DIFFERENTIAL SANDWICH THEOREMS FOR HIGHER-ORDER DERIVATIVES OF \( p \)-VALENT FUNCTIONS INVOLVING A GENERALIZED DIFFERENTIAL OPERATOR

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In the present article, we obtain some applications of first order differential subordination, superordination and sandwich results for higher-order derivatives of \( p \)-valent functions involving a generalized differential operator. Some of our results improve and generalize previously known results.

1. Introduction

Let \( H(U) \) be the class of analytic functions in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) and let \( H[a,p] \) be the subclass of \( H(U) \) consisting of functions of the form:

\[
f(z) = a + apz^p + ap+1z^{p+1} \ldots \ (a \in \mathbb{C}; \ p \in \mathbb{N} = \{1, 2, \ldots \}).
\]

For simplicity \( H[a] = H[a, 1] \). Also, let \( A(p) \) be the subclass of \( H(U) \) consisting of functions of the form:

\[
f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N}), \tag{1}
\]

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\]

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which are \( p \)-valent in \( U \). We write \( A(1) = A \).

If \( f, g \in H(U) \), we say that \( f \) is subordinate to \( g \) or \( g \) is superordinate to \( f \), written \( f(z) \prec g(z) \) if there exists a Schwarz function \( w \), which (by definition) is analytic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) for all \( z \in U \), such that \( f(z) = g(w(z)) \), \( z \in U \). Furthermore, if the function \( g \) is univalent in \( U \), then we have the following equivalence, (cf., e.g., [12], [21] and [22]):

\[
f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).
\]

Let \( \phi : \mathbb{C}^2 \times U \to \mathbb{C} \) and \( h \) be univalent function in \( U \). If \( \beta \) is analytic function in \( U \) and satisfies the first order differential subordination:

\[
\phi \left( \beta(z), z\beta'(z); z \right) \prec h(z), \quad (2)
\]

then \( \beta \) is a solution of the differential subordination (2). The univalent function \( q \) is called a dominant of the solutions of the differential subordination (2) if \( \beta(z) \prec q(z) \) for all \( \beta \) satisfying (2). A univalent dominant \( \tilde{q} \) that satisfies \( \tilde{q} \prec q \) for all dominants of (2) is called the best dominant. If \( \beta \) and \( \phi \) are univalent functions in \( U \) and if satisfies first order differential superordination:

\[
h(z) \prec \phi \left( \beta(z), z\beta'(z); z \right), \quad (3)
\]

then \( \beta \) is a solution of the differential superordination (3). An analytic function \( q \) is called a subordinant of the solutions of the differential superordination (3) if \( q(z) \prec \beta(z) \) for all \( \beta \) satisfying (3). A univalent subordinant \( \tilde{q} \) that satisfies \( q(z) \prec \tilde{q}(z) \) for all subordinants of (3) is called the best subordinant.

Using the results of Miller and Mocanu [22], Bulboaca [11] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [12]. Ali et al. [1], have used the results of Bulboaca [11] to obtain sufficient conditions for normalized analytic functions \( f \in A \) to satisfy:

\[
q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),
\]

where \( q_1 \) and \( q_2 \) are given univalent functions in \( U \) with \( q_1(0) = q_2(0) = 1 \). Also, Tuneski [30] obtained a sufficient condition for starlikeness of \( f \in A \) in terms of the quantity \( \frac{f''(z)f(z)}{(f'(z))^2} \). Recently, Shanmugam et al. [28] obtained sufficient conditions for the normalized analytic function \( f \in A \) to satisfy

\[
q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)
\]
and
\[ q_1(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} \prec q_2(z). \]

For functions \( f \in A(p) \) given by (1) and \( g \in A(p) \) given by
\[ g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \quad (p \in \mathbb{N}), \tag{4} \]
the Hadamard product (or convolution) of \( f \) and \( g \) is given by
\[ (f \ast g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g \ast f)(z). \tag{5} \]

Upon differentiating both sides of (5) \( j \)–times with respect to \( z \), we have
\[ (f \ast g)^{(j)}(z) = \delta(p; j) z^{p-j} + \sum_{k=p+1}^{\infty} \delta(k; j) a_k b_k z^{k-j}, \tag{6} \]
where
\[ \delta(p; j) = \frac{p!}{(p-j)!} \quad (p > j; p \in \mathbb{N}; j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{7} \]

For functions \( f, g \in A(p) \), Aouf et al. [6] (see also [7]) define the linear operator
\[ D_{\lambda, p}^n (f \ast g)^{(j)} : A(p) \to A(p) \]
by
\[ D_{\lambda, p}^n (f \ast g)^{(j)}(z) = \delta(p; j) z^{p-j} + \sum_{k=p+1}^{\infty} \left( \frac{p-j+\lambda(k-p)}{p-j} \right)^n \delta(k; j) a_k b_k z^{k-j} \quad (\lambda \geq 0; p > j; p \in \mathbb{N}; j, n \in \mathbb{N}_0; z \in U). \tag{8} \]

From (8), we can easily deduce that
\[ \frac{\lambda z}{p-j} \left( D_{\lambda, p}^n (f \ast g)^{(j)}(z) \right)' = D_{\lambda, p}^{n+1} (f \ast g)^{(j)}(z) - (1-\lambda) D_{\lambda, p}^n (f \ast g)^{(j)}(z) \quad (\lambda > 0; p > j; p \in \mathbb{N}; n, j \in \mathbb{N}_0; z \in U). \tag{9} \]

