)\overline{z} - \frac{1}{2^n \Gamma_2(\alpha_1)} \left[\frac{(1-\gamma)}{(3-\gamma)} - \frac{(3+\gamma)}{(3-\gamma)} |b_1|\right]^2 \overline{z}^2, \tag{21}$$

and

$$f(z) = (1 + b_1)\overline{z} + \frac{1}{2^n \Gamma_2(\alpha_1)} \left[\frac{(1-\gamma)}{(3-\gamma)} - \frac{(3+\gamma)}{(3-\gamma)} |b_1|\right] \overline{z}^2, \tag{22}$$

where $|b_1| \leq \dfrac{1-\gamma}{3+\gamma}$.

Proof. Let $f(z) \in \overline{N}_H([\alpha_1], n, \gamma)$, then we have

$$\begin{aligned}
|f(z)| &\geq (1 - |b_1|)r - \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^k \\
&\geq (1 - |b_1|)r - r^2 \sum_{k=2}^{\infty} (|a_k| + |b_k|) \\
&\geq (1 - |b_1|)r - \frac{(1-\gamma)}{2^n (3-\gamma)\Gamma_2(\alpha_1)} r^2. \\
&\cdot \sum_{k=2}^{\infty} k^n \left[\frac{(3-\gamma)}{(1-\gamma)} |a_k| + \frac{(3-\gamma)}{(1-\gamma)} |b_k|\right] \Gamma_k(\alpha_1) \\
&\geq (1 - |b_1|)r - \frac{(1-\gamma)}{2^n (3-\gamma)\Gamma_2(\alpha_1)} r^2. \\
&\cdot \sum_{k=2}^{\infty} k^n \left(\frac{(2k-\gamma-1)}{(1-\gamma)} |a_k| + \frac{(2k+\gamma+1)}{(1-\gamma)} |b_k|\right) \Gamma_k(\alpha_1)
\end{aligned}$$

$$\begin{aligned}
&\geq (1 - |b_1|)r - \frac{(1-\gamma)}{2^n (3-\gamma)\Gamma_2(\alpha_1)} \left[1 - \frac{(3+\gamma)}{(1-\gamma)} |b_1|\right] r^2 \\
&\geq (1 - |b_1|)r - \frac{1}{2^n \Gamma_2(\alpha_1)} \left[\frac{(1-\gamma)}{(3-\gamma)} - \frac{(3+\gamma)}{(3-\gamma)} |b_1|\right] r^2,
\end{aligned}$$

which proves the asserion (19) of Theorem 2.2. The proof of the assertion (20) is similar, thus, we omit it. □

Remark 2.3. (i) Putting $n=1, q=2, s=1, \alpha_1=\alpha_2=1$ and $\beta_1=1$ in Theorem 2.2, we improve the result obtained by Kim et al. [8, Theorem 4.2, with $k=1$], by adding the condition $|b_1| \leq \frac{1-\gamma}{3+\gamma}$;
(ii) Putting $n=0$ in Theorem 2.2, we improve the result obtained by Al-Khal [1, Theorem 2.3], by adding the condition $|b_1| \leq \frac{1-\gamma}{3+\gamma}$.

The following covering result follows the left hand inequality of Theorem 2.2

Corollary 2.4. *Let $f \in \overline{N}_H([\alpha_1], n, \gamma)$, then*

$$\left\{ w : |w| < \frac{(3 \cdot 2^n \Gamma_2(\alpha_1) - 1) - (2^n \Gamma_2(\alpha_1) - 1)\gamma}{2^n (3-\gamma) \Gamma_2(\alpha_1)} \right.$$

$$\left. - \frac{3(2^n \Gamma_2(\alpha_1) - 1) - (2^n \Gamma_2(\alpha_1) + 1)\gamma}{2^n (3-\gamma) \Gamma_2(\alpha_1)} |b_1| \right\} \subset f(U),$$

where $|b_1| \leq \left(\frac{1-\gamma}{3+\gamma} \right)$.