We observe that the linear operator \( D_{\lambda, p}^n (f \ast g)^{(j)}(z) \) reduces to several interesting many other linear operators considered earlier for different choices of \( j, n, \lambda \) and the function \( g \):
(i) For \( j = 0 \), \( D^n_{\lambda,p} (f \ast g)^{(j)} = D^n_{\lambda,p} (f \ast g) \), where the operator \( D^n_{\lambda,p} (f \ast g) \) 
(\( \lambda \geq 0, p \in \mathbb{N}, n \in \mathbb{N}_0 \)) was introduced and studied by Selvaraj et al. [26] (see also [10]) and \( D^n_{\lambda,1} (f \ast g) (z) = D^n_{\lambda} (f \ast g) (z) \), where the operator \( D^n_{\lambda} (f \ast g) \) 
was introduced by Aouf and Mostafa [9];

(ii) For
\[
g(z) = \frac{z^p}{1-z} \quad (p \in \mathbb{N}; z \in U)
\]  
(10)
we have \( D^n_{\lambda,p} (f \ast g)^{(j)} (z) = D^n_{\lambda,p} f^{(j)} (z) \), \( D^n_{\lambda,p} f^{(0)} (z) = D^n_{\lambda,p} f (z) \), where the operator \( D^n_{\lambda,p} \) is the \( p \)-valent Al-Oboudi operator which was introduced by El-Ashwah and Aouf [17], \( D^n_{1,p} f^{(j)} (z) = D^n_{p} f^{(j)} (z) \), where the operator \( D^n_{p} f^{(j)} \) 
\((p > j, p \in \mathbb{N}, n, j \in \mathbb{N}_0)\) was introduced and studied by Aouf [3,4] and \( D^n_{1,p} f^{(0)} \) 
\(= D^n_{p} f \), where the operator \( D^n_{p} \) is the \( p \)-valent Salagean operator which was 
introduced and studied by Kamali and Orhan [18] (see also [8]);

(iii) For
\[
g(z) = z^p + \sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p} \cdots (\alpha_q)_{k-p}}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p} (1)_{k-p}} \frac{z^k}{(z \in U)},
\]  
(11)
(for complex parameters \( \alpha_1, \ldots, \alpha_q \) and \( \beta_1, \ldots, \beta_s \) \( \beta_j \notin \mathbb{Z}_0 = \{0, -1, -2, \ldots\} \), 
\( j = 1, \ldots, s \); \( q \leq s + 1; p \in \mathbb{N}; q, s \in \mathbb{N}_0 \)) where \((v)_k\) is the Pochhammer symbol 
defined in terms to the Gamma function \( \Gamma \), by
\[
(v)_k = \frac{\Gamma(v+k)}{\Gamma(v)} = \begin{cases} 
1 & (k = 0), \\
\frac{v(v+1)(v+2)\ldots(v+k-1)}{(k \in \mathbb{N})} & (k \in \mathbb{N}).
\end{cases}
\]
we have \( D^n_{\lambda,p} (f \ast g)^{(j)} (z) = D^n_{\lambda,p} (H_{p,q,s}(\alpha_1) f)^{(j)} (z) \) and \( D^n_{\lambda,p} (f \ast g)^{(0)} (z) = 
H_{p,q,s}(\alpha_1) f(z) \), where the operator \( H_{p,q,s}(\alpha_1) \) is the Dziok-Srivastava operator 
which was introduced and studied by Dziok and Srivastava [15,16] and which contains in turn many interesting operators such as, \( H_{1,2,1}(a,1;c) = L(a,c) \), 
where the operator \( L(a,c) \) was introduced by Carlson and Shaffer [13];

(iv) For
\[
g(z) = z^p + \sum_{k=p+1}^{\infty} \left( p+l + \alpha(k-p) \right)^m \frac{z^k}{p+l} \quad (\alpha \geq 0; l \geq 0; p \in \mathbb{N}; m \in \mathbb{N}_0; z \in U),
\]  
(12)
we have \( D^n_{\lambda,p} (f \ast g)^{(j)} (z) = D^n_{\lambda,p} (I_p(m,\alpha,l) f)^{(j)} (z) \) and \( D^n_{\lambda,p} (f \ast g)^{(0)} (z) = 
I_p(m,\alpha,l) f(z) \), where the operator \( I_p(m,\alpha,l) \) was introduced and studied by Cătăs [14] which contains in turn many interesting operators such as, \( I_p(m,1,l) = I_p(m,l) \), where the operator \( I_p(m,l) \) was investigated by Kumar et al. [19];
(v) For
\[ g(z) = z^p + \frac{\Gamma(p + \alpha + \beta)}{\Gamma(p + \beta)} \sum_{k=p+1}^{\infty} \frac{\Gamma(k + \beta)}{\Gamma(k + \alpha + \beta)} z^k \]  
(\( \alpha \geq 0; \; p \in \mathbb{N}; \; \beta > -1; z \in U \))

we have \( D_{\lambda,p}^n (f * g)^{(j)} = D_{\alpha \beta}^n (Q_{\alpha \beta}^\gamma f)^{(j)} \) and \( D_{\lambda,p}^0 (f * g)^{(0)} = Q_{\alpha \beta}^\gamma f \), where the operator \( Q_{\alpha \beta}^\gamma \) was introduced and studied by Liu and Owa [20];

(vi) For \( j = 0 \) and \( g \) of the form (11) with \( p = 1 \), we have \( D_{\lambda,1}^n (f * g)(z) = D_{\lambda}^n (\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s)(z) \), where the operator \( D_{\lambda}^n (\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) \) was introduced and studied by Selvaraj and Karthikeyan [25];

(vii) For \( j = 0, p = 1 \) and
\[ g(z) = z + \sum_{k=2}^{\infty} \left[ \frac{\Gamma(k + 1) \Gamma(2 - m)}{\Gamma(k + 1 - m)} \right]^n z^k \]  
(\( n \in \mathbb{N}_0; 0 \leq m < 1; z \in U \))

we have \( D_{\lambda,1}^n (f * g)(z) = D_{\lambda}^n (f)(z) \), where the operator \( D_{\lambda}^n \) was introduced and studied by Al-Oboudi and Al-Amoudi [2].

In this paper, we will derive several subordination, superordination and sandwich results involving the operator \( D_{\lambda,p}^n (f * g)^{(j)} \).

2. Definitions and preliminaries

In order to prove our subordinations and superordinations, we need the following definition and lemmas.

**Definition 2.1** ([22]). Denote by \( Q \), the set of all functions \( f \) that are analytic and injective on \( \overline{U} \setminus E(f) \), where
\[ E(f) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\}, \]
and are such that \( f'(\zeta) \neq 0 \) for \( \zeta \in \partial U \setminus E(f) \).

**Lemma 2.2** ([22]). Let \( q \) be univalent in \( U \) and \( \theta \) and \( \varphi \) be analytic in a domain \( D \) containing \( q(U) \) with \( \varphi(w) \neq 0 \) when \( w \in q(U) \). Set
\[ \psi(z) = z q'(z) \varphi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + \psi(z). \]  
(14)

Suppose that

(i) \( \psi(z) \) is starlike univalent in \( U \),
(ii) \( \Re \left\{ \frac{zh'(z)}{\psi(z)} \right\} > 0 \) for \( z \in U \).

If \( \beta \) is analytic with \( \beta(0) = q(0) \), \( \beta(U) \subset D \) and

\[
\theta \left( \beta(z) \right) + z\beta'(z) \phi \left( \beta(z) \right) \prec \theta \left( q(z) \right) + zq'(z) \phi \left( q(z) \right),
\]

then \( \beta(z) \prec q(z) \) and \( q \) is the best dominant.

Lemma 2.3 ([11]). Let \( q \) be convex univalent in \( U \) and \( \theta \) and \( \phi \) be analytic in a domain \( D \) containing \( q(U) \). Suppose that 

(i) \( \Re \left\{ \frac{\theta'(q(z))}{\phi(q(z))} \right\} > 0 \) for \( z \in U \),

(ii) \( \Psi(z) = zq'(z) \phi(q(z)) \) is starlike univalent in \( U \). If \( \beta(z) \in H[q(0), 1] \cap Q \), with \( \beta(U) \subset D \), and \( \theta \left( \beta(z) \right) + z\beta'(z) \phi \left( \beta(z) \right) \) is univalent in \( U \) and

\[
\theta \left( q(z) \right) + zq'(z) \phi \left( q(z) \right) \prec \theta \left( \beta(z) \right) + zp'(z) \phi \left( \beta(z) \right),
\]

then \( q(z) \prec \beta(z) \) and \( q \) is the best subordinant.

Lemma 2.4 ([24]). The function \( q(z) = (1 - z)^{-2ab} \) \((a, b \in \mathbb{C}^* (\mathbb{C} \setminus \{0\}))\) is univalent in \( U \) if and only if \( |2ab - 1| \leq 1 \) or \( |2ab + 1| \leq 1 \).

3. Main Results

Unless otherwise mentioned, we assume throughout this paper that \( \eta, \gamma_i \in \mathbb{C} \) \((i = 1, 2, 3)\), \( \gamma_, \mu \in \mathbb{C}^* \), \( \lambda > 0 \), \( \delta(p; j) \) is given by (7), \( p > j, p \in \mathbb{N}, n, j \in \mathbb{N}_0 \) and the powers are understood as the principle values.

Theorem 3.1. Let \( q(z) \) be univalent in \( U \) with \( q(0) = 1 \), \( q(z) \neq 1 \) and let \( \frac{zq'(z)}{q(z)} \) be starlike in \( U \). Let \( f \in A(p) \) and assume that \( f \) and \( q \) satisfy the following conditions:

\[
\left[ \frac{D^n_{\lambda, p} (f * g_1)(j)(z)}{\delta(p; j) z^{p-j}} \right] \left[ \frac{\delta(p; j) z^{p-j}}{D^n_{\lambda, p} (f * g_2)(j)(z)} \right]^{\eta} \neq 0,
\]