Remark 2.5. Putting $n=1$ in Theorem 2.2, we obtain of the following corollary:

Corollary 2.6. *Let the function $f = h + \overline{g}$, where h and g are given by (12) be in the class $T_{\mathcal{H}}([\alpha_1, \beta_1], \gamma)$, then*

$$|f(z)| \geq (1 - |b_1|) r - \frac{1}{2\Gamma_2(\alpha_1)} \left[\frac{(1-\gamma)}{(3-\gamma)} - \frac{(3+\gamma)}{(3-\gamma)} |b_1| \right] r^2, \tag{23}$$

and

$$|f(z)| \leq (1 + |b_1|) r + \frac{1}{2\Gamma_2(\alpha_1)} \left[\frac{(1-\gamma)}{(3-\gamma)} - \frac{(3+\gamma)}{(3-\gamma)} |b_1| \right] r^2. \tag{24}$$

The equalities in (23) and (24) are attained for the functions f given by

$$f(z) = (1 - b_1) \overline{z} - \frac{1}{2\Gamma_2(\alpha_1)} \left[\frac{(1-\gamma)}{(3-\gamma)} - \frac{(3+\gamma)}{(3-\gamma)} |b_1| \right]^2 \overline{z}^2, \tag{25}$$

and

$$f(z) = (1 + b_1) \overline{z} + \frac{1}{2\Gamma_2(\alpha_1)} \left[\frac{(1-\gamma)}{(3-\gamma)} - \frac{(3+\gamma)}{(3-\gamma)} |b_1| \right] \overline{z}^2, \tag{26}$$

where $|b_1| \leq \left(\frac{1-\gamma}{3+\gamma} \right)$.

Theorem 2.7. *Let $f = h + \overline{g}$, where h and g are given by (12). Then $f(z) \in clco\,\overline{N}_H([\alpha_1], n, \gamma)$ if and only if*

$$f(z) = \sum_{k=1}^{\infty} \left[X_k h_k(z) + Y_k g_k(z)\right], \tag{27}$$

where

$$h_1(z) = z, \tag{28}$$

$$h_k(z) = z - \frac{(1-\gamma)}{k^n(2k-1-\gamma)\Gamma_k(\alpha_1)} z^k \ (k = 2, 3, \dots), \tag{29}$$

and

$$g_k(z) = z + \frac{(1-\gamma)}{k^n(2k+1+\gamma)\Gamma_k(\alpha_1)} \overline{z}^k \ (k = 1, 2, \dots), \tag{30}$$

where $\sum_{k=1}^{\infty} (X_k + Y_k) = 1$, $X_k \geq 0$ and $Y_k \geq 0$. In particular, the extreme points of the class $\overline{N}_H([\alpha_1], n, \gamma)$ are $\{h_k\}\,(k \geq 2)$and $\{g_k\}\,(k \geq 1)$, respectively.

Proof. For a function $f(z)$ of the form (27), we have

$$f(z) = \sum_{k=1}^{\infty} \left[X_k h_k(z) + Y_k g_k(z)\right]$$

$$\begin{aligned}
&= \sum_{k=1}^{\infty} \left[X_k \left(z - \frac{(1-\gamma)}{k^n(2k-1-\gamma)\Gamma_k(\alpha_1)} z^k\right)\right. \\
&\left. + Y_k \left(z + \frac{(1-\gamma)}{k^n(2k+1+\gamma)\Gamma_k(\alpha_1)} \overline{z}^k\right)\right] \\
&= z - \sum_{k=2}^{\infty} \frac{(1-\gamma)}{k^n(2k-1-\gamma)\Gamma_k(\alpha_1)} X_k z^k \\
&+ \sum_{k=1}^{\infty} \frac{(1-\gamma)}{k^n(2k-1-\gamma)\Gamma_k(\alpha_1)} Y_k \overline{z}^k.
\end{aligned}$$

But,

$$\begin{aligned}
&\sum_{k=2}^{\infty} \left(\frac{k^n(2k-1-\gamma)\Gamma_k(\alpha_1)}{(1-\gamma)} \cdot \frac{(1-\gamma)}{k^n(2k-1-\gamma)\Gamma_k(\alpha_1)} X_k\right) \\
&+ \sum_{k=1}^{\infty} \left(\frac{k^n(2k-1-\gamma)\Gamma_k(\alpha_1)}{(1-\gamma)} \cdot \frac{(1-\gamma)}{k^n(2k-1-\gamma)\Gamma_k(\alpha_1)} Y_k\right) \\
&= \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \leq 1.
\end{aligned}$$