and

\[
\Re \left\{ 1 + \frac{\gamma_2}{\gamma_4} q(z) + \frac{2\gamma_3}{\gamma_4} [q(z)]^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)} \right\} > 0 \ (z \in U). \]
If
\[
\gamma_1 + \gamma_2 \left[ D_{\lambda,p}^n (f \ast g_1)^{(j)} (z) \right] \frac{\delta (p; j) z^{p-j}}{D_{\lambda,p}^n (f \ast g_1)^{(j)} (z)} \right]^{\mu} \left[ \frac{\delta (p; j) z^{p-j}}{D_{\lambda,p}^n (f \ast g_2)^{(j)} (z)} \right]^\eta
\]
\[
+ \gamma_3 \left[ D_{\lambda,p}^n (f \ast g_1)^{(j)} (z) \right] \frac{\delta (p; j) z^{p-j}}{D_{\lambda,p}^n (f \ast g_2)^{(j)} (z)} \right]^{2\mu} \left[ \frac{\delta (p; j) z^{p-j}}{D_{\lambda,p}^n (f \ast g_2)^{(j)} (z)} \right]^{2\eta}
\]
\[
+ \gamma_4 \left( \frac{p - j}{\lambda} \right) \left[ D_{\lambda,p}^{n+1} (f \ast g_1)^{(j)} (z) \right] \frac{D_{\lambda,p}^n (f \ast g_1)^{(j)} (z)}{D_{\lambda,p}^n (f \ast g_1)^{(j)} (z)} - 1
\]
\[
+ \gamma_4 \left( \frac{p - j}{\lambda} \right) \left[ 1 - D_{\lambda,p}^{n+1} (f \ast g_2)^{(j)} (z) \right] \frac{D_{\lambda,p}^n (f \ast g_2)^{(j)} (z)}{D_{\lambda,p}^n (f \ast g_2)^{(j)} (z)}
\]
\[
< \gamma_1 + \gamma_2 q(z) + \gamma_3 [q(z)]^2 + \gamma_4 \frac{zq'(z)}{q(z)},
\]  (19)

then
\[
\left[ D_{\lambda,p}^n (f \ast g_1)^{(j)} (z) \right] \frac{\delta (p; j) z^{p-j}}{D_{\lambda,p}^n (f \ast g_2)^{(j)} (z)} \right]^{\mu} \left[ \frac{\delta (p; j) z^{p-j}}{D_{\lambda,p}^n (f \ast g_2)^{(j)} (z)} \right]^\eta < q(z)
\]  (20)

and \(q(z)\) is the best dominant.

**Proof.** Define a function \(\rho\) by
\[
\rho (z) = \left[ D_{\lambda,p}^n (f \ast g_1)^{(j)} (z) \right] \frac{\delta (p; j) z^{p-j}}{D_{\lambda,p}^n (f \ast g_2)^{(j)} (z)} \right]^{\mu} \left[ \frac{\delta (p; j) z^{p-j}}{D_{\lambda,p}^n (f \ast g_2)^{(j)} (z)} \right]^\eta (z \in U).
\]  (21)

Then the function \(\rho\) is analytic in \(U\) and \(\rho(0) = 1\). Therefore, differentiating (21) logarithmically with respect to \(z\) and using the identity (9) in the resulting equation, we have
\[
\gamma_1 + \gamma_2 \left[ D_{\lambda,p}^n (f \ast g_1)^{(j)} (z) \right] \frac{\delta (p; j) z^{p-j}}{D_{\lambda,p}^n (f \ast g_2)^{(j)} (z)} \right]^{\mu} \left[ \frac{\delta (p; j) z^{p-j}}{D_{\lambda,p}^n (f \ast g_2)^{(j)} (z)} \right]^\eta
\]
\[
+ \gamma_3 \left[ D_{\lambda,p}^n (f \ast g_1)^{(j)} (z) \right] \frac{\delta (p; j) z^{p-j}}{D_{\lambda,p}^n (f \ast g_2)^{(j)} (z)} \right]^{2\mu} \left[ \frac{\delta (p; j) z^{p-j}}{D_{\lambda,p}^n (f \ast g_2)^{(j)} (z)} \right]^{2\eta}
\]
Let \( q \in A \) be univalent in \( U \) with \( q(0) = 1 \), \( q(z) \neq 1 \) and \( \frac{zq'(z)}{q(z)} \) is starlike in \( U \). Let \( f \in A(\lambda) \) such that

\[
\left[ \frac{L(a, c) f(z)}{z} \right]^\mu \left[ \frac{z}{L(a + 1, c) f(z)} \right]^\eta \neq 0,
\]  

(22)
and suppose \( q \) satisfies (18). If

\[
\gamma_1 + \gamma_2 \left[ \frac{L(a,c) f(z)}{z} \right]^\mu \left[ \frac{z}{L(a+1,c) f(z)} \right]^\eta \\
+ \gamma_3 \left[ \frac{L(a,c) f(z)}{z} \right]^{2\mu} \left[ \frac{z}{L(a+1,c) f(z)} \right]^{2\eta} \\
+ \gamma_4 \mu a \left[ \frac{L(a+1,c) f(z)}{L(a,c) f(z)} - 1 \right] + \gamma_4 \eta (a+1) \left[ 1 - \frac{L(a+2,c) f(z)}{L(a+1,c) f(z)} \right]
\]

\[
< \gamma_1 + \gamma_2 q(z) + \gamma_3 [q(z)]^2 + \gamma_4 \frac{z q'(z)}{q(z)}
\]

then

\[
\left[ \frac{L(a,c) f(z)}{z} \right]^\mu \left[ \frac{z}{L(a+1,c) f(z)} \right]^\eta < q(z)
\]

and \( q(z) \) is the best dominant.

Putting \( q(z) = \frac{1+A z}{1+B z} (-1 < B < A < 1) \) in Corollary 3.2, we obtain the following corollary which improves the result of Shanmugam et al. [27, Corollary 3.2].

**Corollary 3.3.** Let \(-1 < B < A < 1\) and assume that

\[
\Re \left\{ \frac{\gamma_2}{\gamma_4} \left[ 1 + A z \right] + \frac{2 \gamma_3}{\gamma_4} \left[ 1 + A z \right]^2 + \frac{1 - A B z^2}{(1 + A z)(1 + B z)} \right\} > 0 \quad (z \in U),
\]

holds. If \( f \in \mathcal{A} \) such that (22) holds and satisfy the following subordination condition:

\[
\gamma_1 + \gamma_2 \left[ \frac{L(a,c) f(z)}{z} \right]^\mu \left[ \frac{z}{L(a+1,c) f(z)} \right]^\eta \\
+ \gamma_3 \left[ \frac{L(a,c) f(z)}{z} \right]^{2\mu} \left[ \frac{z}{L(a+1,c) f(z)} \right]^{2\eta} \\
+ \gamma_4 \mu a \left[ \frac{L(a+1,c) f(z)}{L(a,c) f(z)} - 1 \right] + \gamma_4 \eta (a+1) \left[ 1 - \frac{L(a+2,c) f(z)}{L(a+1,c) f(z)} \right]
\]

\[
< \gamma_1 + \gamma_2 \left[ \frac{1 + A z}{1 + B z} \right] + \gamma_3 \left[ \frac{1 + A z}{1 + B z} \right]^2 + \frac{\gamma_4 (A - B) z}{(1 + A z)(1 + B z)},
\]

then

\[
\left[ \frac{L(a,c) f(z)}{z} \right]^\mu \left[ \frac{z}{L(a+1,c) f(z)} \right]^\eta < \frac{1 + A z}{1 + B z}
\]

and the function \( \frac{1 + A z}{1 + B z} \) is the best dominant.
Putting \( q(z) = \left( \frac{1 + z}{1 - z} \right)^{\arg} (0 < \arg \leq 1) \) in Corollary 3.2, we obtain the following corollary which improves the result of Shanmugam et al. [27, Corollary 3.3].

**Corollary 3.4.** Assume that

\[
\Re \left\{ \gamma_2 \left( \frac{1 + z}{1 - z} \right)^{\arg} + \frac{2\gamma_3}{\gamma_4} \left( \frac{1 + z}{1 - z} \right)^{2\arg} + \frac{1 - 3z^2}{1 - z^2} \right\} > 0 \quad (z \in U),
\]

holds. If \( f \in A \) such that (22) holds and satisfy the following subordination condition:

\[
\gamma_1 + \gamma_2 \left[ \frac{L(a, c) f(z)}{z} \right]^{\mu} \left[ \frac{z}{L(a + 1, c) f(z)} \right]^{\eta} + \gamma_3 \left[ \frac{L(a, c) f(z)}{z} \right]^{2\mu} \left[ \frac{z}{L(a + 1, c) f(z)} \right]^{2\eta} + \gamma_4 \left[ \frac{L(a + 1, c) f(z)}{L(a, c) f(z)} - 1 \right] + \gamma_4 \eta (a + 1) \left[ 1 - \frac{L(a + 2, c) f(z)}{L(a + 1, c) f(z)} \right]
\]

\[
\approx \gamma_1 + \gamma_2 \left( \frac{1 + z}{1 - z} \right)^{\arg} + \gamma_3 \left( \frac{1 + z}{1 - z} \right)^{2\arg} + \frac{2\gamma_4 \vartheta z}{(1 - z)^2},
\]

then

\[
\left[ \frac{L(a, c) f(z)}{z} \right]^{\mu} \left[ \frac{z}{L(a + 1, c) f(z)} \right]^{\eta} \prec \left( \frac{1 + z}{1 - z} \right)^{\arg}
\]

and the function \( \left( \frac{1 + z}{1 - z} \right)^{\arg} \) is the best dominant.

Putting \( q(z) = e^{\mu Az} (|\mu A| < \pi) \) in Corollary 3.2, we obtain the following corollary which improves the result of Shanmugam et al. [27, Corollary 3.4].

**Corollary 3.5.** Assume that

\[
\Re \left\{ 1 + \frac{\gamma_2}{\gamma_4} e^{\mu Az} + \frac{2\gamma_3}{\gamma_4} e^{2\mu Az} \right\} > 0 \quad (z \in U),
\]

holds. If \( f \in A \) such that (22) holds and satisfy the following subordination condition:

\[
\gamma_1 + \gamma_2 \left[ \frac{L(a, c) f(z)}{z} \right]^{\mu} \left[ \frac{z}{L(a + 1, c) f(z)} \right]^{\eta} + \gamma_3 \left[ \frac{L(a, c) f(z)}{z} \right]^{2\mu} \left[ \frac{z}{L(a + 1, c) f(z)} \right]^{2\eta} + \gamma_4 \left[ \frac{L(a + 1, c) f(z)}{L(a, c) f(z)} - 1 \right] + \gamma_4 \eta (a + 1) \left[ 1 - \frac{L(a + 2, c) f(z)}{L(a + 1, c) f(z)} \right]
\]

\[
\approx \gamma_1 + \gamma_2 e^{\mu Az} + \gamma_3 e^{2\mu Az} + \gamma_4 \mu Az,
\]
\[
\left[ \frac{L(a,c) f(z)}{z} \right]^\mu \left[ \frac{z}{L(a+1,c) f(z)} \right]^\eta \prec e^{\mu A z}
\]

and the function \( e^{\mu A z} \) is the best dominant.