Thus $f(z) \in clco\, \overline{N}_H([\alpha_1], n, \gamma)$.
Conversely, assume that $f(z) \in clco\, \overline{N}_H([\alpha_1], n, \gamma)$. Set

$$X_k = \frac{k^n(2k-1-\gamma)\Gamma_k(\alpha_1)}{(1-\gamma)}\, |a_k| \;\; (k = 2, 3, \dots),$$
$$Y_k = \frac{k^n(2k+1+\gamma)\Gamma_k(\alpha_1)}{(1-\gamma)}\, |b_k| \;\; (k = 1, 2, \dots)$$

Then by using (17), we have

$$0 \le X_k \le 1 \;\; (k = 2, 3, \dots) \text{ and } 0 \le Y_k \le 1 (k = 1, 2, 3, \dots).$$

Define $X_1 = 1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k$. Thus we obtain $f(z) = \sum_{k=1}^{\infty} [X_k h_k(z) + Y_k g_k(z)]$.
This completes the proof of Theorem 2.7. □

Finally, we discusse the convolution properties, convex combination and integral operator.
Let the functions $f_m(z)$ defined by

$$f_m(z) = z - \sum_{k=2}^{\infty} |a_{k,m}| z^k + \sum_{k=1}^{\infty} |b_{k,m}| \overline{z}^k \, (m = 1, 2) \tag{31}$$

are in the class $\overline{N}_H([\alpha_1], n, \gamma)$, we denote by $(f_1 * f_2)(z)$ the convolution (or Hadamard Product) of the functions $f_1(z)$ and $f_2(z)$, that is,

$$(f_1 * f_2)(z) = z - \sum_{k=2}^{\infty} |a_{k,1}|\, |a_{k,2}| z^k + \sum_{k=1}^{\infty} |b_{k,1}|\, |b_{k,2}| \overline{z}^k. \tag{32}$$

We first show that the class $\overline{N}_H([\alpha_1], n, \gamma)$ is closed under convolution.

Theorem 2.8. *For $0 \le \delta \le \gamma < 1$, let the functions $f_1 \in \overline{N}_H([\alpha_1], n, \gamma)$ and $f_2 \in \overline{N}_H([\alpha_1], n, \delta)$. Then*

$$(f_1 * f_2)(z) \in \overline{N}_H([\alpha_1], n, \gamma) \subset \overline{N}_H([\alpha_1], n, \delta). \tag{33}$$

Proof. Let $f_m(z)$ $(m = 1, 2)$ are given by (31), where $f_1(z)$ be in the class $\overline{N}_H([\alpha_1], n, \gamma)$ and $f_2(z)$ be in the class $\overline{N}_H([\alpha_1], n, \delta)$. We wish to show that the coefficients of $(f_1 * f_2)(z)$ satisfy the required condition given in (17). For $f_2 \in \overline{N}_H([\alpha_1], n, \delta)$, we note that $|a_{k,2}| < 1$ and $|b_{k,2}| < 1$. Now for the convolution functions $(f_1 * f_2)(z)$, we obtain

$$\sum_{k=1}^{\infty} k^n \left(\frac{2k-1-\delta}{1-\delta} |a_{k,1}| |a_{k,2}| + \frac{2k+1+\delta}{1-\delta} |b_{k,1}| |b_{k,2}| \right) \Gamma_k(\alpha_1)$$
$$\leq \sum_{k=1}^{\infty} k^n \left(\frac{2k-1-\delta}{1-\delta} |a_{k,1}| + \frac{2k+1+\delta}{1-\delta} |b_{k,1}| \right) \Gamma_k(\alpha_1)$$
$$\leq \sum_{k=1}^{\infty} k^n \left(\frac{2k-1-\gamma}{1-\gamma} |a_{k,1}| + \frac{2k+1+\gamma}{1-\gamma} |b_{k,1}| \right) \Gamma_k(\alpha_1)$$
$$\leq 2,$$

since $0 \leq \delta \leq \gamma < 1$ and $f_1 \in \overline{N}_H([\alpha_1], n, \gamma)$. Thus $(f_1 * f_2)(z) \in \overline{N}_H([\alpha_1], n, \gamma) \subset \overline{N}_H([\alpha_1], n, \delta)$. This completes the proof of Theorem 2.8. □

Theorem 2.9. *The class $\overline{N}_H([\alpha_1], n, \gamma)$ is closed under convex combinations.*

Proof. For $i = 1, 2, \ldots$, let $f_i \in \overline{N}_H([\alpha_1], n, \gamma)$, where

$$f_i(z) = z - \sum_{k=2}^{\infty} |a_{k,i}| z^k + \sum_{k=1}^{\infty} |b_{k,i}| \overline{z}^k,$$

then from (17), for $\sum_{i=1}^{\infty} t_i = 1,\ 0 \leq t_i < 1$, the convex combination of f_i can be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{k,i}| \right) z^k + \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{k,i}| \right) \overline{z}^k. \tag{34}$$