Taking \( \gamma_1 = p = \lambda = 1, \gamma_2 = \gamma_3 = n = j = \eta = 0, g_1 = z + \sum_{k=2}^{\infty} z^k \), \( q(z) = \frac{1}{(1-z)^{2ab}} (a,b \in \mathbb{C}^*) \), \( \mu = a \) and \( \gamma_4 = \frac{1}{ab} \) in Theorem 3.1, then combining this to gather with Lemma 2.4 we obtain the following corollary obtained by Obradović et al. [23, Theorem 1].

**Corollary 3.6.** Let \( a, b \in \mathbb{C}^* \) such that \(|2ab - 1| \leq 1 \) or \(|2ab + 1| \leq 1 \). Let \( f \in \mathcal{A} \) and suppose that \( \frac{f(z)}{z} \neq 0 \) (\( z \in U \)). If

\[
1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \prec \frac{1+z}{1-z}
\]

then

\[
\left( \frac{f(z)}{z} \right)^a \prec \frac{1}{(1-z)^{2ab}}
\]

and the function \( \frac{1}{(1-z)^{2ab}} \) is the best dominant.

**Remark 3.7.** For \( a = 1 \), Corollary 3.6 reduces to the recent result obtained by Srivastava and Lashin [29, Theorem 3].

Taking \( \gamma_1 = p = \lambda = 1, \gamma_2 = \gamma_3 = n = j = \eta = 0, g_1 = z + \sum_{k=2}^{\infty} z^k \), \( q(z) = (1-z)^{-2ab \cos \tau e^{-i\tau}} (a,b \in \mathbb{C}^*, |\tau| < \frac{\pi}{2}) \), \( \mu = a \) and \( \gamma_4 = \frac{e^{i\tau}}{ab \cos \tau} \) in Theorem 3.1, then combining this to gather with Lemma 2.4 we obtain the following corollary obtained by Aouf et al. [5, Theorem 1].

**Corollary 3.8.** Let \( a, b \in \mathbb{C}^*, |\tau| < \frac{\pi}{2} \) and suppose that \(|2ab \cos \tau e^{-i\tau} - 1| \leq 1 \) or \(|2ab \cos \tau e^{-i\tau} + 1| \leq 1 \). Let \( f \in \mathcal{A} \) and suppose that \( \frac{f(z)}{z} \neq 0 \) (\( z \in U \)). If

\[
1 + \frac{e^{i\tau}}{b \cos \tau} \left( \frac{zf'(z)}{f(z)} - 1 \right) \prec \frac{1+z}{1-z}
\]

then

\[
\left( \frac{f(z)}{z} \right)^a \prec (1-z)^{-2ab \cos \tau e^{-i\tau}}
\]

and the function \( (1-z)^{-2ab \cos \tau e^{-i\tau}} \) is the best dominant.
Theorem 3.9. Let \( q \) be convex univalent in \( U \) with \( q(0) = 1 \) and \( \frac{zq'(z)}{q(z)} \) is starlike in \( U \). Further assume that

\[
\Re \left( (\gamma_2 + 2\gamma_3 q(z)) \frac{q(z)q'(z)}{\gamma_4} \right) > 0.
\]

Let \( f \in A(p) \) such that

\[
0 \neq \left[ \frac{D^n_{\lambda,p} (f*g_1)(j)(z)}{\delta(p;j)z^{p-j}} \right]^{\mu} \left[ \frac{\delta(p;j)z^{p-j}}{D^n_{\lambda,p} (f*g_2)(j)(z)} \right]^{\eta} \in H[ q(0), 1] \cap Q.
\]

If

\[
\gamma_1 + \gamma_2 \left[ \frac{D^n_{\lambda,p} (f*g_1)(j)(z)}{\delta(p;j)z^{p-j}} \right]^{\mu} \left[ \frac{\delta(p;j)z^{p-j}}{D^n_{\lambda,p} (f*g_2)(j)(z)} \right]^{\eta} \\
+ \gamma_3 \left[ \frac{D^n_{\lambda,p} (f*g_1)(j)(z)}{\delta(p;j)z^{p-j}} \right]^{2\mu} \left[ \frac{\delta(p;j)z^{p-j}}{D^n_{\lambda,p} (f*g_2)(j)(z)} \right]^{2\eta} \\
+ \gamma_4 \mu \left( \frac{p-j}{\lambda} \right) \left[ \frac{D^{n+1}_{\lambda,p} (f*g_1)(j)(z)}{D^n_{\lambda,p} (f*g_1)(j)(z)} - 1 \right] \\
+ \gamma_4 \eta \left( \frac{p-j}{\lambda} \right) \left[ 1 - \frac{D^{n+1}_{\lambda,p} (f*g_2)(j)(z)}{D^n_{\lambda,p} (f*g_2)(j)(z)} \right]
\]

is univalent in \( U \) and satisfies the following superordination condition

\[
\gamma_1 + \gamma_2 q(z) + \gamma_3 [q(z)]^2 + \gamma_4 \frac{zq'(z)}{q(z)} < \gamma_1 + \gamma_2 \left[ \frac{D^n_{\lambda,p} (f*g_1)(j)(z)}{\delta(p;j)z^{p-j}} \right]^{\mu} \left[ \frac{\delta(p;j)z^{p-j}}{D^n_{\lambda,p} (f*g_2)(j)(z)} \right]^{\eta} \\
+ \gamma_3 \left[ \frac{D^n_{\lambda,p} (f*g_1)(j)(z)}{\delta(p;j)z^{p-j}} \right]^{2\mu} \left[ \frac{\delta(p;j)z^{p-j}}{D^n_{\lambda,p} (f*g_2)(j)(z)} \right]^{2\eta} \\
+ \gamma_4 \mu \left( \frac{p-j}{\lambda} \right) \left[ \frac{D^{n+1}_{\lambda,p} (f*g_1)(j)(z)}{D^n_{\lambda,p} (f*g_1)(j)(z)} - 1 \right]
\]
If starlike in $U$

Corollary 3.10. Let $q$ be convex univalent in $U$ with $q(0) = 1$ and $q' / q$ is starlike in $U$. Further assume that (23) holds. Let $f \in A(p)$ such that

$$
0 \neq \left[ \frac{L(a, c) f(z)}{z} \right]^\mu \left[ \frac{z}{L(a + 1, c) f(z)} \right]^\eta \in H[q(0), 1] \cap Q.
$$

If

$$
\begin{align*}
\gamma_1 + \gamma_2 & \left[ \frac{L(a, c) f(z)}{z} \right]^\mu \left[ \frac{z}{L(a + 1, c) f(z)} \right]^\eta \\
+ \gamma_3 & \left[ \frac{L(a, c) f(z)}{z} \right]^{2\mu} \left[ \frac{z}{L(a + 1, c) f(z)} \right]^{2\eta} \\
+ \gamma_4 & \mu \alpha \left[ \frac{L(a + 1, c) f(z)}{L(a, c) f(z)} - 1 \right] + \gamma_4 \eta (a + 1) \left[ 1 - \frac{L(a + 2, c) f(z)}{L(a + 1, c) f(z)} \right]
\end{align*}
$$

holds, then

$$
q(z) \prec \left[ \frac{D^n_{\lambda, p} (f \ast g_1)^{(j)} (z)}{D^n_{\lambda, p} (f \ast g_2)^{(j)} (z)} \right]^{\mu} \left[ \frac{\delta(p; j) z^{p-j}}{D^n_{\lambda, p} (f \ast g_2)^{(j)} (z)} \right]^{\eta}
$$

and $q$ is the best subordinant.

Proof. By setting

$$
\theta(w) = \gamma_1 + \gamma_2 w + \gamma_3 w^2 \text{ and } \varphi(w) = \frac{\gamma_4}{w},
$$

it can be easily observed that $\theta$ is analytic function in $\mathbb{C}$, $\varphi$ is analytic function in $\mathbb{C}^*$ and $\varphi(w) \neq 0$. From the assumption of Theorem 3.9, we see that

$$
\Re \left\{ \frac{\theta'(q(z))}{\varphi(q(z))} \right\} = \Re \left( \frac{\gamma_2 + 2\gamma_3 q(z)}{\gamma_4} \frac{q(z) q'(z)}{q(z)} \right) > 0 \text{ for } z \in U,
$$

Therefore, Theorem 3.9 now follows by applying Lemma 2.3. \hfill \Box

Taking $p = \lambda = 1$, $n = j = 0$, $g_1 = z + \sum_{k=2}^{\infty} \frac{(a)_k - 1}{(c)_{k-1}} z^k$, $g_2 = z + \sum_{k=2}^{\infty} \frac{(a+1)_k - 1}{(c)_{k-1}} z^k$, and $c \notin \mathbb{Z}_0$ in Theorem 3.9, we obtain the following corollary which improves the result of Shanmugam et al. [27, Theorem 3.11].
is univalent in $U$ and satisfies the following superordination condition

$$
\gamma_1 + \gamma_2 q(z) + \gamma_3 [q(z)]^2 + \gamma_4 \frac{zq^\prime(z)}{q(z)} < \gamma_1 + \gamma_2 \left[ \frac{L(a,c) f(z)}{z} \right]^\mu \left[ \frac{z}{L(a+1,c) f(z)} \right]^{\eta} + \gamma_3 \left[ \frac{L(a,c) f(z)}{z} \right]^{2\mu} \left[ \frac{z}{L(a+1,c) f(z)} \right]^{2\eta} + \gamma_4 \mu a \left[ \frac{L(a+1,c) f(z)}{L(a,c) f(z)} - 1 \right] + \gamma_4 \eta (a+1) \left[ 1 - \frac{L(a+2,c) f(z)}{L(a+1,c) f(z)} \right],
$$

holds, then

$$q(z) < \left[ \frac{L(a,c) f(z)}{z} \right]^\mu \left[ \frac{z}{L(a+1,c) f(z)} \right]^{\eta}$$

and $q$ is the best subordinant.

Combining Theorems 3.1 and 3.9, we get the following sandwich theorem for the linear operator $D^n_{\lambda,p} (f * g)^{(j)}$.

**Theorem 3.11.** Let $q_i$ be convex univalent in $U$ with $q_i(0) = 1$ and let $\frac{zq_i^\prime(z)}{q_i(z)}$ ($i = 1, 2$) be starlike in $U$. Suppose that $q_1$ satisfies (23) and $q_2$ satisfies (18). Let $f \in A(p)$ such that

$$0 \neq \left[ \frac{D^n_{\lambda,p} (f * g_1)^{(j)} (z)}{\delta(p;j) z^{p-j}} \right]^\mu \left[ \frac{\delta(p;j) z^{p-j}}{D^n_{\lambda,p} (f * g_2)^{(j)} (z)} \right]^{\eta} \in H[q(0), 1] \cap Q. \quad (28)$$