Then by (17), we have

$$\sum_{k=1}^{\infty} k^n \left[\frac{(2k-1-\gamma)}{1-\gamma} \left(\sum_{i=1}^{\infty} t_i |a_{k,i}| \right) + \frac{(2k+1+\gamma)}{1-\gamma} \left(\sum_{i=1}^{\infty} t_i |b_{k,i}| \right) \right] \Gamma_k(\alpha_1)$$
$$= \sum_{i=1}^{\infty} t_i \left(\sum_{k=1}^{\infty} k^n \left[\frac{(2k-1-\gamma)}{1-\gamma} |a_{k,i}| + \frac{(2k+1+\gamma)}{1-\gamma} |b_{k,i}| \right] \right) \Gamma_k(\alpha_1)$$
$$\leq 2 \sum_{i=1}^{\infty} t_i = 2.$$

This completes the proof of Theorem 2.9. □

We examine a closure property of the class $\overline{N}_H([\alpha_1], n, \gamma)$ under the generalized Bernardi-Libera-Livingston integral operator (see [3, 9]) $L_c(f(z))$ which is defined by

$$L_c(f(z)) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt,\ c > -1. \tag{35}$$

Theorem 2.10. *Let* $f(z) \in \overline{N}_H([\alpha_1], n, \gamma)$. *Then* $L_c(f(z)) \in \overline{N}_H([\alpha_1], n, \gamma)$.

Proof. From (35), it follows that

$$\begin{aligned}
L_c(f(z)) &= \frac{c+1}{z^c} \int_0^z t^{c-1} \left[h(t) + \overline{g(t)} \right] dt \\
&= \frac{c+1}{z^c} \left[\int_0^z t^{c-1} \left(t - \sum_{k=2}^{\infty} a_k t^k \right) dt + \int_0^z t^{c-1} \left(\overline{\sum_{k=2}^{\infty} b_k t^k} \right) dt \right] \\
&= z - \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} B_k z^k,
\end{aligned}$$

where

$$A_k = \frac{c+1}{c+k} a_k, \; B_k = \frac{c+1}{c+k} b_k.$$

Therefore,

$$\begin{aligned}
&\sum_{k=1}^{\infty} k^n \left[\frac{2k-1-\gamma}{1-\gamma} \left(\frac{c+1}{c+k} \right) |a_k| + \frac{2k+1+\gamma}{1-\gamma} \left(\frac{c+1}{c+k} \right) |b_k| \right] \Gamma_k(\alpha_1) \\
&\leq \sum_{k=1}^{\infty} k^n \left[\frac{2k-1-\gamma}{1-\gamma} |a_k| + \frac{2k+1+\gamma}{1-\gamma} |b_k| \right] \Gamma_k(\alpha_1) \leq 2.
\end{aligned}$$

Since $f(z) \in \overline{N}_H([\alpha_1], n, \gamma)$, by Theorem 2.1, $L_c(f(z)) \in \overline{N}_H([\alpha_1], n, \gamma)$. This completes the proof of Theorem 2.10. □

Remark 2.11. (i) Putting $n = 1$, $q = 2, s = 1, \alpha_1 = \alpha_2 = 1$ and $\beta_1 = 1$ in the above results, we obtain the corresponding results obtained by Kim et al. [8, with $k = 1$];
(ii) Putting $n = 0$ in all the above results, we obtain the corresponding results obtained by Al-Khal [1];
(iii) Putting $n = 1$ in all the above results, we obtain the corresponding results obtained by Chandrashekar et al. [4];
(iv) Putting $n = 0$ in all the above results, we obtain the corresponding results obtained by El-Ashwah et al. [7, with $k = 1$ and $\gamma = 0$].

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M. K. AOUF
Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt
e-mail: `mkaouf127@yahoo.com`

R. M. EL-ASHWAH
Department of Mathematics, Faculty of Science, Damietta University,
New Damietta Egypt 34517, Egypt
e-mail: `r_elashwah@yahoo.com`

F. M. ABDULKAREM
Department of Mathematics, Faculty of Science, Damietta University,
New Damietta Egypt 34517, Egypt
e-mail: `fammari76@gmail.com`