If

$$
\gamma_1 + \gamma_2 \left[ \frac{D^n_{\lambda,p} (f * g_1)^{(j)} (z)}{\delta(p;j) z^{p-j}} \right]^\mu \left[ \frac{\delta(p;j) z^{p-j}}{D^n_{\lambda,p} (f * g_2)^{(j)} (z)} \right]^{\eta} + \gamma_3 \left[ \frac{D^n_{\lambda,p} (f * g_1)^{(j)} (z)}{\delta(p;j) z^{p-j}} \right]^{2\mu} \left[ \frac{\delta(p;j) z^{p-j}}{D^n_{\lambda,p} (f * g_2)^{(j)} (z)} \right]^{2\eta} + \gamma_4 \mu \left( \frac{p-j}{\lambda} \right) \left[ \frac{D^{n+1}_{\lambda,p} (f * g_1)^{(j)} (z)}{D^n_{\lambda,p} (f * g_1)^{(j)} (z)} - 1 \right] + \gamma_4 \eta \left( \frac{p-j}{\lambda} \right) \left[ 1 - \frac{D^{n+1}_{\lambda,p} (f * g_2)^{(j)} (z)}{D^n_{\lambda,p} (f * g_2)^{(j)} (z)} \right],
$$

(29)
is univalent in $U$ and

$$
\gamma_1 + \gamma_2 q_1(z) + \gamma_3 [q_1(z)]^2 + \gamma_4 \frac{z q_1'(z)}{q_1(z)} < \gamma_1 + \gamma_2 \left[ \frac{D_{\lambda,p}^n(f \ast g_1)^{(j)}(z)}{\delta(p;j)z^{p-j}} \right]^{\mu} \left[ \frac{\delta(p;j)z^{p-j}}{D_{\lambda,p}^n(f \ast g_2)^{(j)}(z)} \right]^{\eta} \\
+ \gamma_3 \left[ \frac{D_{\lambda,p}^n(f \ast g_1)^{(j)}(z)}{\delta(p;j)z^{p-j}} \right]^{2\mu} \left[ \frac{\delta(p;j)z^{p-j}}{D_{\lambda,p}^n(f \ast g_2)^{(j)}(z)} \right]^{2\eta} \\
+ \gamma_4 \left( \frac{p-j}{\lambda} \right) \left[ \frac{D_{\lambda,p}^{n+1}(f \ast g_1)^{(j)}(z)}{D_{\lambda,p}^n(f \ast g_1)^{(j)}(z)} - 1 \right] \\
+ \gamma_4 \eta \left( \frac{p-j}{\lambda} \right) \left[ 1 - \frac{D_{\lambda,p}^{n+1}(f \ast g_2)^{(j)}(z)}{D_{\lambda,p}^n(f \ast g_2)^{(j)}(z)} \right] \\
< \gamma_1 + \gamma_2 q_2(z) + \gamma_3 [q_2(z)]^2 + \gamma_4 \frac{z q_2'(z)}{q_2(z)}
$$

(30)

holds, then

$$
q_1(z) < \left[ \frac{D_{\lambda,p}^n(f \ast g_1)^{(j)}(z)}{\delta(p;j)z^{p-j}} \right]^{\mu} \left[ \frac{\delta(p;j)z^{p-j}}{D_{\lambda,p}^n(f \ast g_2)^{(j)}(z)} \right]^{\eta} < q_2(z)
$$

(31)

and $q_1$ and $q_2$ are, respectively, the best subordinant and the best dominant.

Taking $p = \lambda = 1$, $n = j = 0$, $g_1 = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k$, $g_2 = z + \sum_{k=2}^{\infty} \frac{(a+1)_{k-1}}{(c)_{k-1}} z^k$, and $c \notin \mathbb{Z}_0^-$ in Theorem 3.11, we obtain the following corollary which improves the result of Shanmugam et al. [27, Theorem 3.12].

**Corollary 3.12.** Let $q_i$ be convex univalent in $U$ with $q_i(0) = 1$ and $\frac{z q_i'(z)}{q_i(z)}$ be starlike in $U$ for $i = 1, 2$. Suppose that $q_1$ satisfies (23) and $q_2$ satisfies (18). Let $f \in \mathcal{A}(p)$ such that

$$
0 \neq \left[ \frac{L(a,c)f(z)}{z} \right]^{\mu} \left[ \frac{z}{L(a+1,c)f(z)} \right]^{\eta} \in H[q(0), 1] \cap Q.
$$
If
\[
\gamma_1 + \gamma_2 \left[ \frac{L(a,c) f(z)}{z} \right]^\mu \left[ \frac{z}{L(a+1,c) f(z)} \right]^{\eta_1} + \gamma_3 \left[ \frac{L(a,c) f(z)}{z} \right]^{2\mu} \left[ \frac{z}{L(a+1,c) f(z)} \right]^{2\eta_1} + \gamma_4 \mu a \left[ \frac{L(a+1,c) f(z)}{L(a,c) f(z)} - 1 \right] + \gamma_4 (a+1) \left[ 1 - \frac{L(a+2,c) f(z)}{L(a+1,c) f(z)} \right]
\]
is univalent in \( U \) and
\[
\gamma_1 + \gamma_2 q_1(z) + \gamma_3 [q_1(z)]^2 + \gamma_4 \frac{z q_1'(z)}{q_1(z)},
\]
holds, then
\[
q_1(z) \prec \left[ \frac{L(a,c) f(z)}{z} \right]^\mu \left[ \frac{z}{L(a+1,c) f(z)} \right]^{\eta_1} \prec q_2(z)
\]
and \( q_1 \) and \( q_2 \) are, respectively, the best subordinant and the best dominant.

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